Projective spectrum, group theory and complex dynamics

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Spectrum

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- A different point of view: $\sigma(a)$ is a reflects how a and I interact.
- linear pencil : $A_1 - \lambda A_2$;
- multiparameter pencil: $A(z) = z_1 A_1 + z_2 A_2 + \cdots + z_n A_n$; (algebraic geometry, differential equations, group theory, math physics, etc.)

Joint spectra

$A_1, A_2, ..., A_n$: elements in $\mathcal{B}$.

Problem. How to define a joint spectrum for these elements?

Preferred properties:
1. Reflects joint behavior of the elements.
2. Reflects interaction of the elements.
3. Reflects algebraic properties of the elements, if any.
4. Computable in many examples.

Known joint spectra:

- 70s, Taylor's spectrum for commuting operators: invertibility of $(A_1 - \lambda_1 I, A_2 - \lambda_2 I, ..., A_n - \lambda_n I)$ through the exactness of Koszul complex.
- 70s, Harte spectrum: invertibility of $\sum B_k(A_k - \lambda_k I)$.
- 80s, HcIntosh-Pryde spectrum: invertibility of $\sum (A_k - \lambda_k I)^2$

Projective spectrum

Let $A(z) = z_1 A_1 + z_2 A_2 + \cdots + z_n A_n$.

Projective joint spectrum:
$P(A) = \{z \in \mathbb{C}^n : \ A(z) \text{ is not invertible in } \mathcal{B}.\}$
$p(A) = \{z = [z_1, \ z_2, \ \ldots, \ z_n] \in \mathbb{P}^{n-1} : \ A(z) \text{ is not invertible in } \mathcal{B}.\}$

Projective resolvent sets: $P^c(A)$ in $\mathbb{C}^n$, $p^c(A)$ in $\mathbb{P}^{n-1}$.

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Features:
1. symmetry: $A_1, ..., A_n$ are treated equally.
2. base-free: parameter is assigned to each A_j. "I can help but I am not central".
3. generality: applicable to noncommuting operators.
4. computability: easy to compute in many examples.

Examples

- 1. Let $\mathcal{B} = M_2(\mathbb{C})$, $A_0 = I$ and $A_1 = i\sigma_1,\ A_2 = i\sigma_2, A_3 = i\sigma_3$, where σ_i are the Pauli matrices, i.e.,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Then $\mathfrak{su}(2) = span\{A_1, A_2, A_3\}$. And $p(A) = \{z \in \mathbb{P}^3 :\ z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\}$ is a compact algebraic manifold.

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- 2 [Bannon, Cade, Y., 2013]. Free group F_n with generators $\{g_1, g_2, \cdots, g_n\}$, and let λ be the regular representation of F_n on $\ell^2(F_n)$. Set $A(z) = z_1\lambda(g_1) + \cdots + z_n\lambda(g_n))$, then

$$P(A) = \bigcap_{j=1}^{n} R_j,$$

  where $R_j = \{z \in \mathbb{C}^n : 2|z_j|^2 \leq \|z\|^2\}, j = 1, 2, \cdots, n.$

- 3 [He, Wang, Y., 2017]. The Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries $S_1, S_2, \dots, S_n$ satisfying

$$\sum_{i=1}^{n} S_i S_i^* = I \quad \text{and} \quad S_i^* S_j = \delta_{ij} I \quad \text{for } 1 \le i, j \le n.$$

Let $S(z) = z_1 S_1 + \cdots z_n S_n$ and $S_*(z) = I + S(z)$. Then it is shown in that
(a) $P(S) = \mathbb{C}^n$.
(b) $P^c(S_*)$ is equal to the unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$.

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- 4 [Stessin, Y., Zhu, 2011; He, Wang, Y., 2017]. If $A_*(z) = I + z_1 A_1 + \cdots + z_n A_n$, where A_j are compact operators on a Hilbert space, then
  (i) $P(A_*)$ is a *thin set*.
  (ii) When $P(A_*)$ is *smooth*, then $E_A = \bigvee_{z \in P(A_*)} \operatorname{Ker} A_*(z)$ is a holomorphic line bundle (kernel bundle) over $P(A_*)$.

Finitely generated groups

$G =< g_1, g_2, \cdots , g_n \mid \cdots >$, $\rho : G \to U(\mathcal{H})$ a unitary representation.
Set $A_\rho(z) = z_0 I + z_1\rho(g_1) + \cdots + z_n\rho(g_n)$.

