Invariant subspaces and operator model theory on noncommutative varieties

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Beurling theorem (Acta Math., 1949)

- $\mathbb{D} := \{z \in \mathbb{C} : \; |z| < 1\}$
- $H^2(\mathbb{D})$ is the Hardy space of all analytic functions on $\mathbb{D}$ with square-summable coefficients.
- S is the unilateral shift defined by $(S\varphi)(z) := zf(z)$.

Classification of the invariant subspaces of S :

Theorem

Any invariant subspace $\mathcal{M} \subset H^2(\mathbb{D})$ of S is of the form

$$\mathcal{M} = \theta H^2(\mathbb{D}),$$

where θ is an inner function.

Beurling theorem (Acta Math., 1949)

- A subspace $\mathcal{L} \subset H^2(\mathbb{D})$ is called wandering subspace of S if $\mathcal{L} \perp S^n\mathcal{L}$ for any $n = 1, 2, \ldots$.
- An equivalent form of Beurling result :

Theorem

If $\{0\} \neq \mathcal{M} \subset H^2(\mathbb{D})$ is an invariant subspace of S, then $\mathcal{W} := \mathcal{M} \ominus S\mathcal{M}$ is a one dimensional wandering subspace spanned by an inner function θ, and

$$\mathcal{M} = \overline{\operatorname{span}}\{S^n\mathcal{W} : \ n = 0, 1, \ldots\}.$$

- Therefore, the invariant subspaces of S are in one-to-one correspondence with the wandering subspaces of S.

Beuling-Lax-Halmos theorem (Acta Math, Crelle)

Theorem

A non-trivial closed subspace $\mathcal{M}$ of the vector-valued Hardy space $H^2(\mathbb{D})\otimes\mathcal{E}$ is invariant under $S\otimes I_{\mathcal{E}}$ if and only if there is a Hilbert space $\mathcal{G}$ and an isometric analytic operator $M_\Theta : H^2(\mathbb{D})\otimes\mathcal{G}\to H^2(\mathbb{D})\otimes\mathcal{E}$, i.e.

$$M_\Theta(S\otimes I_{\mathcal{G}}) = (S\otimes I_{\mathcal{E}})M_\Theta,$$

such that $\mathcal{M} = M_\Theta(H^2(\mathbb{D})\otimes\mathcal{G})$. Moreover, the wandering subspace

$$\mathcal{W} := \mathcal{M}\ominus z\mathcal{M}$$

admits the representation $\mathcal{W} = M_\Theta(\mathcal{G})$ and

$$\mathcal{M} = \overline{\operatorname{span}}\{(S^n\otimes I_{\mathcal{E}})\mathcal{W} :\ n = 0,1,\ldots\}.$$

Universal model operator

- The unilateral shift plays the role of universal contraction on a Hilbert space.

Theorem

Any pure contraction $T \in B(\mathcal{H})$, i.e. $\|T\| \leq 1$ and ${T^*}^n \to 0$ strongly, as $n \to \infty$, has its adjoint unitarily equivalent to $(S^* \otimes I_\mathcal{E})|_\mathcal{N}$, where $\mathcal{N}$ is a co-invariant subspace of $S \otimes I_\mathcal{E}$.

- This result led to the Sz.-Nagy-Foiaş model theory for arbitrary completely non-unitary contractions on Hilbert spaces in terms of the associated characteristic functions.

Generalizations : single variable case

- Shift-invariant subspaces and their wandering subspaces for other classical Hilbert spaces of analytic functions on $\mathbb{D}$.

- Richter (Crelle) for the Dirichlet space.

- Aleman, Richter, Sundberg (Acta Math.) for the Bergman space.

- Shimorin (Crelle) for left invertible operators satisfying some suitable operator inequalities.

- Olofsson, Ball and Bolotnikov, and Sarkar for the weighted Bergman space.

Generalizations : multivariable commutative case

- Shift-invariant subspaces and their wandering subspaces for Hilbert spaces of analytic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$.
- McCullough and Trent (2000) for the Drurry-Arveson space. This also follows also from the Beurling-Lax-Halmos type theorem for the left creation operators (Popescu, 1989) and the noncommutative commutant lifting theorem (Popescu, 1992).
- Eschmeier, Sarkar for the Bergman space and weighted Bergman space over the unit ball $\mathbb{B}_n$.
- Bhattacharjee, Eschmeier, Keshari, and Sarkar for a class of reproducing kernel Hilbert spaces on $\mathbb{B}_n$.

Generalizations : multivariable noncommutative case

- Let H_n be a complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. The full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k\geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$.
- The left creation operators $S_i : F^2(H_n) \to F^2(H_n)$ are defined by

$$S_i\varphi := e_i \otimes \varphi, \qquad \varphi \in F^2(H_n).$$

- $(S_1, \ldots, S_n)$ plays the role of universal model for row contractions :

$$\{T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n : \; T_1 T_1^* + \cdots + T_n T_n^* \leq I\}.$$

Generalizations : multivariable noncommutative case

- A Beurling-Lax-Halmos type theorem for the left creation operators was obtained in 1989 (Popescu).
- Universal model for pure row contractions (Popescu, 1989).

