Perfect strategies for imitation and reflexive games

with
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(5) Reflexive games and operator system quotients

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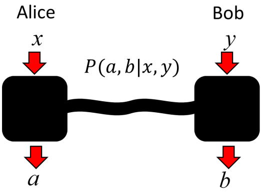
Non-signalling correlations

Let X, Y, A and B be finite sets.

Alice (resp. Bob) receives an input x (resp. y) drawn from the set X (resp. Y) and produces an output a (resp. b) from the set A (resp. B).



The statistics of the answers is observed.

Non-signalling correlations

Let $p(a,b|x,y)$ be the probability that the pair (a,b) is produced, given the input pair (x,y).

For a fixed (x,y), the tuple $(p(a,b|x,y))_{(a,b)\in A\times B}$ is a probability distribution on $A\times B$.

We assume that A and B do not communicate: expressed by the fact that the marginal distributions are well-defined:

$$p(a|x) = \sum_{b\in B} p(a,b|x,y), \quad p(b|y) = \sum_{a\in A} p(a,b|x,y).$$

A non-signalling (NS) correlation is a family

$$\{(p(a,b|x,y))_{(a,b)\in A\times B} : x\in X, y\in Y\}$$

of probability distributions satisfying these conditions.

Notation: $\mathcal{C}_{\rm ns}$.

Classes of NS correlations

A correlation p is called

- deterministic if there exist functions $f : X \to A$ and $g : Y \to B$ such that

$$p(a, b|x, y) = 1 \text{ if and only if } a = f(x) \text{ and } b = g(y).$$

  Notation: $\mathcal{C}_{\mathrm{det}}$.

- local if

$$p(a, b|x, y) = \sum_{k=1}^{m} \lambda_k p_1^k(a|x) p_2^k(b|y),$$

  for some probability distributions p_1^k, p_2^k, and non-negative reals $\lambda_1, \dots, \lambda_m$ with sum 1.

  Notation: $\mathcal{C}_{\mathrm{loc}}$.

Classes of NS correlations

- quantum if

$$p(a,b|x,y) = \langle (E_{x,a} \otimes F_{y,b})\eta, \eta \rangle,$$

where $(E_{x,a})_{a=1}^{c}$ (resp. $(F_{y,b})_{b=1}^{c}$) is a PVM on a finite dimensional Hilbert space.

Notation: $\mathcal{C}_{\mathrm{q}}$.

- spacially quantum if

$$p(a,b|x,y) = \langle (E_{x,a} \otimes F_{y,b})\eta, \eta \rangle,$$

where $(E_{x,a})_{a=1}^{c}$ (resp. $(F_{y,b})_{b=1}^{c}$) is a PVM on a (perhaps infinite dimensional) Hilbert space.

Notation: $\mathcal{C}_{\mathrm{qs}}$.

Classes of NS correlations

- approximately quantum if $p \in \overline{\mathcal{C}_{\rm q}}$.

  Notation: $\mathcal{C}_{\rm qa}$.

- quantum commuting if

$$p(a, b|x, y) := \langle E_{x,a} F_{y,b} \eta, \eta \rangle,$$

  where $(E_{x,a})_{a=1}^{c}$ and $(F_{y,b})_{b=1}^{c}$ are commuting POVM's on a Hilbert space.

  Notation: $\mathcal{C}_{\rm qc}$.

$$\mathcal{C}_{\rm det} \subseteq \mathcal{C}_{\rm loc} \subseteq \mathcal{C}_{\rm q} \subseteq \mathcal{C}_{\rm qs} \subseteq \mathcal{C}_{\rm qa} \subseteq \mathcal{C}_{\rm qc} \subseteq \mathcal{C}_{\rm ns}$$

Non-local games

A non-local game is a tuple $\mathcal{G} = (X, Y, A, B, \lambda)$, where

- X and Y are input sets for players Alice and Bob, respectively;
- A and B are output sets for players Alice and Bob, respectively, and
- $\lambda : X \times Y \times A \times B \to \{0, 1\}$ is a rule function.

Alice and Bob play cooperatively against a verifier R.

Upon receiving inputs (x, y), Alice and Bob reply with certain outputs (a, b).

They win if $\lambda(x, y, a, b) = 1$, and lose otherwise.

Alice and Bob know the rule function but are not allowed to communicate after the game commences. However, they are allowed to decide on a joint strategy beforehand.

Strategies for non-local games

A deterministic strategy is given by two functions $f : X \to A$ and $g : Y \to B$.

It is a perfect (or winning) strategy if

$$\lambda(x, y, f(x), g(y)) = 1, \quad x \in X, y \in Y.$$

However, Alice and Bob may employ randomness in their choices of outputs, deciding their outputs according to a probability distribution.

Let $p(a, b|x, y)$ be the probability that Alice and Bob give outputs (a, b) when they are given inputs (x, y).

Then $p(\cdot, \cdot|x, y)$ is a probability distribution for each pair (x, y), and since the players are not allowed to communicate, the family p is non-signalling.

Winning strategies for non-local games

Definition

Let $\mathrm{x} \in \{\mathrm{det}, \mathrm{loc}, \mathrm{q}, \mathrm{qs}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$.