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Theorem (Y.)

Let $\lambda_G : G \to \ell^2(G)$ be the left regular representation. Then G is amenable if and only if $P(A_\lambda)$ contains the hyperplane H_1.

Proposition

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Problem: Does the converse hold for irreducible representations?

Infinite dihedral group $D_\infty =< a, t \mid a^2 = t^2 = 1 >$

For a fixed $\theta \in [0,\ 2\pi)$, consider two-dimensional representation ρ_θ given by

$$\rho_\theta(a) = \begin{bmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix}, \quad \rho_\theta(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Known: every irreducible repr. of D_∞ is either one dimensional or two dimensional.

Halmos: Every irreducbile 2-dim. repr. of D_∞ is of the form ρ_θ for some $\theta \in (0, \pi)$.

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Fact: $P(A_{\rho_\theta}) = \{z \in \mathbb{C}^3 :\ z_0^2 - z_1^2 - z_2^2 - 2z_1 z_2 \cos\theta = 0\}$.

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Theorem (Grigorchuk, Y., 2017)

If $\lambda : D_\infty \longrightarrow U(l^2(D_\infty))$ is the left regular representation, then

$$P(A_\lambda) = \bigcup_{0 \le \theta < 2\pi} P(A_{\rho_\theta}).$$

Proof

The left regular representation of D_∞ is equivalent to the following representation λ of D_∞ on $L^2(\mathbb{T}, \frac{d\theta}{2\pi}) \oplus L^2(\mathbb{T}, \frac{d\theta}{2\pi})$:

$$\lambda(a) = \begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix}, \quad \lambda(t) = \begin{bmatrix} 0 & I_0 \\ I_0 & 0 \end{bmatrix},$$

where T is the bilateral shift operator $L^2(\mathbb{T})$, i.e., $Tf(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$. If we let $T = \int_0^{2\pi} e^{i\theta} dE(e^{i\theta})$. Then

$$\lambda(a) = \int_0^{2\pi} \begin{bmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix} dE(e^{i\theta}) = \int_0^{2\pi} \rho_\theta(a) dE(e^{i\theta});$$

$$\lambda(t) = \int_0^{2\pi} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} dE(e^{i\theta}) = \int_0^{2\pi} \rho_\theta(t) dE(e^{i\theta}).$$

Group of intermediate growth

Let $S = \{g_1, ..., g_n\}$ be a symmetric generating set of group G.
Word length: $|x| = \min\{k | x = x_1 x_2 \cdots x_k,\ x_j \in S\}$, $|e| = 0$.
Set $B_r(G) = \{x \in G \,||x| \le r\}, \quad r \ge 0$.

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- G has polynomial growth if $|B_r| \leq \alpha r^\beta$ for some fixed $\alpha, \beta > 0$.
- G has exponential growth if $|B_r| \geq \alpha \beta^r$ for some fixed $\alpha > 0, \beta > 1$.
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Grigorchuk (80, 84): Yes! $\mathcal{G} =< a, b, c, d >$, where

$$a^2 = b^2 = c^2 = d^2 = bcd = 1$$
$$\sigma^k((ad)^4) = \sigma^k((adacac)^4) = 1,\ k = 0,\ 1,\ 2,\ \cdots,$$

where $\sigma :\ a \to aca,\ b \to d,\ c \to b,\ d \to c$ is a substitution.
$(e^{\sqrt{r}} \prec |(B_r(\mathcal{G})| \prec e^{0.991})$.

Theorem (Grigorchuk, Y., 2017)

Let $\rho : \mathcal{G} \to L^2(T)$ be the Koopman repr., where T is the rooted binary tree, and let $M = \frac{1}{4}\big(\rho(a) + \rho(b) + \rho(c) + \rho(d)\big)$ be the Markov operator of $\mathcal{G}$. Then $\sigma(M) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$.

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1. Set $u = \frac{b+c+d-1}{2} \in \mathbb{C}[\mathcal{G}]$ and observe $u^2 = 1$.