Theorem

If $T = (T_1, \ldots, T_n)$ is a pure row contraction, then its characteristic function $\Theta_T : F^2(H_n) \otimes \mathcal{D}_{T^*} \to F^2(H_n) \otimes \mathcal{D}_T$ is an isometric multi-analytic operator, i.e

$$\Theta_T(S_i \otimes I_{\mathcal{D}_{T^*}}) = (S_i \otimes I_{\mathcal{D}_T})\Theta_T,$$

and

$$T_i^* = (S_i^* \otimes I_{\mathcal{D}_T})|_{\mathcal{M}^\perp},$$

where $\mathcal{M} := \Theta_T(F^2(H_n) \otimes \mathcal{D}_{T^*})$.

- Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0.
- If $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$.
- Let $Z_1, \ldots, Z_n$ be noncommutative indeterminates. A formal power series $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha,\ a_\alpha \in \mathbb{C}$, is called free holomorphic function on the noncommutative ball

$$[B(\mathcal{H})^n]_\rho = \{(T_1, \ldots, T_n) \in B(\mathcal{H})^n : \ \|T_1 T_1^* + \cdots + T_n T_n^*\| < \rho^2\},$$

if the series $\sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha$ is convergent in the operator norm topology for any $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_\rho$, and any $\mathcal{H}$.

- f is called positive regular free holomorphic function if $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$ if $i = 1, \ldots, n$.

Noncommutative domains

- We define the noncommutative regular domain $\mathcal{D}_f^m(\mathcal{H})$, $m = 1, 2, \ldots$, to be the set of all $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ such that

$$(id - \Phi_{f,X})^k(I) \geq 0 \text{ for } 1 \leq k \leq m,$$

where $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by

$$\Phi_{f,X}(Y) := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha Y X_\alpha^*, \quad Y \in B(\mathcal{H}),$$

and the convergence is in the weak operator topology.

- Define $b_{g_0}^{(m)} := 1$ and

$$b_\alpha^{(m)} := \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \ldots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \binom{j+m-1}{m-1} \qquad \text{if } |\alpha| \geq 1.$$

Universal model

- Let $D_i^{(m)} : F^2(H_n) \to F^2(H_n)$, $i \in \{1, \ldots, n\}$, be the diagonal operators defined by setting

$$D_i^{(m)} e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}} e_\alpha, \qquad \alpha \in \mathbb{F}_n^+,$$

where $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is the orthonormal basis of $F^2(H_n)$.

- The n-tuple $(W_1^{(m)}, \ldots, W_n^{(m)})$ of weighted shifts, $W_i^{(m)} := S_i D_i^{(m)}$, associated with the noncommutative domain $\mathcal{D}_f^m$, plays the role of universal model for the pure elements of $\mathcal{D}_f^m$ (Popescu, Mem. AMS, 2010, JFA, 2008).

Noncommutative varieties $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$

- Let $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n\rangle$ be a fixed set of noncommutative polynomials such that $q(0) = 0$ for any $q \in \mathcal{Q}$
- Define the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$, to be the set

$$\{(X_1, \ldots, X_n) \in \mathcal{D}^m_f(\mathcal{H}) : \ q(X_1, \ldots, X_n) = 0 \quad \text{for any} \quad q \in \mathcal{Q}\}.$$

- The universal model $(B_1^{(m)}, \ldots, B_n^{(m)})$ associated with $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ is given by

$$B_i^{(m)*} = W_i^{(m)*}|_{\mathcal{N}^m_{f,\mathcal{Q}}}, \quad 1 = 1, \ldots, n,$$

acting on a model space $\mathcal{N}^m_{f,\mathcal{Q}} \subset F^2(H_n)$ which is a joint invariant subspace under the adjoints $W_1^{(m)*}, \ldots, W_n^{(m)*}$.

Single variable case : $n = 1$ and $Q = 0$

- If $m = 1$ and $p = Z$, the corresponding domain $\mathcal{D}_p^m(\mathcal{H})$ coincides with

$$[B(\mathcal{H})]_1 := \{X \in B(\mathcal{H}) : \|X\| \leq 1\}.$$

In this case, the universal model is the unilateral shift S acting on the Hardy space $H^2(\mathbb{D})$.

- If $m \geq 2$ and $p = Z$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler, Olofsson, Ball-Bolotnikov. The corresponding universal model is the unilateral shift acting on the weighted Bergman space, which is a reproducing kernel Hilbert space corresponding to the kernel $k_m(z, w) = \frac{1}{(1-z\bar{w})^m}$, $z, w \in \mathbb{D}$.

Multivariable commutative case : $n \geq 2$

Case : $\mathcal{Q} := \{Z_i Z_j - Z_j Z_i, i, j = 1, \ldots, n\}$

- If $m \geq 1$ and $p = Z_1 + \cdots + Z_n$ the corresponding commutative variety was studied by Drurry, Arveson, Bhattacharyya-Eschmeier-Sarkar, Popescu (when $m = 1$), and Athavale, Müller, Müler-Vasilescu, and Curto-Vasilescu (when $m \geq 2$).
  The corresponding universal model is the n-tuple $(M_{z_1}, \ldots, M_{z_n})$ of multipliers by the coordinate functions, acting on the reproducing kernel Hilbert space corresponding to the kernel

$$k_m(\mathbf{z}, \mathbf{w}) = \frac{1}{(1 - z_1 \bar{w}_1 - \cdots - z_n \bar{w}_n)^m}, \qquad \mathbf{z}, \mathbf{w} \in \mathbb{B}_n,$$

on the unit ball of $\mathbb{C}^n$.