A winning, or perfect, x-strategy for a game $\mathcal{G} = (X, Y, A, B, \lambda)$ is an element $p \in \mathcal{C}_{\mathrm{x}}$ such that

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0.$$

$\mathcal{C}_{\mathrm{x}}(\lambda)$: the set of all perfect x-strategies for $\mathcal{G} = (X, Y, A, B, \lambda)$.

The elements of $\mathcal{C}_{\mathrm{loc}}(\lambda)$ are called classical winning strategies.

Examples of non-local games

- The synchronicity game has $X = Y$, $A = B$, and $\lambda(x, y, a, b) = 0$ if and only if $x = y$ and $a \neq b$.

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- Let $G = (V(G), E(G))$ be a graph. The graph colouring game for G has $X = Y = V(G)$, $A = B$, and $\lambda(x, y, a, b) = 1$ unless

  either $x = y$ and $a \neq b$, *or* $(x, y) \in E(G)$ and $a = b$.

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- Let G and H be graphs. The graph homomorphism game $G \to H$ has $X = Y = V(G)$, $A = B = V(H)$, and $\lambda(x, y, a, b) = 1$ unless

$$\text{either } x = y \text{ and } a \neq b, \text{ or } (x, y) \in E(G) \text{ and } (a, b) \notin E(H).$$

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  either $x = y$ and $a \neq b$, *or* $(x, y) \in E(G)$ *and* $(a, b) \notin E(H)$.

- The graph isomorphism game $G \simeq H$.

Winning classically vs quantumly

Games that can be won using a quantum strategy only:

- Colouring the Hadamard graph Ω_N on $\{1, -1\}^N$ using N colours (*Avis-Hasegawa-Kikuchi-Sasaki*).

  (v, w) is an edge if and only if $v \cdot w = 0$.

- Filling successfully the Mermin-Peres magic square.

  Alice receives a row of a 3 by 3 square, Bob a column, and they are required to assign 1 or -1 to the entries, the product of Alice's entries being 1, the product of Bob's entries being -1, and assigning the same value to the common entry of the selected row and column.

The C*-algebra $\mathcal{A}(X, A)$

We let $\mathcal{A}(X, A)$ be the free product of $|X|$ copies of $\ell^\infty(A)$, amalgamated over the unit:

$$\mathcal{A}(X, A) = \underbrace{\ell^\infty(A) *_1 \cdots *_1 \ell^\infty(A)}_{|X| \text{ times}}.$$

Let $e_{x,a}$ be the canonical basis vectors of the x-th copy of $\ell^\infty(A)$.

Thus, $e_{x,a}$ is a projection in $\mathcal{A}(X, A)$ for all $x \in X$ and all $a \in A$, and

$$\sum_{a \in A} e_{x,a} = 1, \quad x \in X.$$

A dense spanning set for $\mathcal{A}(X, A)$ is formed by the words $e_{x_1,a_1} \dots e_{x_k,a_k}$.

Representations of synchronous correlations

Let $\tau : \mathcal{A}(X, A) \to \mathbb{C}$ be a trace. Setting

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}), \quad x, y \in X, a, b \in A,$$

we obtain a winning qc-strategy for the synchronicity game.

The converse is also true:

Representations of synchronous correlations

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The converse is also true:

Theorem (Severini-Stahlke-Paulsen-T-Winter)

If p is a winning qc-strategy for the synchronicity game then there exists a trace τ on $\mathcal{A}(X, A)$ such that

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}), \quad x, y \in X, a, b \in A.$$

Write $p = p_\tau$.

Representations of synchronous correlations

Theorem (Kim-Paulsen-Schafhauser, S-S-P-T-W)

Suppose that $p \in \mathcal{C}_{\rm qc}$ is a synchronous correlation.

- $p \in \mathcal{C}_{\rm qa}$ if and only if there exists an amenable trace $\tau : \mathcal{A}(X, A) \to \mathbb{C}$ with $p = p_\tau$;
- $p \in \mathcal{C}_{\rm q}$ if and only if there exists a finite dimensional *-representation $\pi : \mathcal{A}(X, A) \to \mathcal{M}$ and a trace $\tau' : \mathcal{M} \to \mathbb{C}$ such that $p = p_\tau$, where $\tau = \tau' \circ \pi$;
- $p \in \mathcal{C}_{\rm loc}$ if and only if there exists an abelian *-representation $\pi : \mathcal{A}(X, A) \to \mathcal{D}$ and a trace $\tau' : \mathcal{D} \to \mathbb{C}$ such that $p = p_\tau$, where $\tau = \tau' \circ \pi$.

