Wold decompositions and Bergman shifts

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Joint work with Sebastian Langendörfer

Classical Wold decomposition

- $H^2(\mathbb{D}) = H(\frac{1}{1-z\overline{w}})$ Hardy space
- $M_z^* M_z = 1_{H^2(\mathbb{D})}$ Hardy space shift

Up to unitaries and multiplicity there are no other isometries:

Theorem (Wold decomposition)

Let $T \in L(H)$. Then $T^*T = 1_H$ iff

$$T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)$$

with

- $T_0 = T|H_0$ *unitary and*
- $T_1 = T|H_1 \cong M_z \in L(H^2(\mathbb{D}, \mathcal{D}))$ *for some Hilbert space* $\mathcal{D}$.

In this case:

$$H_0 = \bigcap_{k \geq 0} T^k H \quad \text{and} \quad H_1 = \bigvee_{k \geq 0} T^k (H \ominus \mathrm{Im}\, T)$$

Call T pure if $H_0 = \{0\}$. Define $W(T) = H \ominus \mathrm{Im}\, T$ (Wandering subspace)

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Generalized Bergman spaces

Let $H_m(\mathbb{D})$ be the functional Hilbert space with kernel

$$K = K_m : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, K_m(z, w) = \frac{1}{(1 - z\overline{w})^m}.$$

Then

- $H_1(\mathbb{D}) = H^2(\mathbb{D})$ is the Hardy space,
- $H_2(\mathbb{D}) = L^2_a(\mathbb{D})$ is the Bergman space and
- $H_{2+m}(\mathbb{D}) = L^2_a(\mathbb{D}, (1 - |z|^2)^m d\lambda)$ are the weighted Bergman spaces.

Question: Is there an algebraic operator identity characterizing:

Bergman shifts (with multiplicity) $\oplus$ 'nice operators' ?

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General scheme

Let c_k be the coefficients determined by

$$(1 - z\overline{w})^m = \sum_k c_k (z\overline{w})^k \qquad (m = 1 : \quad c_0 = 1 = -c_1)$$

Define $\qquad \sigma_T(X) = TXT^*$

- $\frac{1}{K}(T) = \sum_k c_k \sigma_T^k(1_H) \qquad (\overset{m=1}{=} 1_H - TT^*)$
- $\Delta_T \quad = \sum_k (-c_{k+1}) \sigma_T^k(1_H) \qquad (\overset{m=1}{=} 1_H)$

Theorem (Wold decomposition: $m = 1$)

Let $T \in L(H)$ be left invertible. Then

$$(T^*T)^{-1} = \Delta_T$$

if and only if $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)$ with

- *$T_0 = T|H_0$ invertible with $\frac{1}{K}(T_0) = 0$ (invertible coisometry)*
- *$T_1 = T|H_1 \cong M_z \in L(H(K) \otimes \mathcal{D})$ for some $\mathcal{D}$.*

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Bergman-type Wold decompositions

Let $H_m(\mathbb{D})$ be the functional Hilbert space with kernel

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Call $T \in L(H)$ an *m-coisometry* if $\frac{1}{K}(T) = 0$.

Theorem (Giselsson-Olofsson '12)

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Invertible m-coisometry = unitary for $m = 1, 2$, but not for $m \geq 3$ (Agler-Stankus 1995)

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Ball case

Let $T \in L(H)^n$ be commuting. Regard T as a row operator

$$T : H^n \to H, (x_i) \mapsto \sum_{1 \leq i \leq n} T_i x_i$$

and write $T^* : H \to H^n, x \mapsto (T_i^* x)$ for its adjoint. Suppose that

$$\mathrm{Im}\ T = \sum_{i=1}^{n} T_i H \subset H \quad \text{is closed}$$

$$\Rightarrow T^* T : \mathrm{Im}\ T^* \to \mathrm{Im}\ T^* \quad \text{is invertible}$$

Define $L = (T^* T)^{-1} T^* \in L(H, H^n)$ and

$$P : B_{\frac{1}{\|L\|}}(0) \to L(H), P(z) = (T - z)L(1 - zL)^{-1}$$

Lemma

$P : B_{\frac{1}{\|L\|}}(0) \to L(H)$ is analytic with $P(z)^2 = P(z)$ and $\mathrm{Im}\ P(z) \subset \mathrm{Im}(T - z)$,

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Regularity replaces left invertibility

$T \in L(H)^n_{\mathrm{com}}$ is called regular if for $|z|$ small enough

$$H = \mathrm{Im}(T - z) \oplus W(T) \text{ algebraic direct sum of closed subspaces,}$$

or equivalently, if $\mathrm{Im}\ T$ is closed and $\mathrm{Im}(T - z) \cap W(T) = \{0\}$.

Lemma

T regular $\Rightarrow$ $\mathrm{Im}\, P(z) = \mathrm{Im}(T - z)$ for all $|z| < 1/\|L\|$

Examples of regular tuples: $T \in L(H)^n_{\mathrm{com}}$ with

- $\mathrm{Im}\ T \subset H$ closed and $H^{n-1}(T, H) = \{0\}$
- $\mathrm{Im}\ T = H$
- $\dim H/\mathrm{Im}(T - z) = N < \infty$ for $|z| < \epsilon$.