Multivariable commutative case : $n \geq 2$

Case : $\mathcal{Q} := \{Z_iZ_j - Z_jZ_i, i, j = 1, \ldots, n\}$

- When $m \geq 1$ and p is a positive regular commutative polynomial, the commutative variety $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{K})$ was studied by S. Pott . In this case, the universal model $(M_{z_1}, \ldots, M_{z_n})$ acts on a reproducing kernel Hilbert space of holomorphic functions on a Reinhardt domain in $\mathbb{C}^n$ uniquely determined by p.

Multivariable noncommutative case

Case : $n \geq 2$ and $\mathcal{Q} = 0$

- When $m = 1$, $p = Z_1 + \cdots + Z_n$, the noncommutative domain $\mathcal{D}_p^m(\mathcal{H})$ coincides with the closed unit ball $[B(\mathcal{H})^n]_1$, the study of which has generated a free analogue of Sz.-Nagy-Foiaş theory. The corresponding universal model is the n-tuple of left creation operators $(S_1, \ldots, S_n)$.

- When $m \geq 1$, $n \geq 1$, and f is any positive regular free holomorphic function the domain $\mathcal{D}_f^m(\mathcal{H})$ was studied by Popescu (Mem. AMS, 2010 and JFA 2008). In this case, the corresponding universal model is the n-tuple of weighted left creation operators $(W_1^{(m)}, \ldots, W_n^{(m)})$ acting on the full Fock space.

Multivariable noncommutative case

Case : $n \geq 2$, $m \in \mathbb{N}$, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$

- The study of general noncommutative varieties $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ in $B(\mathcal{H})^n$, where $m \geq 1$, f is a positive regular free holomorphic function, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$ is any set of noncommutative polynomials such that $q(0) = 0$ for any $q \in \mathcal{Q}$, was initiated in 2006 ($m = 1$, $f = Z_1 + \cdots + Z_n$).

• G. Popescu, Operator theory on noncommutative varieties, *Indiana Univ. Math. J.* **56** (2006), 389–442.

• G. Popescu, Noncommutative Berezin transforms and multivariable operator model theory, *J. Funct. Anal.* **254** (2008), no. 4, 1003–1057 (no characteristic functions).

• G. Popescu, Operator theory on noncommutative domains, *Mem. Amer. Math. Soc.* **205** (2010), no. 964, vi+124 pp. Case ($m = 1$).

Our goals

Case : $n \geq 2$, $m \in \mathbb{N}$, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$

- To provide a Beurling type characterization of the joint invariant subspaces under the universal model $(B_1^{(m)}, \ldots, B_n^{(m)})$, when $m \geq 2$, and to parameterize the corresponding wandering subspaces.

- To characterize the elements in the noncommutative variety $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ which admit characteristic functions, develop an operator model theory for the completely non-coisometric elements, and prove that the characteristic function is a complete unitary invariant for this class of elements.

Our goals

All our the results hold in the commutative case :

$$n \geq 2, m \in \mathbb{N}, \textit{and } \mathcal{Q} = \{Z_iZ_j - Z_iZ_j : \; i,j \in \{1, \ldots, n\}\}$$

- In this case, the universal model is $(M_{z_1}, \ldots, M_{z_n})$ acting on the reproducing kernel Hilbert space with kernel $\kappa_{f,m} : \mathcal{D}^1_{f,\circ}(\mathbb{C}) \times \mathcal{D}^1_{f,\circ}(\mathbb{C}) \to \mathbb{C}$ defined by

$$\kappa_{f,m}(\mu, \lambda) := \frac{1}{\left(1 - \sum_{|\alpha| \geq 1} a_\alpha \mu_\alpha \overline{\lambda}_\alpha\right)^m} \quad \text{for all } \lambda, \mu \in \mathcal{D}^1_{f,\circ}(\mathbb{C}),$$

where

$$\mathcal{D}^1_{f,\circ}(\mathbb{C}) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{|\alpha| \geq 1} a_\alpha |\lambda_\alpha|^2 < 1 \right\}.$$

REFERENCES

Case : $n \geq 2$, $m \in \mathbb{N}$, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n\rangle$

- G.Popescu, *Invariant subspaces and operator model theory on noncommutative varieties*, Math. Ann. **372** (2018), no. 1-2, 611–650.
- G.Popescu, *Noncommutative hyperballs, wandering subspaces and inner functions*, J. Funct. Anal., to appear in 2019.

Noncommutative Berezin kernel

- Let $T := (T_1, \ldots, T_n) \in \mathcal{D}_f^m(\mathcal{H})$ and let $K_{f,T}^{(m)} : \mathcal{H} \to F^2(H_n) \otimes \overline{\Delta_{f,m,T}(\mathcal{H})}$ be the map defined by

$$K_{f,T}^{(m)} h := \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes \Delta_{f,m,T} T_\alpha^* h, \qquad h \in \mathcal{H},$$

where $\Delta_{f,m,T} := \left[(I - \Phi_{f,T})^m (I)\right]^{1/2}$ and

$$\Phi_{f,T}(Y) = \sum_{|\alpha| \geq 1} a_\alpha T_\alpha Y T_\alpha^*, \quad Y \in B(\mathcal{H})$$

Noncommutative Berezin kernel

Definition

The *constrained noncommutative Berezin kernel* associated with the n-tuple $T \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ is the bounded operator $K^{(m)}_{f,T,\mathcal{Q}} : \mathcal{H} \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \overline{\Delta_{f,m,T}(\mathcal{H})}$ defined by

$$K^{(m)}_{f,T,\mathcal{Q}} := (P_{\mathcal{N}^m_{f,\mathcal{Q}}} \otimes I_{\overline{\Delta_{f,m,T}(\mathcal{H})}})K^{(m)}_{f,T},$$

where $K^{(m)}_{f,T}$ is the Berezin kernel associated with $T \in \mathcal{D}^m_f(\mathcal{H})$ and $\mathcal{N}^m_{f,\mathcal{Q}} \subset F^2(H_n)$ is the model space on which the universal model $(B^{(m)}_1, \ldots, B^{(m)}_n)$ is acting.