Imitation games – definition

Definition

$\mathcal{G} = (X, Y, A, B, \lambda)$ is called an imitation game if

- for every $x \in X$ and $a, a' \in A$ with $a \neq a'$, there exists $y \in Y$ such that
$$\sum_{b \in B} \lambda(a, b|x, y)\, \lambda(a', b|x, y) = 0;$$
- for every $y \in Y$ and $b, b' \in B$ with $b \neq b'$, there exists $x \in X$ such that
$$\sum_{a \in A} \lambda(a, b|x, y)\, \lambda(a, b'|x, y) = 0.$$

Imitation games – definition

Set

$$E_{x,y} = \{(a,b) \in A \times B : \lambda(x,y,a,b) = 1\},$$

$$E^a_{x,y} = \{b \in B : \lambda(x,y,a,b) = 1\},$$

and

$$E^b_{x,y} = \{a \in A : \lambda(x,y,a,b) = 1\}.$$

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$$E_{x,y} = \{(a,b) \in A \times B : \lambda(x,y,a,b) = 1\},$$

$$E_{x,y}^{a} = \{b \in B : \lambda(x,y,a,b) = 1\},$$

and

$$E_{x,y}^{b} = \{a \in A : \lambda(x,y,a,b) = 1\}.$$

For imitation games,

- for all $x \in X$, and all possible answers $a \neq a'$ of Alice, $\exists\ y \in Y$ such that $E_{x,y}^{a} \cap E_{x,y}^{a'} = \emptyset$, and
- for all $y \in X$, and all possible answers $b \neq b'$ of Bob, $\exists\ x \in X$ such that $E_{x,y}^{b} \cap E_{x,y}^{b'} = \emptyset$.

Imitation games – definition

Set

$$E_{x,y} = \{(a,b) \in A \times B : \lambda(x,y,a,b) = 1\},$$

$$E^a_{x,y} = \{b \in B : \lambda(x,y,a,b) = 1\},$$

and

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- for all $x \in X$, and all possible answers $a \neq a'$ of Alice, $\exists\ y \in Y$ such that $E^a_{x,y} \cap E^{a'}_{x,y} = \emptyset$, and
- for all $y \in X$, and all possible answers $b \neq b'$ of Bob, $\exists\ x \in X$ such that $E^b_{x,y} \cap E^{b'}_{x,y} = \emptyset$.

Thus, the answers Bob gives when he is asked y are “determined” by the answers of Alice when asked x, and vice versa.

Imitation games – examples

- Every synchronous game is an imitation game. Indeed, $E^a_{x,x} = \{a\}$, and so, given $x \in X$, we can take $y = x$, having $E^a_{x,x} \cap E^{a'}_{x,x} = \emptyset$.

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- $\mathcal{G}$ is called unique if, for every $(x, y) \in X \times Y$, the set $E_{x,y}$ (of "allowed" pairs (a, b)) is the graph of a bijection $f : A \to B$. Thus, $E^a_{x,y} = \{f(a)\}$ and hence every unique game is an imitation game.

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- $\mathcal{G}$ is called a mirror game if there exist functions $\xi : X \to Y$ and $\eta : Y \to X$ such that

$$E^a_{x,\xi(x)} \cap E^{a'}_{x,\xi(x)} = \emptyset, \quad x \in X, \; a \neq a',$$

and

$$E^b_{\eta(y),y} \cap E^{b'}_{\eta(y),y} = \emptyset, \quad y \in Y, \; b \neq b'.$$

Every mirror game is an imitation game.

Imitation games – examples

- **Cleve-Mittal**: a binary constraint system (BCS) game has $Y = \{v_1, \dots, v_n\}$, a set of variables that take values in $\{1, -1\}$.

  A constraint is an equation $f((v)_{v\in V}) = 1$, where $V \subseteq Y$ and $f : \{1, -1\}^V \to \{1, -1\}$ is a function.

  X is a set of constraints, say (V_x, f_x), $x \in X$.

  $A = \cup_{x\in X}\{1, -1\}^{V_x}$ and $B = \{1, -1\}$.

  Given $x \in X$, $y \in Y$, $a \in A$ and $b \in B$, writing $a = (a_z)_{z\in V}$, we let $\lambda(x, y, a, b) = 1$ precisely when

$$V = V_x, \quad f_x(a) = 1 \quad \text{and} \quad a_y = b.$$

Every BCS game is an imitation game.

Imitation games – examples

- Let V be a set of n variables, and C be a finite set of possible values of these variables.

  In a variable assignment game upon V and C,
  - X and Y are sets of subsets of V;
  - for every $v \in V$ there exist $x \in X$ and $y \in Y$ with $v \in x \cap y$;
  - $A = B = \cup_{W \subseteq V} C^W$.
  - $\lambda(x, y, (a_v)_{v \in W}, (b_v)_{v \in W'}) = 1$ implies that $x = W$, $y = W'$ and $a_v = b_v$ for every $v \in x \cap y$.

Example: Peres-Mermin square

Every variable assignment game is an imitation game.

The C*-algebra of an imitation game

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be an imitation game.

The C*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$ is the universal unital C*-algebra generated by elements $(p_{x,a})_{x \in X, a \in A}$ and $(q_{y,b})_{y \in Y, b \in B}$ satisfying the following relations:

1. for every $x \in X$, $(p_{x,a})_{a \in A}$ are pairwise orthogonal projections with $\sum_{a \in A} p_{x,a} = 1$;
2. for every $y \in Y$, $(q_{y,b})_{b \in B}$ are pairwise orthogonal projections with $\sum_{b \in A} q_{y,b} = 1$;
3. If $\lambda(x, y, a, b) = 0$ then $p_{x,a} q_{y,b} = 0$.