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Multivariable Shimorin model

Let $T \in L(H)^n$ be regular. Define $\Omega = B_{1/\|L\|}(0)$

Theorem ($n = 1$: Shimorin 2001)

The mapping $V : H \to \mathcal{O}(\Omega, W(T))$,

$$(Vx)(z) = (1_H - P(z))x = P_{W(T)}(1_H - zL)^{-1}x$$

is continuous linear with $VT_i = M_{z_i}V$ *and*

$$\ker V = \bigcap_{k \in \mathbb{N}} \sum_{|\alpha|=k} T^\alpha H = \bigcap_{z \in \Omega} \mathrm{Im}(T - z).$$

Theorem

$\mathcal{H} = VH \subset \mathcal{O}(\Omega, W(T))$ *is a functional Hilbert space with kernel*

$$K(z, w) = P_{W(T)}(1_H - zL)^{-1}(1_H - L^*w^*)^{-1}|W(T)$$

and

$$T/\ker V \cong M_z \in L(\mathcal{H})^n$$

Call T analytic if $\ker V = \{0\}$.

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Generalized Bergman spaces on the unit ball

Let $H_m(\mathbb{B})$ be the functional Hilbert space with kernel

$$K_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m}$$

Then

$$H_m(\mathbb{B}) = \{\sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B});\ \|f\|^2 = \sum_{\alpha\in\mathbb{N}^n} \frac{|f_\alpha|^2}{\rho_m(\alpha)} < \infty\}$$

and

$$H_1(\mathbb{B}) = \text{Drury-Arveson space}$$

$$H_n(\mathbb{B}) = \{f \in \mathcal{O}(\mathbb{B});\ \sup_{0<r<1} \int_S |f(r\xi)|^2 d\xi < \infty\} \text{ Hardy space}$$

$$H_{n+1}(\mathbb{B}) = L^2_a(\mathbb{B},\mathcal{D}) = \{f \in \mathcal{O}(\mathbb{B});\ \int_{\mathbb{B}} |f|^2 dz < \infty\} \text{ Bergman space}$$

$$H_{n+1+k}(\mathbb{B}) = \{f \in \mathcal{O}(\mathbb{B});\ \int_{\mathbb{B}} |f|^2 (1-|z|^2)^k dz < \infty\}$$

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The Bergman shifts $M_z = (M_{z_1}, \dots, M_{z_n}) \in L(H(K_m))^n$ are regular with

$$(M_z^* M_z)^{-1} = (\oplus \Delta_{M_z})|\,\mathrm{Im}\ M_z^*$$

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$\Delta_{M_z}(\sum f_k) = \sum \frac{a_{k+1}}{a_k} f_k$ is a diagonal operator wrt the homogeneous expansion.

General scheme

Let c_k be the coefficients determined by

$$(1 - \langle z, w\rangle)^m = \sum_k c_k \langle z, w\rangle^k$$

For $T \in L(H)^n_{\mathrm{com}}$ define $\quad (\sigma_T(X) = \sum_{1 \le i \le n} T_i X T_i^*)$

- $\frac{1}{K}(T) = \sum_k c_k \sigma_T^k(1_H)$,
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Wold decomposition on the unit ball

Call $T \in L(H)^n$ an m-coisometry if $\frac{1}{K_m}(T) = 0$.

Theorem

Let $T \in L(H)^n$ be a regular . Then

$$(T^*T)^{-1} = (\oplus \Delta_T)|\,\mathrm{Im}\ T^*$$

if and only if $T = T_0 \oplus T_1 \in L(H_0 \oplus H_1)^n$ *with*

- $T_0 = T|H_0$ *is an m-coisometry*
- $T_1 = T|H_1 \cong M_z \in L(H(K_m) \otimes \mathcal{D})^n$

In this case

$$H_0 = \bigcap_{k \geq 0} \Big(\sum_{|\alpha|=k} T^\alpha H \Big) \quad \text{and} \quad H_1 = \bigvee_\alpha T^\alpha W(T)$$

Call T analytic if $H_0 = \{0\}$.

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Theorem

Let $T \in L(H)^n$ be a regular commuting tuple with

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Equivalent are:

- *T is analytic*
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- *$T \cong M_z \in L(H(K_m) \otimes \mathcal{D})^n$*

Special case: (almost Richter-Sundberg 2010) If $m = 1$, then $\Delta_T = 1_H$ and

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Brown-Halmos theorem

Brown-Halmos '63: For $T \in L(H^2(\mathbb{D}))$

$$T = T_f \text{ with } f \in L^\infty(\mathbb{T}) \quad \Leftrightarrow \quad M_z^* T M_z = T$$

Englis '92: No such characterization on $L_a^2(\mathbb{D})$!

Louhichi-Olofsson ' 08: On $H_m(\mathbb{D})$ define

$$M_z' = M_z(M_z^* M_z)^{-1} \quad (\overset{m=1}{=} M_z) \qquad \text{Cauchy dual of } M_z$$

Then for $T \in L(H_m(\mathbb{D}))$

$$T = T_f \quad \text{with} \quad f \in h_\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C};\, f \text{ bounded harmonic}\}$$

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Toeplitz operators with pluriharmonic symbols

Define: $\mathrm{Ph}_\infty(\mathbb{B}) = \{f : \mathbb{B} \to \mathbb{C};\ f$ bounded pluriharmonic $\}$ and

$$T_f = T_g + T_h^* \quad \text{for } f = g + \overline{h} \in \mathrm{Ph}_\infty(\mathbb{B}) \text{ with } g, h \in \mathcal{O}(\mathbb{B})$$

Theorem (Langendörfer-E. '19)

For $T \in L(H_m(\mathbb{B}))$ equivalent are:

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- $\widetilde{T} : \mathbb{B} \to \mathbb{C}, z \mapsto \langle T k_z, k_z \rangle$ *is pluriharmonic*

In this case: $f = \widetilde{T}$ *and* $\|f\|_\infty \leq \|T\|_e$ *(equality for* $m \geq n$*).*

On the Drury-Arveson space $H_1(\mathbb{B})$: $\Delta_{M_z}(T) = T$ and $M_z' = M_z$

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Thank you! That's all.