Geometry of the set of synchronous quantum correlations

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Correlations

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Figure: Alice

Figure: Bob

$n =$ number of experiments,
$m =$ number of possible outcomes.

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Figure: Bob

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A **correlation** is a tuple

$$\{p(i,j|x,y) \geq 0\}; \quad i,j \leq m, \quad x,y \leq n$$

satisfying

$$\sum_{i,j} p(i,j|x,y) = 1.$$

Quantum correlation sets

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Let $\mathfrak{A}$ be a C^*-algebra with a state ϕ. Assume

$$\{E_{x,i}\}_{i=1}^{m}, \{F_{y,j}\}_{j=1}^{m} \subseteq \mathfrak{A}$$

where $E_{x,i}F_{y,j} = F_{y,j}E_{x,i}$. Then

$$p(i,j|x,y) = \phi(E_{x,i}F_{y,j})$$

defines a **quantum-commuting** correlation.

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Set of all qc correlations: $C_{qc}(n,m)$.

Set of all quantum correlations: $C_q(n,m)$.

Set of all local correlations: $C_{loc}(n,m)$.

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Each $C_*(n, m)$ is convex and satisfies:

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Theorem
(Junge-Navascues-Palazuelos-Perez-Garcia-Scholz-Werner, Fritz, Ozawa)

Connes' embedding conjecture is true if and only if $\overline{C_q(n, m)} = C_{qc}(n, m)$ for every n, m.

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Theorem (Paulsen-Severini-Stahlke-Todorov-Winter)

A correlation $p \in C^s_{qc}(n, m)$ iff there exists a C^*-algebra $\mathfrak{A}$, $\{E_{x,i}\}_{i=1}^m \subset \mathfrak{A}$, and a tracial state $\tau : \mathfrak{A} \to \mathbb{C}$ such that

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Theorem (Dykema-Paulsen)

Connes' embedding conjecture is true if and only if $\overline{C_q^s(n,m)} = C_{qc}^s(n,m)$ for every n,m.

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Theorem (R.)

The set $C_q^s(3,2)$ is closed. Moreover, if $p \in C_q^s(3,2)$, then there exists a $\mathfrak{A} \subset \mathbb{M}_{16}$, projection valued measures $\{E_{x,i}\} \subset \mathfrak{A}$ and a trace τ such that

$$p(i,j|x,y) = \tau(E_{x,i}E_{y,j}).$$

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When $m=2$,

$$p(i,j|x,x) = \begin{pmatrix} r_x & 0 \\ 0 & 1-r_x \end{pmatrix},$$

$$p(i,j|x,y) = \begin{pmatrix} w_{x,y} & r_x - w_{x,y} \\ r_y - w_{x,y} & w_{x,y} + (1 - r_x - r_y) \end{pmatrix}.$$

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$$p \cong \begin{pmatrix} r_1 & w_{1,2} & w_{1,3} \\ w_{2,1} & r_2 & w_{2,3} \\ w_{3,1} & w_{3,2} & r_3 \end{pmatrix}, \quad w_{x,y} = w_{y,x}$$

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For each $(r_1,r_2,r_3)\in[0,1]^3$, we will determine the corresponding set of $\{(w_{1,2},w_{1,3},w_{2,3})\}\subseteq\mathbb{R}^3$, denoted $S_{\vec{r}}[C_q^s(3,2)]$.

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$$C^s_{max}(n,m) = \{p(i,j|x,y) = \frac{1}{d} Tr(E_{x,i} F_{y,j})\}.$$

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$\overline{C_{max}^s(n,m)} = \overline{C_q^s(n,m)}$ *for all* n, m.

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$$\begin{aligned} S_d(n_1,n_2,n_3) &:= \{\tfrac{1}{d}(Tr(E_1E_2), Tr(E_1E_3), Tr(E_2E_3)) : Tr(E_x) = n_x\} \\ &\subseteq \mathbb{R}^3 \end{aligned}$$

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Goal: Describe $S_d(n_1,n_2,n_3) \subseteq S_{(n_1/d, n_2/d, n_3/d)}[C_q^s(3,2)]$.

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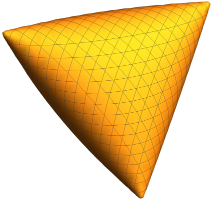
$$\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \mapsto (x, y, z)$$



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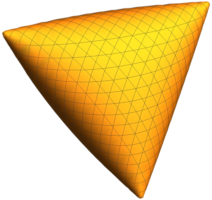

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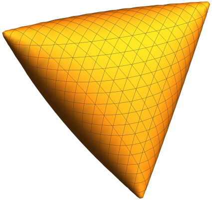

Set $S_d(n) := S_d(n, n, n)$.

Theorem

For every n, $S_{2n}(n) = S_2(1) = S_{(.5,.5,.5)}[C_q^s(3, 2)]$ is an affine image of the 3×3 elliptope.

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Need to determine the geometry of $S_d(n_1, n_2, n_3)$, for all $n_1, n_2, n_3 \leq d$.

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Lemma

Assume $n_1 + n_2 < d$. Then

$$S_d(n_1, n_2, n_3) \subseteq \tfrac{d-1}{d} co\{S_{d-1}(n_1, n_2, n_3), S_{d-1}(n_1, n_2, n_3 - 1)\}.$$

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1. Apply lemma many times to find geometry of $S_d(n_1, n_2, n_3)$.
2. Calculate the closure of $\cup S_d(n_1, n_2, n_3)$. This is equal to $\overline{C_q^s(3,2)}$.
3. Observe that every correlation in $\overline{C_q^s(3,2)}$ can be realized with $\mathfrak{A} \subseteq \mathbb{M}_{16}$.

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Assume $r_1 \leq r_2 \leq r_3 \leq 1/2$, $\vec{r} = (r_1, r_2, r_3) \in [0,1]^3$. *Then*

$$S_{\vec{r}}[C_q^s(3,2)] = co\{C_1(\vec{r}), C_2(\vec{r}), C_3(\vec{r})\}$$

where

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C_1(\vec{r}) &= 2\max(0, r_1 + r_2 + r_3 - 1)S_2(1) \\
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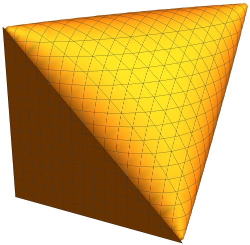

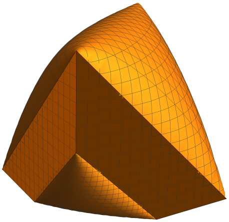

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Thanks for your attention!