2. Prove that $< a, u >$ is isomorphic to D_∞.

3. Prove that Koopman repr. and the regular repr. of D_∞ are weakly equivalent.

4. Write $M - \alpha I = (\frac{1}{4} - \alpha)I + \frac{1}{4}\rho(a) + \frac{1}{2}\rho(u)$ and use the projective spectrum of D_∞.

Hermitian metric on $P^c(A)$ (with Douglas), preliminary

$\mathcal{B}$: unital C^*-alg., $A_1, \ldots, A_n \in \mathcal{B}$, $\phi \in \mathcal{B}^*$.
$\omega_A(z) := A^{-1}(z)dA(z) = \sum_j A^{-1}(z)A_j dz_j, z \in P^c(A)$.

Fundamental form: $\Omega_A(z) = -\omega_A^* \wedge \omega_A$.

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Fundamental form: $\Omega_A(z) = -\omega_A^* \wedge \omega_A$.

$$\begin{aligned}\phi(\Omega_A(z)) &= -\phi(\omega_A^* \wedge \omega_A) \\ &= \phi[A_k^*(A^{-1}(z))^* A^{-1}(z)A_j] dz_j \wedge d\bar{z}_k \\ &:= g_{jk}(z) dz_j \wedge d\bar{z}_k, \quad z \in P^c(A).\end{aligned}$$

Observe: When $g_\phi(z) := \big(g_{jk}(z)\big)$ is positive definite on $P^c(A)$, it defines an inner product on the holomorphic tangent bundle of $P^c(A)$ through $(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k})_z = g_{jk}(z)$.

Theorem

Let ϕ be a state *on $\mathcal{B}$ ($\phi(a^*a) \geq 0$, $\phi(I) = 1$). Then $\phi(\Omega_A)$ defines a Hermitian metric on $P^c(A)$ if and only if ϕ is faithful on span$\{A_1, ..., A_n\}$.*

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$G =< g_1, ..., g_n | \cdots >$: finitely generated group. Let $C_r^*(G)$ be the reduced C^*-subalg. in $B(\ell^2(G))$. Define linear functional $\phi(a) =< a\delta_e, \delta_e >$. Then ϕ is a faithful tracial state on $C_r^*(G)$.

Corollary

Let $A_\lambda(z) = z_0 I + z_1\lambda(g_1) + \cdots + z_n\lambda(g_n)$. *Then* $\phi(\Omega_{A_\lambda})$ *defines a* G*-invariant Hermitian metric on* $P^c(A_\lambda)$.

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Theorem (Goldberg, Y., 2018)

Consider D_∞ *and set* $A_*(z) = I + z_1\lambda(a) + z_2\lambda(t)$. *Then the metric defined by* $\phi(\Omega_{A_*})$ *is imcomplete, and the completion* $[P^c(A_*)] = \mathbb{C}^2 \setminus \{(\pm 1, 0), (0, \pm 1)\}$.

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Corollary

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Note: $\sigma(\lambda(a)) = \sigma(\lambda(t)) = \{\pm 1\}$.

Def.

- The metric on $P^c(A)$ defined by $\phi(\Omega_A(z))$ is said to be Kähler if $d\phi(\Omega_A(z)) = 0$.
- The metric on $P^c(A)$ defined by $\phi(\Omega_A(z))$ is said to be flat if

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \det g_\phi(z) = 0, \quad \forall z \in P^c(A), \ \forall j, k.$$

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Example. If $A_1, ..., A_n$ is a basis of $M_k(\mathbb{C})$ $(n = k^2)$, then $\mathrm{Tr}(\Omega_A)$ defines a complete, non-Kähler, flat and GL_k-invariant metric on $P^c(A) \cong GL_k$.

A connection with complex dynamics

Def. A unitary representation $(\pi, \mathcal{H})$ is self-similar if there exists a $d \in \mathbb{N}$ and a unitary map $W : \mathcal{H} \to \mathcal{H}^d$ such that for all $g \in G$ every entry in the $d \times d$ block matrix $W\pi(g)W^*$ is either 0 or of the form $\pi(x)$ for some $x \in G$. Observe that in this case exactly one entry in every row or column of $W\pi(g)W^*$ is nonzero.

