Uniqueness of Solutions Versus Convex Integration for Conservation Laws in One Space Dimension

Sam G. Krupa
Joint work with Alexis F. Vasseur and László Székelyhidi, Jr.

The University of Texas at Austin

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Introduction

Today we will be thinking about...

- Hyperbolic Systems of Conservation Laws in One Space Dimension
- Entropies
- Uniqueness of solutions

Two parts of the talk: the positive side of uniqueness and the negative side (convex integration).

The Positive Side for Uniqueness

Theory of uniqueness

Theory of uniqueness is largely open. Best theory so far is

Bressan, Crasta, and Piccoli ('00) .

Progress on developing theory of uniqueness has been slow: systems often only admit one entropy.

We look for new ideas.

In this talk, we will lay out a newly developed framework for proving uniqueness of solutions.

We use

- relative entropy method
- the existence of only a single entropy
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Our methods have no small data restrictions.

Plan for the positive part of talk

The plan for the positive part of the talk...

- briefly introduce Burgers equation and the scalar conservation laws
- lay out framework
- discuss briefly the tools in the framework
- apply framework to prove uniqueness for solutions to Burgers verifying only one entropy condition
  - Proven first by Panov ('94). Proven again by De Lellis, Otto and Westdickenberg ('04).
- why we hope framework will work on systems

The scalar conservation laws

The System: Scalar Conservation Law in One Space Dimension

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- *classical* (strong) and *weak* solutions

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- $A \in C^2(\mathbb{R})$ and strictly convex is the given *flux function*.
- *classical* (strong) and *weak* solutions
- Burgers equation $\implies A(u) = \frac{u^2}{2}$

Resolution of non-uniqueness

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- In particular, for scalar conservation laws in one space dimension: a pair of functions $\eta, q : \mathbb{R} \to \mathbb{R}$ are called an *entropy* and *entropy flux*, respectively, if

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- A solution u is then *entropic* for the entropy η if it satisfies the *entropy inequality*

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

  in a distributional sense, where q is any corresponding entropy flux.

Resolution of non-uniqueness – continued

Two standard theories for existence and uniqueness for bounded weak solutions

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- Oleĭnik ('57) $\implies$ *"condition E."* A solution u satisfies condition E if

$$\begin{cases} \text{There exists a constant } C > 0 \text{ such that} \\ \qquad u(x+z,t) - u(x,t) \leq \frac{C}{t} z \\ \text{for all } t > 0 \text{, almost every } z > 0 \text{, and almost every } x \in \mathbb{R}. \end{cases}$$

Resolution of non-uniqueness – continued

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Kruzhkov $\iff$ Oleĭnik's condition E

Framework for showing uniqueness

Framework for showing uniqueness of weak solutions entropic for a single entropy

1. *Construct a modified weak-strong estimate*

   We start with the famous Dafermos/DiPerna weak-strong estimates for conservation laws.

2. *Approximate the weak solution by a sequence of more regular solutions*
   - *use the above estimate*
3. *Detect structure in weak solution*
4. *Uniqueness follows from the additional structure*

Tools used in the framework

The method of relative entropy (Dafermos and DiPerna 1979)

Given an entropy η, the method of relative entropy considers the quantity

$$\eta(a|b) := \eta(a) - \eta(b) - \eta'(b)(a-b) \text{ for all } a, b \in \mathbb{R}.$$

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$$\eta(a|b) := \eta(a) - \eta(b) - \eta'(b)(a-b) \text{ for all } a, b \in \mathbb{R}.$$

For $\eta \in C^2(\mathbb{R})$ strictly convex, $\eta(a|b)$ is locally quadratic in $a-b$: for all a and b in a fixed compact set,

$$c^*(a-b)^2 \leq \eta(a|b) \leq c^{**}(a-b)^2 \text{ for constants } c^*, c^{**} > 0.$$

Method of relative entropy – fundamentally L^2 theory.

The method of relative entropy (Dafermos and DiPerna 1979)

How does

$$\left\|u(\cdot,t)-\bar{u}(\cdot,t)\right\|_{L^2}$$

grow in time, where

$\bar{u}$ – classical (strong) solution

u – weak solution.

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Weak-strong estimates proved by turning the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

into the *relative* entropy inequality

$$\partial_t \eta(u|\bar{u}) + \partial_x q(u;\bar{u}) \leq 0$$

and taking time derivative of $\int \eta(u|\bar{u})\,dx$.

Beyond classical weak-strong: put discontinuities into $\bar{u}$

In weak-strong stability,

$$\int \eta(\overbrace{u}^{\text{an entropic weak solution goes here}} \mid \underbrace{\bar{u}}_{\text{a continuous solution goes here}})\,dx$$

Discontinuity in continuous solution $\implies$ growth in L^2.