Generalises the C*-algebra of a synchronous game (**Ortiz-Paulsen, Helton-Meyer-Paulsen-Satriano**).

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Description in special cases?

The C*-algebra of a variable assignment game

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a variable assignment game with a set of variables V and a set of values C.

Let $\mathfrak{C}(\mathcal{G})$ be the universal C*-algebra generated by projections $e_{v,c}$, with $v \in V$, $c \in C$, and with relations

1. $\sum_{c \in C} e_{v,c} = 1$, $v \in V$;
2. If $v, w \in x$ for some $x \in X$ or $v, w \in y$ for some $y \in Y$, then $e_{v,c} e_{w,d} = e_{w,d} e_{v,c}$ for all c and d;
3. If $\lambda(x, y, (a_v)_{v \in x}, (b_w)_{w \in y}) = 0$ then

$$\left(\prod_{v \in x} e_{v,a_v}\right)\left(\prod_{w \in y} e_{w,b_w}\right) = 0.$$

The C*-algebra of a variable assignment game

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a variable assignment game with a set of variables V and a set of values C. Then $C^*(\mathcal{G}) \cong \mathfrak{C}(\mathcal{G})$ canonically.

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Proof

Consider the assignment

$$p_{x,(a_v)_{v\in V}} \mapsto \begin{cases} \prod_{v\in x} e_{v,a_v} & \text{if } x = V, \\ 0 & \text{otherwise} \end{cases}$$

and

$$q_{y,(b_v)_{v\in V}} \mapsto \begin{cases} \prod_{v\in y} e_{v,a_v} & \text{if } y = V, \\ 0 & \text{otherwise} \end{cases}.$$

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$$q_{y,(b_v)_{v\in V}} \mapsto \begin{cases} \prod_{v\in y} e_{v,a_v} & \text{if } y = V, \\ 0 & \text{otherwise} \end{cases}.$$

This assignment extends to a *-homomorphism $\pi : C^*(\mathcal{G}) \to \mathfrak{C}(\mathcal{G})$.

The C*-algebra of a variable assignment game

Proof continued

Suppose $p_{x,(a_v)_{v\in x}}$ and $q_{y,(b_v)_{v\in y}}$ are the canonical generators of $C^*(\mathcal{G})$. For $x \in X$, $v \in x$, $y \in Y$, $w \in y$, and $c \in C$, let

$$a^x_{v,c} = \sum_{a\in C^x, a_v = c} p_{x,a}, \quad b^y_{w,c} = \sum_{b\in C^y, b_w = c} q_{y,b}.$$

Note $\sum_{c\in C} a^x_{v,c} = \sum_{d\in C} b^y_{w,d} = 1, \ \ v \in x, \ w \in y$.

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Note $\sum_{c\in C} a^x_{v,c} = \sum_{d\in C} b^y_{w,d} = 1, \ \ v \in x, \ w \in y$.

Since we have that $p_{x,a}q_{y,b} = 0$ whenever $a_v \neq b_v$, we have that $a^x_{v,c}b^y_{v,d} = 0$ if $v \in x \cap y$ and $c \neq d$. So

$$\sum_{c\in C} a^x_{v,c}b^y_{v,c} = 1.$$

The C*-algebra of a variable assignment game

Proof continued

Thus, if $\xi \in \mathcal{H}$ is a unit vector, then

$$\sum_{c \in C} \langle \xi | a^x_{v,c} b^y_{v,c} | \xi \rangle = \sum_{c \in C} \langle \xi | a^*_{v,c} a_{v,c} | \xi \rangle = \sum_{c \in C} \langle \xi | b^*_{v,c} b_{v,c} | \xi \rangle = 1.$$

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So $\beta = (b^y_{v,c}\xi)_{c \in C}$ and $\alpha = (a^x_{v,c}\xi)_{c \in C}$ are unit vectors in $\oplus_{c \in C} \mathcal{H}$ with $\langle \alpha | \beta \rangle = 1$; thus $\alpha = \beta$.

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Hence $a^x_{v,c} = b^y_{v,c}$ for every $v \in x \cap y$ and $c \in C$.

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Proof continued

Thus, if $\xi \in \mathcal{H}$ is a unit vector, then

$$\sum_{c \in C} \langle \xi | a^x_{v,c} b^y_{v,c} | \xi \rangle = \sum_{c \in C} \langle \xi | a^*_{v,c} a_{v,c} | \xi \rangle = \sum_{c \in C} \langle \xi | b^*_{v,c} b_{v,c} | \xi \rangle = 1.$$

So $\beta = (b^y_{v,c}\xi)_{c \in C}$ and $\alpha = (a^x_{v,c}\xi)_{c \in C}$ are unit vectors in $\oplus_{c \in C} \mathcal{H}$ with $\langle \alpha | \beta \rangle = 1$; thus $\alpha = \beta$.

Hence $a^x_{v,c} = b^y_{v,c}$ for every $v \in x \cap y$ and $c \in C$.