Noncommutative Berezin kernel

Main properties :

- $K_{f,T,\mathcal{Q}}^{(m)} T_i^* = (B_i^{(m)*} \otimes I) K_{f,T,\mathcal{Q}}^{(m)}, \qquad i \in \{1, \ldots, n\}.$
- When T is a pure n-tuple, i.e. $\Phi_{f,T}^k(I) \to 0$, as $k \to \infty$, the constrained noncommutative Berezin kernel $K_{f,T,\mathcal{Q}}^{(m)}$ is an isometry.

Beurling type factorization result

- Beurling type factorization result

Theorem

Let $X = (X_1, \ldots, X_n)$ be a pure n-tuple of operators in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$ and let $Y \in B(\mathcal{K})$ be a self-adjoint operator. Then the following statements are equivalent :

(i) *There is a Hilbert space $\mathcal{E}$ and $\Psi : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E} \to \mathcal{K}$ such that*

$$Y = \Psi\Psi^* \quad \text{and} \quad \Psi(B_i^{(1)} \otimes I_{\mathcal{E}}) = X_i \Psi, \qquad i \in \{1, \ldots, n\},$$

where $(B_1^{(1)}, \ldots, B_n^{(1)})$ is the universal model of $\mathcal{V}^1_{f,\mathcal{Q}}$.

(ii) $\Phi_{f,X}(Y) \leq Y$.

Beurling-Lax-Halmos type representation

Theorem

Let $X = (X_1, \ldots, X_n)$ be a pure n-tuple of operators in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$. The following statements are equivalent.

(i) *$\mathcal{M} \subset \mathcal{K}$ is a joint invariant subspace under $X_1, \ldots, X_n$.*

(ii) *There is a Hilbert space $\mathcal{E}$ and a partial isometry $\Psi : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E} \to \mathcal{K}$ such that*

$$\mathcal{M} = \Psi\left(\mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E}\right) \quad \text{and} \quad \Psi(B_i^{(1)} \otimes I_{\mathcal{E}}) = X_i \Psi,$$

where $(B_1^{(1)}, \ldots, B_n^{(1)})$ is the universal model of the variety $\mathcal{V}^1_{f,\mathcal{Q}}$.

Beurling-Lax-Halmos type representation

Theorem

Let $(B_1^{(m)}, \ldots, B_n^{(m)})$ *be the universal model of the noncommutative variety* $\mathcal{V}_{f,\mathcal{Q}}^m$*, acting on the model space* $\mathcal{N}_{f,\mathcal{Q}}^m$*. The following statements are equivalent.*

(i) $\mathcal{M} \subset \mathcal{N}_{f,\mathcal{Q}}^m \otimes \mathcal{K}$ *is a joint invariant subspace under* $B_1^{(m)} \otimes I_\mathcal{K}, \ldots, B_n^{(m)} \otimes I_\mathcal{K}$.

(ii) *There is a Hilbert space* $\mathcal{E}$ *and a partial isometry* $\Psi : \mathcal{N}_{f,\mathcal{Q}}^1 \otimes \mathcal{E} \to \mathcal{N}_{f,\mathcal{Q}}^m \otimes \mathcal{K}$ *such that*

$$\mathcal{M} = \Psi\left(\mathcal{N}_{f,\mathcal{Q}}^1 \otimes \mathcal{E}\right) \quad \text{and} \quad \Psi(B_i^{(1)} \otimes I_\mathcal{E}) = (B_i^{(m)} \otimes I_\mathcal{K})\Psi.$$

Beurling-Lax-Halmos type representation

Proposition

A bounded operator $M : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E}_1 \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{E}_2$ *satisfies the relation*

$$M(B_i^{(1)} \otimes I_{\mathcal{E}_1}) = (B_i^{(m)} \otimes I_{\mathcal{E}_2})M, \qquad i \in \{1, \dots, n\},$$

if and only if there is $\Phi : F^2(H_n) \otimes \mathcal{E}_1 \to F^2(H_n) \otimes \mathcal{E}_2$ *satisfying the relation*

$$\Phi(W_i^{(1)} \otimes I_{\mathcal{E}_1}) = (W_i^{(m)} \otimes I_{\mathcal{E}_2})\Phi, \qquad i \in \{1, \dots, n\},$$