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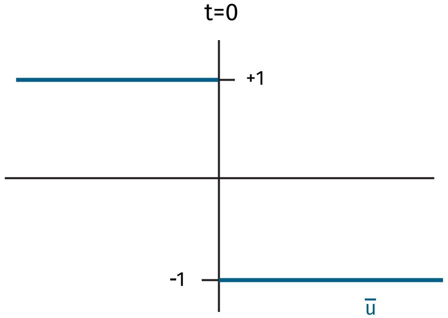
Consider Burgers equation.



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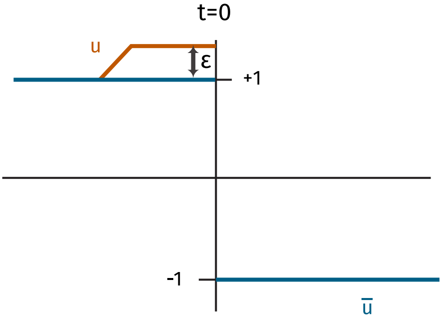

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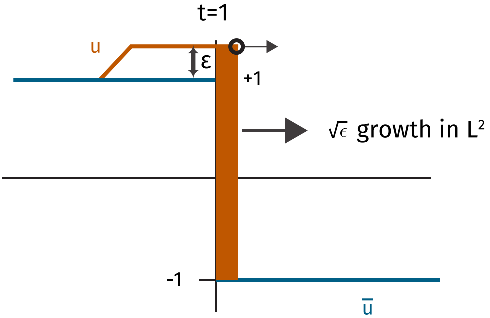
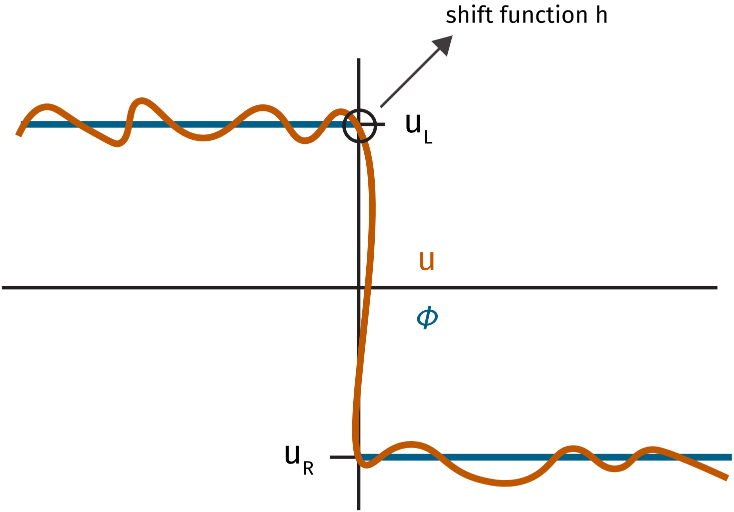


Theory of shifts

Program of stability up to a shift was initiated by Vasseur ('08).

The first result was by Leger ('11) for scalar conservation laws in one space dimension.

Theory of shifts: diagram of Leger's result for scalar



Applying the framework for uniqueness to Burgers

A theorem: Single Entropy Condition for Burgers

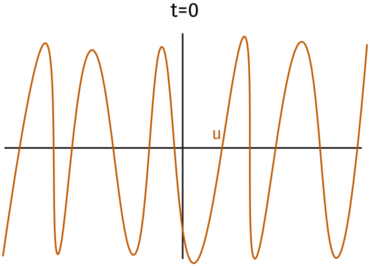
Theorem (K.-Vasseur – JHDE '19)

Let $u \in L^\infty(\mathbb{R} \times [0, \infty))$ *be a weak solution with initial data* $u^0 \in L^\infty(\mathbb{R})$ *to the scalar conservation law in one space dimension with flux* $A \in C^2(\mathbb{R})$ *strictly convex. Assume* u *satisfies the entropy inequality for at least one strictly convex entropy* $\eta \in C^2(\mathbb{R})$. *Further, assume* u *satisfies a strong trace property.*

Then u *is the unique solution to the conservation law verifying Oleĭnik's condition E and with initial data* u^0.

Framework

1. Construct a modified weak-strong estimate
2. Approximate weak solution by a sequence of more regular solutions
3. Detect structure
4. Structure $\implies$ uniqueness



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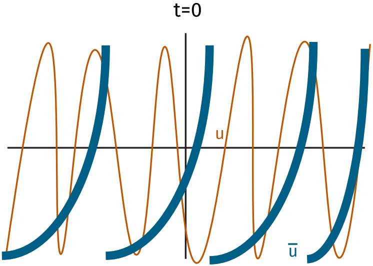

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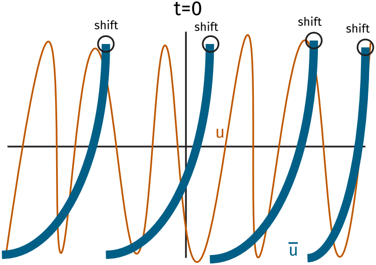

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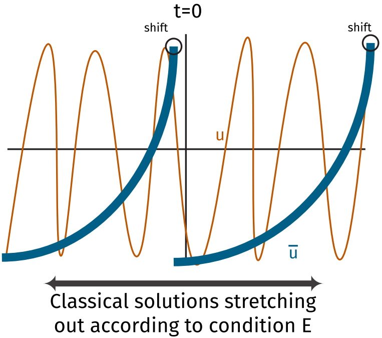