Thus $a^x_{v,c} = a^{x'}_{v,c} = a_{v,c}$ for any $x, x' \in X$, $v \in x \cap x'$ and $c \in C$. Hence the map

$$e_{v,c} \mapsto a_{v,c}$$

extends to a *-hom. $\rho : \mathfrak{C}(\mathcal{G}) \to C^*(\mathcal{G})$ with $\rho \circ \pi = \pi \circ \rho = \mathrm{id}$.

Linear BCS games

Cleve-Liu-Slofstra: BCS $\mathcal{S}$ with constraints $f:\{1,-1\}^W \to \{1,-1\}$ of the form

$$f\left((\lambda_v)_{v\in W}\right) = (-1)^\rho \prod_{v\in W} \lambda_v, \quad \text{where } \rho \in \{0,1\}.$$

The solution group $\Gamma(\mathcal{S})$ associated to such a linear BCS is generated by involutions $u_1,\dots,u_n,J$ subject to the relations:

- J commutes with $u_1,\dots,u_n$;
- u_v, u_w commute whenever the constraint $(-1)^\rho \prod_{i\in x} \lambda_i = 1$ belongs to the system with $v,w\in x$, in which case $J^\rho \prod_{i\in x} u_i = 1$.

Let $\mathcal{G}_\mathcal{S}$ be the corresponding BCS game.

Proposition

$$C^*(\mathcal{G}_\mathcal{S}) \cong C^*(\Gamma(\mathcal{S}))/\langle J+1\rangle.$$

Quantum commuting strategies for imitation games

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be an imitation game and

$$p : A \times B \times X \times Y \to [0,1]$$

be a non-signalling correlation. The following are equivalent:

- $p \in \mathcal{C}_{\mathrm{qc}}(\lambda)$;
- $C^*(\mathcal{G})$ is non-zero, and there exists a tracial state

$$\tau : C^*(\mathcal{G}) \to \mathbb{C}$$

  such that

$$p(a, b|x, y) = \tau(p_{x,a} q_{y,b}), \quad \text{for all } x, y, a, b.$$

Quantum spacial strategies for imitation games

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be an imitation game and

$$p : A \times B \times X \times Y \to [0, 1]$$

be a non-signalling correlation. The following are equivalent:

- $p \in \mathcal{C}_{\rm qs}(\lambda)$;
- $p \in \mathcal{C}_{\rm q}(\lambda)$;
- $C^*(\mathcal{G})$ is non-zero, and there exists a finite dimensional C*-algebra $\mathcal{M}$ with a tracial state τ and a unital *-homomorphism $\pi : C^*(\mathcal{G}) \to \mathcal{M}$ such that

$$p(a, b|x, y) = (\tau \circ \pi)(p_{x,a} q_{y,b}).$$

Ingredients of the proof

- A spacial winning strategy for $\mathcal{G}$ has the form

$$p(a,b|x,y) = \langle (P_{x,a} \otimes Q_{y,b})\xi, \xi \rangle,$$

for some PVM's $(P_{x,a})_{a\in A}$ and $(Q_{y,b})_{b\in B}$ on H and K and a unit vector $\xi \in H \otimes K$.

- Use Schmidt decomposition to write

$$\xi = \sum_{i=1}^{\infty} \alpha_i \phi_i \otimes \psi_i,$$

where $(\phi_i)_{i\in\mathbb{N}}$ and $(\psi_i)_{i\in\mathbb{N}}$ are orthonormal families.

- For a given α, set $I_\alpha = \{i : \alpha_i = \alpha\}$, $H_\alpha = \mathrm{span}\{\phi_i : i \in I_\alpha\}$, $K_\alpha = \mathrm{span}\{\psi_i : i \in I_\alpha\}$.
- for $x \in X, b \in B, y \in Y$, let

$$\Pi^x_{y,b} = \sum_{a\in A, \lambda(x,y,a,b)=1} P_{x,a}.$$

Ingredients of the proof

- Show that $(\Pi^x_{y,b} \otimes I)\xi = (I \otimes Q_{y,b})\xi$.
- Show that $\Pi^x_{y,b}$ leave H_α invariant, $P_{x,a}$ leave H_α invariant, and

$$H_\alpha = \mathrm{span}\{\Pi^x_{y,b}\phi_i : i \in I_\alpha, y, b, x\} = \mathrm{span}\{P_{x,a}\phi_i : i \in I_\alpha, x, a\}.$$

- Letting $\Pi^{x,\alpha}_{y,b}$ be the restriction of $\Pi^x_{y,b}$ to H_α, and similarly for $P^\alpha_{x,a}$, show that $\Pi^{x,\alpha}_{y,b}$ does not depend on x, so set $\Pi^\alpha_{y,b} = \Pi^{x,\alpha}_{y,b}$.
- For each α, the families $\{P^\alpha_{x,a}\}$ and $\{\Pi^\alpha_{y,b}\}$ determine a *-representation π_α of $C^*(\mathcal{G})$ into $\mathcal{B}(\mathbb{C}^{|I_\alpha|})$.
- In addition,

$$p(a, b|x, y) = \sum_\alpha \mu_\alpha(\tau_\alpha \circ \pi_\alpha)(P^\alpha_{x,a}\Pi^\alpha_{y,b}).$$

- Use the fact that in finite dimensional vector spaces every infinite convex combination of vectors is a finite one.