and such that

$$M = P_{\mathcal{N}^m_{f,\mathcal{Q}}} \Phi|_{\mathcal{N}^1_{f,\mathcal{Q}}}.$$

Uniqueness of representation

- Fix an n-tuple $Y := (Y_1, \ldots, Y_n) \in B(\mathcal{K})^n$. A bounded linear operator $M : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{H} \to \mathcal{K}$ is called multi-analytic with respect to $B^{(1)} := (B^{(1)}_1, \ldots, B^{(1)}_n)$ and $Y := (Y_1, \ldots, Y_n)$ if
$$M(B^{(1)}_i \otimes I_{\mathcal{H}}) = Y_i M, \qquad i \in \{1, \ldots, n\}.$$
- The support of M, $\operatorname{supp} M$, is the smallest reducing subspace $\mathcal{N}$ under $B^{(1)}_1 \otimes I_{\mathcal{H}}, \ldots, B^{(1)}_n \otimes I_{\mathcal{H}}$ such that $M|_{\mathcal{N}^\perp} = 0$.
- We have
$$\operatorname{supp} M = \bigvee_{\alpha \in \mathbb{F}_n^+} (B^{(1)}_\alpha \otimes I_{\mathcal{H}})(\overline{M^*(\mathcal{K})}) = \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{G}, \quad \text{where}$$
$$\mathcal{G} := (P_{\mathbb{C}} \otimes I_{\mathcal{H}}) \left(\overline{M^*(\mathcal{K})}\right), \text{ and } MM^* = (M|_{\operatorname{supp} M})(M|_{\operatorname{supp} M})^*.$$

Uniqueness of representation

Corollary

Let $(X_1, \ldots, X_n)$ be a pure n-tuple in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$. The Beurling-Lax-Halmos type representation for the joint invariant subspace under $X_1, \ldots, X_n$ is essentially unique. More precisely, if

$$\Psi_1\left(\mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E}_1\right) = \Psi_2\left(\mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E}_2\right),$$

where $\Psi_j : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E}_j \to \mathcal{K}$, $j = 1, 2$, are partially isometric multi-analytic operators, then there is a partial isometry $V : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\Psi_1 = \Psi_2(I_{\mathcal{N}^1_{f,\mathcal{Q}}} \otimes V)$. In particular, $\Psi_1|_{\operatorname{supp}\Psi_1}$ coincides with $\Psi_2|_{\operatorname{supp}\Psi_2}$.

Wandering subspaces

- Let $X = (X_1, \ldots, X_n)$ be an n-tuple of operators on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{W} \subset \mathcal{H}$ is called *wandering subspace* for X if

$$\mathcal{W} \perp X_\alpha(\mathcal{W}), \qquad \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1.$$

  If, in addition,

$$\mathcal{H} = \bigvee_{\alpha \in \mathbb{F}_n^+} X_\alpha(\mathcal{W}) \qquad \text{(closed linear span)},$$

  we say that $\mathcal{W}$ is a *generating wandering subspace* for X.
- If $\mathcal{W}$ is a generating wandering subspace for X, then

$$\mathcal{W} = \mathcal{H} \ominus \sum_{i=1}^{n} X_i(\mathcal{H}),$$

  which shows that $\mathcal{W}$ is uniquely determined by X.

Wandering subspaces

Theorem

Let $X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ *and let* $\Theta : \mathcal{N}^1_{f,Q} \otimes \mathcal{E} \to \mathcal{H}$ *be a partial isometry such that* $\Theta(B_i^{(1)} \otimes I_{\mathcal{E}}) = X_i\Theta$, $i \in \{1, \ldots, n\}$, *where* $(B_1^{(1)}, \ldots, B_n^{(1)})$ *is the universal model of* $\mathcal{V}^1_{f,\mathcal{Q}}$. *Then*

(i) *the closed subspace* $\mathcal{M} := \Theta\left(\mathcal{N}^1_{f,Q} \otimes \mathcal{E}\right)$ *is invariant under* $X_1, \ldots, X_n$.

(ii) $\mathcal{W} := \mathcal{M} \ominus \sum X_i(\mathcal{M})$ *is the wandering subspace for* $X_1|_{\mathcal{M}}, \ldots, X_n|_{\mathcal{M}}$.

(iii) $\mathcal{W} = \Theta(\mathcal{L})$, *where* $\mathcal{L} := (\mathrm{range}\, \Theta^*) \cap \mathcal{E}$.

(iv) $\bigvee_{\alpha \in \mathbb{F}_N^+} X_\alpha(\mathcal{W}) = \overline{\Theta\left(\mathcal{N}^1_{f,Q} \otimes \mathcal{L}\right)}$.

Wandering subspaces

- Let $X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ and let $\Psi : \mathcal{N}^1_{f,Q} \otimes \mathcal{E} \to \mathcal{H}$ be a bounded operator such that

$$\Psi(B_i^{(1)} \otimes I_\mathcal{E}) = X_i \Psi, \qquad i \in \{1, \ldots, n\}.$$

  We say that Ψ is *$(B^{(1)}, X)$-quasi-inner* if $\|\Psi(1 \otimes x)\| = \|x\|$ for all $x \in \mathcal{E}$ and

$$\Psi(1 \otimes \mathcal{E}) \perp X_\alpha \left(\Psi(1 \otimes \mathcal{E})\right), \qquad \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1.$$

- Ψ is uniquely determined by its restriction $\Psi|_\mathcal{E} : \mathcal{E} \to \mathcal{H}$, which can be seen as the symbol of Ψ.

Wandering subspaces

Theorem

Let $X = (X_1, \ldots, X_n)$ be a pure element in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ and let $\psi : \mathcal{E} \to \mathcal{H}$ be an isometry such that

$$\psi(\mathcal{E}) \perp X_\alpha(\psi(\mathcal{E})), \qquad \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1.$$

Then ψ has a unique extension to a bounded operator $\Psi : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E} \to \mathcal{H}$ such that

$$\Psi(B_i^{(1)} \otimes I_\mathcal{E}) = X_i \Psi, \qquad i \in \{1, \ldots, n\}.$$

Moreover, Ψ is a contraction.