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Facts:

- The Koopman repr. ρ of D_∞ is self-similar. In fact,

$$\rho(a) \cong \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \rho(t) \cong \begin{bmatrix} \rho(a) & 0 \\ 0 & \rho(t) \end{bmatrix},$$

  hence

$$A_\rho(z) = z_0 + z_1\rho(a) + z_2\rho(t) \cong \begin{bmatrix} z_0 + z_2\rho(a) & z_1 \\ z_1 & z_0 + z_2\rho(t) \end{bmatrix}.$$

- ρ is weakly equivalent to λ, hence $P(A_\rho) = P(A_\lambda)$.

When $z_0 \neq \pm z_2$, $A_\rho(z)$ is invertible if and only if $z_0 + z_2 t - z_1^2(z_0 - z_2 a)(z_0^2 - z_2^2)^{-1}$ is invertible. Define

$$F(z_0, z_1, z_2) = \left(z_0(z_0^2 - z_1^2 - z_2^2), z_1^2 z_2, z_2(z_0^2 - z_2^2) \right).$$

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Lem: $F : P(A_\lambda) \to P(A_\lambda)$.

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Def.

- Indeterminacy sets: $I_n = \{z \in \mathbb{P}^2 | F^n(z) = (0,0,0)\}$, $n \geq 1$.
- Extended indeterminacy set: $E = \overline{\cup_{n\geq 1} I_n}$.
- Fatou point: z has a nbd. V_z such that $\{F^n\}$ is a normal family on V_z.
- Fatou set $\mathcal{F}(F)$: the set of Fatou points for F.
- Julia set $\mathcal{J}(F)$: $\mathbb{P}^2 \setminus \mathcal{F}(F)$.

Let $T(x) = 2x^2 - 1, x \in \hat{\mathbb{C}}$ (Chebyshev poly.).
$\tau(z) = \frac{z_0^2 - z_1^2 - z_2^2}{2z_1 z_2} : \mathbb{P}^2 \to \hat{\mathbb{C}}$. Note: $z \in p(A_\lambda)$ iff $\tau(z) \notin [-1, 1]$.

Fact: $\mathcal{J}(T) = [-1, 1]$.

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Fact: $\mathcal{J}(T) = [-1, 1]$.

Theorem (Grigorchuk, Y., 2017)

The following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^2 \setminus l_1 & \xrightarrow{F} & \mathbb{P}^2 \\ \downarrow \tau & & \downarrow \tau \\ \hat{\mathbb{C}} & \xrightarrow{T} & \hat{\mathbb{C}}. \end{array}$$

the Julia set

For $z \in p^c(A_\lambda)$, define
$f_n(z) = \frac{1}{2\tau(z)} + \frac{1}{2^2 T(\tau(z))\tau(z)} + \cdots \frac{1}{2^{n-1} T^{n-1}(\tau(z)) \cdots T(\tau(z))\tau(z)}.$

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Lemma (Goldberg, Y.)

(a) $\{f_n\}$ *converges normally (to* f*) on* $p^c(A_\lambda)$.
(b) $I_{n+1} \subset \{z_2 = \pm z_0 + z_1 f_n(z)\}$.
(c) $\lim_n F^n(z) = [z_0 : 0 : z_2 + z_1 f(z)], \; z \in p^c(A) \setminus E$

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Theorem (Goldberg, Y.)

Consider the map $F : \mathbb{P}^2 \to \mathbb{P}^2$ *derived from the self-similarity of the Koopman repr. of* D_∞*. Then* $\mathcal{J}(F) = p(A_\lambda) \cup E$.

Definition

Consider operator-valued 1-form $\omega_A(z) = -(A-z)^{-1}dz$.
For $x \in \mathcal{H}$ with $\|x\| = 1$, let ϕ_x be the vector state on $\mathcal{B}$ such that $\phi_x(A) =< Ax, x >, \quad A \in \mathcal{B}$.

Define metric g_x through

$$g_x(z)dz \wedge d\bar{z} = \phi_x\big(\omega_A^*(z) \wedge \omega_A(z)\big) = \|(A-z)^{-1}x\|^2 dz \wedge d\bar{z}.$$

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Notes:
1. g_x defines a non-Euclidean metric on $\rho(A)$ that may have singularities at $\sigma(A)$.
2. g_x depends on A and x.