Local strategies for imitation games

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be an imitation game and

$$p : A \times B \times X \times Y \to [0, 1]$$

be a non-signalling correlation. The following are equivalent:

- $p \in \mathcal{C}_{\rm loc}(\lambda)$;
- $C^*(\mathcal{G})$ is non-zero, and there exists a finite dimensional abelian C*-algebra $\mathcal{D}$ with a tracial state τ and a unital *-homomorphism $\pi : C^*(\mathcal{G}) \to \mathcal{D}$ such that

$$p(a, b|x, y) = (\tau \circ \pi)(p_{x,a} q_{y,b}).$$

The operator system $\mathcal{S}(X, A)$

What can we say for general games?

The operator system $\mathcal{S}(X, A)$

What can we say for general games?

Recall that $\mathcal{A}(X, A)$ is the universal C*-algebra generated by projections $e_{x,a}$, $x \in X$, $a \in A$, subject to the relations

$$\sum_{a \in A} e_{x,a} = 1, \quad x \in X.$$

We define

$$\mathcal{S}_{X,A} = \mathrm{span}\{e_{x,a} : x \in X, a \in A\}.$$

$\mathcal{S}_{X,A}$ is an operator system, its matrix order structure being inherited from $\mathcal{A}(X, A)$.

Reason for passing to $\mathcal{S}_{X,A}$: richer tensor theory.

Tensor products of operator systems

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\mathcal{S} \otimes \mathcal{T}$ be the vector space tensor product.

- The minimal tensor product: $\mathcal{S} \otimes_{\min} \mathcal{T} \subseteq \mathcal{B}(H \otimes K)$.
- The commuting tensor product: $X \in M_n(\mathcal{S} \otimes_{\rm c} \mathcal{T})^+$ if $(\phi \cdot \psi)^{(n)}(X) \geq 0$ for all cp $\phi : \mathcal{S} \to \mathcal{B}(H)$ and $\psi : \mathcal{T} \to \mathcal{B}(H)$ with commuting ranges.

  Here $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$.
- The maximal tensor product: $M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ is the Archimedeanisation of the cone of $A^*(X \otimes Y)A$, where $X \in M_k(\mathcal{S})^+$, $Y \in M_l(\mathcal{T})^+$, $A \in M_{kl,n}(\mathbb{C})$.

$\mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{S} \otimes_{\rm c} \mathcal{T} \to \mathcal{S} \otimes_{\min} \mathcal{T}$ completely positive.

Winning strategies for general non-local games

For $s \in (\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^{\rm d}$, set

$$p_s(a,b|x,y) = s(e_{x,a} \otimes e_{y,b}), \quad (x,y) \in X \times Y, (a,b) \in A \times B.$$

The collection p_s is non-signalling.

Conversely, given a non-signalling p, let $s_p \in (\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^{\rm d}$ be given by

$$s_p(e_{x,a} \otimes e_{y,b}) = p(a,b|x,y), \quad (x,y) \in X \times Y, (a,b) \in A \times B.$$

$p \to s_p$ is a bijection between $(\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^{\rm d}$ and the set of all non-signalling collections on (X, Y, A, B).

Winning strategies for general non-local games

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a non-local game.

$$J(\lambda) = \mathrm{span}\{e_{x,a} \otimes e_{y,b} : \lambda(x, y, a, b) = 0\} \subseteq \mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B}.$$

For $\tau \in \{\max, \mathrm{c}, \min\}$ and $J \subseteq \mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B}$, let

$$\mathcal{P}_\tau(J) = \{s \in (\mathcal{S}_{X,A} \otimes_\tau \mathcal{S}_{Y,B})^d : \text{ a state with } J \subseteq \ker(s)\}.$$

Theorem

The map $p \to s_p$ is a continuous affine isomorphism between

(i) $\mathcal{C}_{\mathrm{ns}}(\lambda)$ and $\mathcal{P}_{\max}(J(\lambda))$;

(ii) $\mathcal{C}_{\mathrm{qc}}(\lambda)$ and $\mathcal{P}_{\mathrm{c}}(J(\lambda))$;

(iii) $\mathcal{C}_{\mathrm{qa}}(\lambda)$ and $\mathcal{P}_{\min}(J(\lambda))$;

$\rightsquigarrow$ a complete description of the classes of non-signalling correlations (trivial game) via states on op. sys. tensor products.

Harder games

For $\lambda : X \times Y \times A \times B \to \{0, 1\}$, let

$$N(\lambda) = \{(x, y, a, b) : \lambda(x, y, a, b) = 0\}.$$

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If $\mathcal{G}_1 = (X, Y, A, B, \lambda_1)$ and $\mathcal{G}_2 = (X, Y, A, B, \lambda_2)$ are games, we say that

$\mathcal{G}_1$ is harder than $\mathcal{G}_2$ if $\lambda_1 \leq \lambda_2$, that is, if $N(\lambda_2) \subseteq N(\lambda_1)$.