Wandering subspaces

Theorem

Let $X = (X_1, \ldots, X_n)$ *be a pure element in* $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ *and let* $\mathcal{M} := \Theta\left(\mathcal{N}^1_{f,Q} \otimes \mathcal{E}\right)$ *be a joint invariant subspace for* $X_1, \ldots, X_n$, *where* $\Theta : \mathcal{N}^1_{f,Q} \otimes \mathcal{E} \to \mathcal{H}$ *is a partially isometric multi-analytic operator with respect to* $B^{(1)}$ *and* X. *If* $\mathcal{L} := \mathcal{E} \cap \operatorname{range} \Theta^*$ *and* $\Psi := \Theta|_{\mathcal{N}^1_{f,Q} \otimes \mathcal{L}} : \mathcal{N}^1_{f,Q} \otimes \mathcal{L} \to \mathcal{H}$, *then*

(i) Ψ *is a* $(B^{(1)}, X)$*-quasi-inner multi-analytic operator.*

(ii) *The wandering subspace* $\mathcal{W} := \mathcal{M} \ominus \sum_{i=1}^n X_i(\mathcal{M})$ *of the n-tuple* $(X_1|_{\mathcal{M}}, \ldots, X_n|_{\mathcal{M}})$ *satisfies the relation* $\mathcal{W} = \Psi(\mathcal{L})$.

(iii) *The wandering subspace* $\mathcal{W}$ *is generating for* $(X_1|_{\mathcal{M}}, \ldots, X_n|_{\mathcal{M}})$ *if and only if* $\operatorname{range} \Theta^* \perp \mathcal{N}^1_{f,Q} \otimes \mathcal{L}$.

Characteristic functions

Definition

An n-tuple $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ is said to have constrained characteristic function if there is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{E} \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{D}^m_{f,T}$, i.e.

$$\Psi(B_i^{(1)} \otimes I_{\mathcal{E}}) = (B_i^{(m)} \otimes I_{\mathcal{D}^m_{f,T}})\Psi, \qquad i \in \{1, \ldots, n\},$$

such that

$$K^{(m)}_{f,T,\mathcal{Q}} \left(K^{(m)}_{f,T,\mathcal{Q}}\right)^* + \Psi\Psi^* = I,$$

where $K^{(m)}_{f,T,\mathcal{Q}}$ is the constrained noncommutative Berezin kernel of $T \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ and $\mathcal{D}^m_{f,T} := \overline{\Delta_{f,m,T}(I)(\mathcal{H})}$.

Characteristic functions

Proposition

An n-tuple $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ *admits a constrained characteristic function if and only if the defect operator* $I - K^{(m)}_{f,T,\mathcal{Q}} \left(K^{(m)}_{f,T,\mathcal{Q}}\right)^*$ *is a solution of the inequation*

$$\Phi_{f,B^{(m)}\otimes I_{\mathcal{D}^m_{f,T}}}(Y) \leq Y, \qquad Y \in B(\mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{D}^m_{f,T}),$$

where $B^{(m)} \otimes I_{\mathcal{D}^m_{f,T}} := (B^{(m)}_1 \otimes I_{\mathcal{D}^m_{f,T}}, \ldots, B^{(m)}_n \otimes I_{\mathcal{D}^m_{f,T}})$ *and* $K^{(m)}_{f,T,\mathcal{Q}}$ *is the constrained Berezin kernel of the n-tuple* T*.*

Operator model theory

- Any pure n-tuple $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ admits a constrained characteristic function.
- If $m = 1$, any n-tuple $T \in \mathcal{V}^1_{f,\mathcal{Q}}(\mathcal{H})$ admits characteristic function.
- We say that $T := (T_1, \ldots, T_n) \in \mathbf{D}^m_f(\mathcal{H})$ is completely non-coisometric if there is no nonzero vector $h \in \mathcal{H}$ such that $\left\langle (id - \Phi^k_{f,T})(I_{\mathcal{H}})h, h \right\rangle = 0$ for any $k \in \mathbb{N}$.
- We can develop an operator model theory for the completely non-coisometric elements in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ which admit characteristic functions and prove that the characteristic function is a complete unitary invariant for this class of elements.

Operator model theory

- Particular case : pure elements in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$

Theorem

Let $T = (T_1, \ldots, T_n)$ be an element in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$. Then T is pure if and only if the constrained characteristic function $\Theta_{f,T,\mathcal{Q}}$ is a partially isometric multi-analytic operator. Moreover, in this case T is unitarily equivalent to $G = (G_1, \ldots, G_n)$, where

$$G_i := P_{\mathbf{H}_{f,T,\mathcal{Q}}} \left(B_i^{(m)} \otimes I_{\mathcal{D}} \right) |_{\mathbf{H}_{f,T,\mathcal{Q}}},$$

$\mathcal{D} := \overline{\Delta_{f,m,T}(I)(\mathcal{H})}$*, and* $P_{\mathbf{H}_{f,T,\mathcal{Q}}}$ *is the orthogonal projection of* $\mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{D}$ *onto* $\mathbf{H}_{f,T,\mathcal{Q}} := \left(\mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{D} \right) \ominus \operatorname{range} \Theta_{f,T,\mathcal{Q}}$.