- If $A(z) = A - zI$, then $\omega_A(z) = -(A - zI)^{-1} dz$.
- Maurer-Cartan form $g^{-1}dg$.
- For a linear functional ϕ on $\mathcal{B}$, $\phi(\omega_A(z)) = \sum_{j=1}^{n} \phi(A^{-1}(z)A_j)dz_j$ is a holomorphic 1-form on $P^c(A)$. For a k-linear functional F, $\kappa(F) := F(\omega_A(z),\ \omega_A(z),\ \ldots,\ \omega(z))$ is a holomorphic k-form on $P^c(A)$.

Definition. A k-linear functional F on $\mathcal{B}$ is said to be invariant if

$$F(a_1,\ a_2,\ ,\ ...,\ a_k) = F(ga_1g^{-1},\ ga_2g^{-1},\ ...,\ ga_kg^{-1})$$

for all $a_1,\ a_2,\ ,\ ...,\ a_k$ in $\mathcal{B}$ and every invertible $g \in \mathcal{B}$.

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Proposition

If the k-linear functional F is invariant, then $\phi(\omega_A(z), \omega_A(z),\ ..., \omega_A(z))$ is closed.

Hochschild q-cochain: $(q+1)$-linear functionals ϕ on $\mathcal{B}$.
ϕ is a *cyclic cocycle* if for all elements $a_0,\ a_1,\ ...,\ a_q$ in $\mathcal{B}$,

(1) $\phi(a_0,\ a_1,\ ...,\ a_q) = (-1)^q \phi(a_q,\ a_0,\ ...,\ a_{q-1})$, and
(2) $(b\phi) := \sum_{j=0}^{q} (-1)^j \phi(a_0,\ ...,\ a_j a_{j+1},\ ...,\ a_{q+1}) +$
$(-1)^{q+1} \phi(a_{q+1} a_0,\ a_1,\ ...,\ a_q) = 0$,

$HC^q(\mathcal{B})$: the space of q-cyclic cocycles.

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$HC^q(\mathcal{B})$: the space of q-cyclic cocycles.

Proposition

(Cade and Y., 2012) Let $\mathcal{B}$ be a topological algebra and ϕ be a continuous cyclic q-cocycle on $\mathcal{B}$. Then for any tuple A, $\kappa(\phi) := \phi(\omega_A(z), \omega_A(z), ..., \omega_A(z))$ *is a closed holomorphic $q+1$ form on $P^c(A)$. In fact,*

$$\frac{q}{q+1}\kappa(b\phi) = -d\kappa(\phi).$$

Examples

- Let $\mathcal{B} = M_k(\mathbb{C})$. Then Jacobi's formula:
  $Tr(\omega_A(z)) = d \log detA(z), \quad z \in P^c(A).$

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- If $\mathcal{B}$ is a topological algebra with a continuous trace tr, then for q even

  $$\phi(a_0,\ a_1,\ a_2,\ \ldots,\ a_q) := tr(a_0 a_1 a_2 \cdots a_q)$$

  is a cyclic q-cocycle, and $tr(\omega^{q+1}A(z))$ is closed.

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Proposition

(Cade and Y., 2009) If A is a 4-tuple in a Banachl algebra with a continuous trace tr, then $tr(\omega_A^3) = \phi(z)S(z)$, where $S(z) = z_1 dz_2 dz_3 dz_4 - z_2 dz_1 dz_3 dz_4 + z_3 dz_1 dz_2 dz_4 - z_4 dz_1 dz_2 dz_3$, and $\phi(z)$ is a holomorphic function on $P^c(A)$.

(A higher order form of Jacobi's formula!)

On $M_2(\mathbb{C})$

Let $A_1 = I$ and $A_2,\ A_3,\ A_4$ be the Pauli matrices.

$$A(z) = \sum_{k=1}^{4} z_k A_k, \quad \text{and} \quad \omega_A(z) = A^{-1}(z) dA(z).$$

Then

1) $P(A) = \{z \in \mathbb{C}^4 :\ z_1^2 - z_2^2 - z_3^2 - z_4^2 = 0\}$.
2) $P^c(A) \cong GL(2,\ \mathbb{C})$.
3) $Tr(\omega_A^3(z)) = 12iD^{-2}S(z)$, where $D = z_1^2 - z_2^2 - z_3^2 - z_4^2$.
4) $SU(2,\ \mathbb{C})) \cong S^3 = \{(x_1,\ ix_2,\ ix_3,\ ix_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$,
and $Tr(\omega_A^3(z))|_M = -12S(x)$ is the standard 3-form on S^3.