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$$\mathcal{G}_1 \text{ is harder than } \mathcal{G}_2 \text{ if } \lambda_1 \leq \lambda_2, \text{ that is, if } N(\lambda_2) \subseteq N(\lambda_1).$$

For $\Sigma \subseteq \mathcal{C}_{\rm ns}$, let

$$\lambda_\Sigma : X \times Y \times A \times B \to \{0,1\}$$

be defined by

$$N(\lambda_\Sigma) = \cap_{p \in \Sigma} N(p).$$

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$$N(\lambda_\Sigma) = \cap_{p \in \Sigma} N(p).$$

λ_Σ is the rule function of the hardest game for which every element of Σ is a winning strategy.

Winning harder games with no extra effort

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a game. Set $\lambda_{\rm x} = \lambda_{\mathcal{C}_{\rm x}(\lambda)}$; thus,

$$\lambda_{\rm x}(x, y, a, b) = 0 \iff p(a, b|x, y) = 0 \text{ for every } p \in \mathcal{C}_{\rm x}(\lambda).$$

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Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a game. Set $\lambda_{\rm x} = \lambda_{\mathcal{C}_{\rm x}(\lambda)}$; thus,

$$\lambda_{\rm x}(x, y, a, b) = 0 \iff p(a, b|x, y) = 0 \text{ for every } p \in \mathcal{C}_{\rm x}(\lambda).$$

Note the inequalities

$$\lambda_{\rm loc} \leq \lambda_{\rm q} \leq \lambda_{\rm qs} \leq \lambda_{\rm qa} \leq \lambda_{\rm qc} \leq \lambda_{\rm ns} \leq \lambda.$$

Set

$$\mathrm{Ref}_{\rm x}(\mathcal{G}) = (X, Y, A, B, \lambda_{\rm x})$$

and call it the reflexive ${\rm x}$-cover of $\mathcal{G}$.

Call $\mathcal{G}$ ${\rm x}$-reflexive if $\mathrm{Ref}_{\rm x}(\mathcal{G}) = \mathcal{G}$.

Example

Consider the graph colouring game for the graph $G = \{(1, 2), (2, 3), (3, 4)\}$. Then every 2-colouring of G is also a 2-colouring of the 4-cycle.

Winning strategies for reflexive games

Theorem

The spaces $J_{\rm x}(\lambda)$ are kernels, and

(i) the winning strategies for $\mathrm{Ref}_{\rm ns}(\mathcal{G})$ are in one-to-one correspondence with the states of $(\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})/J_{\max}(\lambda)$;

(ii) the winning strategies for $\mathrm{Ref}_{\rm qc}(\mathcal{G})$ are in one-to-one correspondence with the states of $(\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B})/J_{\rm c}(\lambda)$;

(iii) the winning strategies for $\mathrm{Ref}_{\rm qa}(\mathcal{G})$ are in one-to-one correspondence with the states of $(\mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B})/J_{\min}(\lambda)$.

Mirror games

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a game. Recall

$E^a_{x,y} = \{b \in B : \lambda(x, y, a, b) = 1\}$ and $E^b_{x,y} = \{a \in A : \lambda(x, y, a, b) = 1\}$.

$\mathcal{G}$ is a mirror game if there exist functions

$$\xi : X \to Y \quad \text{and} \quad \eta : Y \to X$$

such that

$$E^a_{x,\xi(x)} \cap E^{a'}_{x,\xi(x)} = \emptyset, \quad x \in X, \ a \neq a',$$

and

$$E^b_{\eta(y),y} \cap E^{b'}_{\eta(y),y} = \emptyset, \quad y \in Y, \ b \neq b'.$$

Quantum commuting strategies revisited

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a mirror game, $p \in \mathcal{C}_{\rm qc}(\lambda)$ and $s \in S(\mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B))$ be such that $p = p_s$. Then

(i) the functional $\tau : \mathcal{A}(X, A) \to \mathbb{C}$ given by $\tau(z) = s(z \otimes 1)$, $z \in \mathcal{A}(X, A)$, is a tracial state, and

(ii) there exists a set $\mathcal{Q} = \{q_{y,b} : y \in Y, b \in B\}$ of projections in $\mathcal{A}(X, A)$ such that $\sum_{b \in B} q_{y,b} = 1$ for all $y \in Y$, and

$$p(a, b|x, y) = \tau(e_{x,a} q_{y,b}), \quad x \in X, y \in Y, a \in A, b \in B.$$

Quantum commuting strategies revisited

Theorem

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a mirror game, $p \in \mathcal{C}_{\rm qc}(\lambda)$ and $s \in S(\mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B))$ be such that $p = p_s$. Then

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$$p(a, b|x, y) = \tau(e_{x,a} q_{y,b}), \quad x \in X, y \in Y, a \in A, b \in B.$$

For $s \in S(\mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B))$ we get precisely amenable traces.