Operator model theory

- We say that two multi-analytic operators $F : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{K}_1 \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{K}_2$ and $F' : \mathcal{N}^1_{f,\mathcal{Q}} \otimes \mathcal{K}'_1 \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{K}'_2$ *coincide* if there are two unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}'_j)$, $j = 1, 2$, such that

$$F'(I_{\mathcal{N}^1_{f,\mathcal{Q}}} \otimes \tau_1) = (I_{\mathcal{N}^m_{f,\mathcal{Q}}} \otimes \tau_2)F.$$

Theorem

Let $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ and $T' := (T'_1, \ldots, T'_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H}')$ be two completely non-coisometric n-tuples which admit characteristic functions. Then T and T' are unitarily equivalent if and only if their characteristic functions $\Theta_{f,T,\mathcal{Q}}$ and $\Theta_{f,T',\mathcal{Q}}$ coincide.

Minimal dilations, uniqueness

- If $\mathcal{E}$ is an arbitrary Hilbert space, we say that $(B_1 \otimes I_\mathcal{E}, \ldots, B_n \otimes I_\mathcal{E})$ is a dilation of $(T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ if there is an isometry $V : \mathcal{H} \to \mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{E}$ such that

$$VT_i^* = (B_i^{(m)*} \otimes I_\mathcal{E})V, \qquad i \in \{1, \ldots, n\}.$$

If, in addition,

$$\mathcal{N}^m_{f,\mathcal{Q}} \otimes \mathcal{E} = \bigvee_{\alpha \in \mathbb{F}_n^+} (B_\alpha^{(m)} \otimes I_\mathcal{E})V(\mathcal{H}),$$

then the dilation is called *minimal*. We say that $\mathcal{V}^m_{f,\mathcal{Q}}$ has *unique minimal dilation* if each pure element in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ has a unique minimal dilation up to an isomorphism.

Minimal dilations, uniqueness

Theorem

Let q be a positive regular noncommutative polynomial such that

$$\lim_{|\gamma|\to\infty}\left(\frac{b^{(m)}_{g_p\gamma}}{b^{(m)}_{g_i g_p\gamma}} - \frac{b^{(m)}_{\gamma}}{b^{(m)}_{g_p\gamma}}\right) = 0,$$

for any $i, p \in \{1, \ldots, n\}$. Then any pure n-tuple $T \in \mathcal{D}^m_q(\mathcal{H})$ has a unique minimal dilation up to an isomorphism.

- The noncommutative *m*-hyperball $\mathcal{D}^m(\mathcal{H})$ (which corresponds to $q = Z_1 + \cdots + Z_n$) has the unique minimal dilation property. In this case, $b^{(m)}_{g_0} = 1$ and $b^{(m)}_{\alpha} = \binom{|\alpha| + m - 1}{m - 1}$ if $\alpha \in \mathbb{F}_n^+, |\alpha| \geq 1$.

Wandering subspaces

- If $\mathcal{M}$ is an arbitrary joint invariant subspace under $B_1^{(m)} \otimes I_{\mathcal{G}}, \ldots, B_n^{(m)} \otimes I_{\mathcal{G}}$ and the variety $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ has the unique minimal dilation property, we can provide an explicit description of the wandering subspace

$$\mathcal{W} := \mathcal{M} \ominus \sum_{i=1}^{n} (B_i^{(m)} \otimes I_{\mathcal{G}})\mathcal{M}$$

and obtain a characterization of the *quasi-inner* multi-analytic operators from $\Theta : \mathcal{N}_{f,\mathcal{Q}}^1 \otimes \mathcal{G}_* \to \mathcal{N}_{f,\mathcal{Q}}^m \otimes \mathcal{G}$, i.e. $\Theta|_{\mathcal{G}_*}$ is an isometry and

$$\Theta(\mathcal{G}_*) \perp (B_\alpha \otimes I_{\mathcal{G}})\Theta(\mathcal{G}_*), \qquad \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1,$$

where $\mathbb{F}_n^+$ is the free semigroup with n generators.

Noncommutative m-hyperball $\mathcal{D}^m(\mathcal{H})$

Theorem

Let $W^{(m)} = (W_1^{(m)}, \ldots, W_n^{(m)})$ be the universal model of $\mathcal{D}^m(\mathcal{H})$. Then $\psi : \mathcal{G}_* \to F^2(H_n) \otimes \mathcal{G}$ is an $(W^{(1)}, W^{(m)})$-quasi-inner operator if and only if there exist a Hilbert space $\mathcal{H}$, a pure element T in $\mathcal{D}^m(\mathcal{H})$, and a matrix operator $\begin{pmatrix} T^* & E \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{G}_* \to \mathcal{H}^{(n)} \oplus \mathcal{G}$ such that its entries satisfy some natural conditions and such that ψ is equal to

$$\left(I \otimes D + (I \otimes C) \sum_{k=1}^{m} \left(I - \sum_{i=1}^{n} \Lambda_i \otimes T_i^* \right)^k \mathbf{\Lambda}(I \otimes E) \right) |_{\mathcal{G}_*},$$

where $\mathbf{\Lambda} := [\Lambda_1 \otimes I_{\mathcal{H}} \cdots \Lambda_n \otimes I_{\mathcal{H}}]$.

Noncommutative m-hyperball $\mathcal{D}^m(\mathcal{H})$

- This extends to our noncommutative setting the corresponding result obtained by Olofsson (when $n = 1$) and by Eschmeier in the multivariable commutative case.

Polydomains and varieties

- Let $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be the set of all tuples $\mathbf{X} := (X_1, \ldots, X_k)$ in $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$.
- Let $\mathbf{f} = (f_1, \ldots, f_k)$ be positive regular free holomorphic functions, $\mathbf{m} := (m_1, \ldots, m_k)$, $\mathbf{n} := (n_1, \ldots, n_k)$. The *regular polydomain* $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ is the set of all k-tuples $\mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ such that

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{X}}^{\mathbf{p}}(I) \geq 0 \ \text{ for } \ \mathbf{0} \leq \mathbf{p} \leq \mathbf{m}, \quad \mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{N}^k$$

where

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{X}}^{p} := (id - \Phi_{f_1,X_1})^{m_1} \circ \cdots \circ (id - \Phi_{f_k,X_k})^{m_k}$$

Polydomains and varieties

- For each $i \in \{1, \ldots, k\}$, let $Z_i := (Z_{i,1}, \ldots, Z_{i,n_i})$ be an n_i-tuple of noncommutative indeterminates and assume that, for any $t, s \in \{1, \ldots, k\}$, $s \neq t$, the entries of Z_t are commuting with the entries if Z_s.
- We study noncommutative varieties in the polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$, given by

$$\mathcal{V}_{\mathbf{f},\mathcal{Q}}^\mathbf{m}(\mathcal{H}) := \{\mathbf{X} \in \mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H}) : \ g(\mathbf{X}) = 0 \text{ for all } g \in \mathcal{Q}\},$$

where $\mathcal{Q}$ is a set of polynomials in noncommutative indeterminates $Z_{i,j}$, which generates a nontrivial ideal in $\mathbb{C}[Z_{i,j}]$.

Polydomains and varieties

- Let $\mathcal{Q}_i \subset \mathbb{C}\langle Z_i\rangle$ be a set of noncommutative polynomials such that $q(0) = 0$ for any $q \in \mathcal{Q}_i$, and set $\mathcal{Q} := \cup_{i=1}^k \mathcal{Q}_i \subset \mathbb{C}\left\langle Z_{i,j}\right\rangle$.
- Let $\{\mathbf{B}_{i,j}^{(\mathbf{m})}\}$ be the universal model of the noncommutative variety $\mathcal{V}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}}$, acting on the appropriate model space $\mathcal{N}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}}$.
- For each $i \in \{1, \ldots, k\}$, let $(B_{i,1}^{(1)}, \ldots, B_{i,n_i}^{(1)})$ be the universal model of the variety $\mathcal{V}_{f_i,\mathcal{Q}_i}^{1}$, acting on the model space $\mathcal{N}_{f_i,\mathcal{Q}_i}^{1}$.

Polydomains and varieties

- We can obtain a Beurling-Lax-Halmos type characterization for the joint invariant subspaces $\mathcal{M} \subset \mathcal{N}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}} \otimes \mathcal{K}$ under the operators $\mathbf{B}^{(\mathbf{m})}_{i,j} \otimes I_{\mathcal{K}}$.

Theorem

$\mathcal{M}$ is an invariant subspace under the operators $\mathbf{B}^{(\mathbf{m})}_{i,j} \otimes I_{\mathcal{K}}$ if and only if there are Hilbert spaces $\mathcal{E}_i$ and partial isometries $\psi_i : \mathcal{N}^{1}_{f_i,\mathcal{Q}_i} \otimes \mathcal{E}_i \to \mathcal{N}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}} \otimes \mathcal{K}$ such that $\mathcal{M} = \psi_i(\mathcal{N}^{1}_{f_i,\mathcal{Q}_i} \otimes \mathcal{E}_i)$ and

$$\psi_i(B^{(1)}_{i,j} \otimes I_{\mathcal{E}_i}) = (\mathbf{B}^{(\mathbf{m})}_{i,j} \otimes I_{\mathcal{K}})\psi_i$$

for any $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$. Therefore, we have

$$P_{\mathcal{M}} = \psi_1\psi_1^* = \cdots = \psi_k\psi_k^*.$$

Polydomains and varieties

- If $\mathbf{T} \in \mathcal{V}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}}(\mathcal{H})$ is pure, then we can find multi-analytic operators $\theta_i : \mathcal{N}^1_{f_i,\mathcal{Q}_i} \otimes \mathcal{E}_i \to \mathcal{N}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}} \otimes \mathcal{D}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}$ such that

$$K^{(\mathbf{m})}_{\mathbf{f},\mathbf{t},\mathcal{Q}}(K^{(\mathbf{m})}_{\mathbf{f},\mathbf{t},\mathcal{Q}})^* + \psi_i\psi_i^* = I, \quad i \in \{1,\ldots,k\},$$

  where $\mathcal{D}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}$ is an appropriate defect space associated with $\mathbf{T}$ and $K^{(\mathbf{m})}_{\mathbf{f},\mathbf{t},\mathcal{Q}}$ is the corresponding Berezin kernel.

- The k-tuple $\Theta_{\mathbf{T}} := (\theta_1,\ldots,\theta_k)$ can be viewed as a *characteristic function* of $\mathbf{T}$.
- As in the case of the regular domains, $\Theta_{\mathbf{T}}$ can be defined for a larger class of tuples in $\mathcal{V}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}}(\mathcal{H})$ (namely, the completely non-coisometric elements).

Polydomains and varieties

OPEN PROBLEM

- It remains to be seen if $\Theta_{\mathbf{T}}$ can be used to provide an operator model that enjoys properties similar to those from the classical case or the regular domains.

THANK YOU