A Hilbert-space-free proof

We may assume that

$$\cup_{a\in A} E^{a}_{x,\xi(x)} = B \text{ and } \cup_{b\in B} E^{b}_{\eta(y),y} = A, \quad x \in X, y \in Y.$$

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For $x \in X, y \in Y, a \in A$ and $b \in B$, let

$$p_{x,a} = \sum_{b\in E^a_{x,\xi(x)}} f_{\xi(x),b}, \quad q_{y,b} = \sum_{a\in E^b_{\eta(y),y}} e_{\eta(y),a}.$$

A Hilbert-space-free proof

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For $u_1, u_2 \in \mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B)$, write

$$u_1 \sim u_2 \text{ if } s(u_1 - u_2) = 0.$$

Then $\sim$ is an equivalence relation.

Proof continued

Fix $x \in X$ and $a \in A$. Then

$$\begin{aligned} s(e_{x,a} \otimes 1) &= \sum_{b \in B} s(e_{x,a} \otimes f_{\xi(x),b}) = \sum_{b \in E^a_{x,\xi(x)}} s(e_{x,a} \otimes f_{\xi(x),b}) \\ &= s(e_{x,a} \otimes p_{x,a}). \end{aligned}$$

Proof continued

Fix $x \in X$ and $a \in A$. Then

$$s(e_{x,a} \otimes 1) = \sum_{b \in B} s(e_{x,a} \otimes f_{\xi(x),b}) = \sum_{b \in E^a_{x,\xi(x)}} s(e_{x,a} \otimes f_{\xi(x),b})$$
$$= s(e_{x,a} \otimes p_{x,a}).$$

If $a' \neq a$ then

$$E^{a'}_{x,\xi(x)} \cap E^a_{x,\xi(x)} = \emptyset$$

so $s(e_{x,a'} \otimes f_{\xi(x),b}) = 0$ whenever $b \in E^a_{x,\xi(x)}$.
Thus

$$s(e_{x,a'} \otimes p_{x,a}) = \sum_{b \in E^a_{x,\xi(x)}} s(e_{x,a'} \otimes f_{\xi(x),b}) = 0.$$

Proof continued

$$\implies \quad s(1 \otimes p_{x,a}) = \sum_{a' \in A} s(e_{x,a'} \otimes p_{x,a}) = s(e_{x,a} \otimes p_{x,a}).$$

Proof continued

$$\implies \quad s(1 \otimes p_{x,a}) = \sum_{a' \in A} s(e_{x,a'} \otimes p_{x,a}) = s(e_{x,a} \otimes p_{x,a}).$$

$$\implies \quad e_{x,a} \otimes 1 \sim e_{x,a} \otimes p_{x,a} \sim 1 \otimes p_{x,a}, \quad x \in X, a \in A.$$

Proof continued

$$\Longrightarrow \quad s(1 \otimes p_{x,a}) = \sum_{a' \in A} s(e_{x,a'} \otimes p_{x,a}) = s(e_{x,a} \otimes p_{x,a}).$$

$$\Longrightarrow \quad e_{x,a} \otimes 1 \sim e_{x,a} \otimes p_{x,a} \sim 1 \otimes p_{x,a}, \quad x \in X, a \in A.$$

Set $h_{x,a} = e_{x,a} \otimes 1 - 1 \otimes p_{x,a}$. Then $h_{x,a} = h_{x,a}^*$ and

$$h_{x,a}^2 = e_{x,a} \otimes 1 - e_{x,a} \otimes p_{x,a} - e_{x,a} \otimes p_{x,a} + 1 \otimes p_{x,a};$$

thus,

$$h_{x,a}^2 \sim 0.$$

Proof continued

The Cauchy-Schwarz inequality implies

$$uh_{x,a} \sim 0 \text{ and } h_{x,a}u \sim 0, \quad x \in X, a \in A$$

for every $u \in \mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B)$.

Proof continued

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for every $u \in \mathcal{A}(X, A) \otimes_{\max} \mathcal{A}(Y, B)$.

In particular,

$$ze_{x,a} \otimes 1 \sim z \otimes p_{x,a} \sim e_{x,a}z \otimes 1, \quad x \in X, a \in A, \ z \in \mathcal{A}(X, A).$$

Proof continued

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In particular,

$$ze_{x,a} \otimes 1 \sim z \otimes p_{x,a} \sim e_{x,a}z \otimes 1, \quad x \in X, a \in A, \; z \in \mathcal{A}(X,A).$$

Similarly,

$$zq_{y,b} \otimes 1 \sim z \otimes f_{y,b} \sim q_{y,b}z \otimes 1, \quad y \in Y, b \in B, \; z \in \mathcal{A}(X,A).$$

Proof continued

Let z and w be words on $\mathcal{E} := \{e_{x,a} : x \in X, a \in A\}$. We show that

$$zw \otimes 1 \sim wz \otimes 1,$$

from where it follows that τ is a trace.

Proof continued

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Induction on $|w|$: for $|w| = 1$, the claim is already proved.

Proof continued

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$$zw \otimes 1 \sim wz \otimes 1,$$

from where it follows that τ is a trace.

Induction on $|w|$: for $|w| = 1$, the claim is already proved.

Let $|w| = n$ and write $w = w'e$, where $e \in \mathcal{E}$. Then

$$zw \otimes 1 = zw'e \otimes 1 \sim ezw' \otimes 1 \sim w'ez \otimes 1 = wz \otimes 1.$$

Thank you very much!