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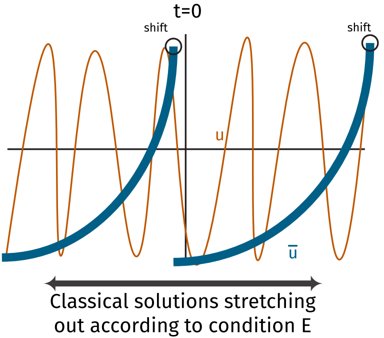

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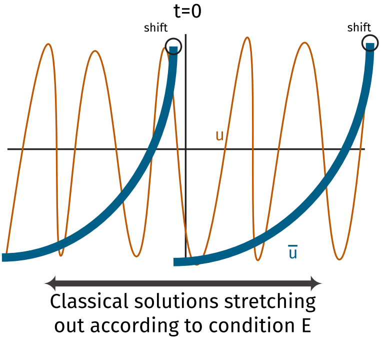

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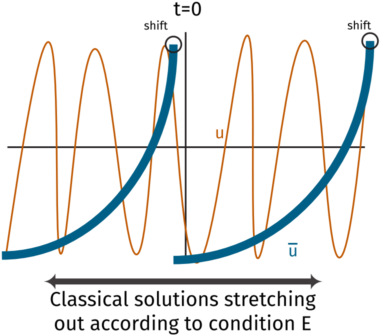

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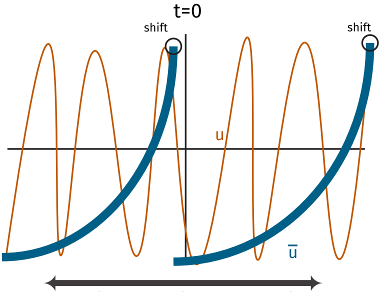

Classical solutions stretching out according to condition E

Hope for systems?

Why we have hope framework will work on systems

an entropic weak solution goes here

$$\int \eta(\boxed{u} \mid \boxed{\bar{u}})\, dx$$

run front tracking in this slot

Hope for systems: L^2 Stability for the Riemann Problem for Systems

Theorem (K. – arXiv:1905.04347)
$\bar{v}$ – classical Riemann solution (a fan) where any shocks are extremal.

u – rough solution with traces. Entropic for a strictly convex entropy. Can have shocks from any family.

Then,

$$\int_{-R}^{R} \left| u(x,t_0) - \Psi_{\bar{v}}(x,t_0) \right|^2 \, dx \leq C \int_{-R-rt_0}^{R+rt_0} \left| u^0(x) - \bar{v}(x,0) \right|^2 \, dx,$$

for $R, t_0 > 0$. $r =$ speed of info.

$\Psi_{\bar{v}}$ is the shifted $\bar{v}$ – we shift each shock.

- For 2×2 systems, all shocks are extremal shocks.
- This theorem applies to the full Euler system.
- The theorem holds for a large class of systems.

Difficulty in the Proof of L^2 Stability for Riemann Problem

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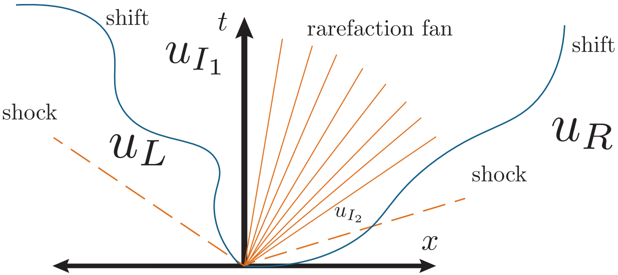

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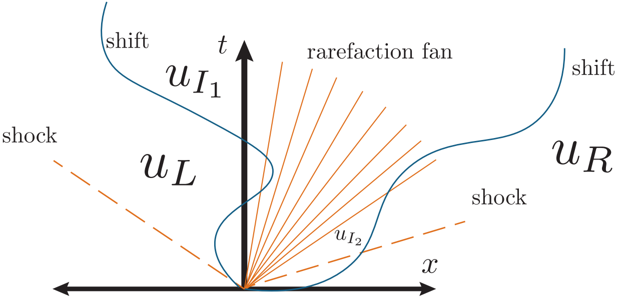

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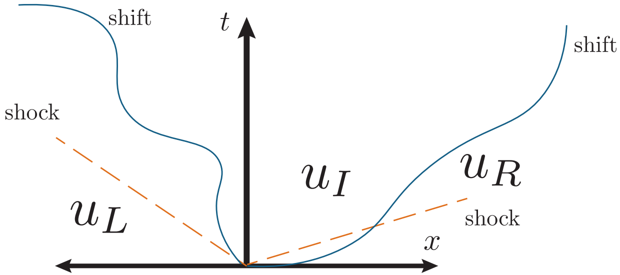

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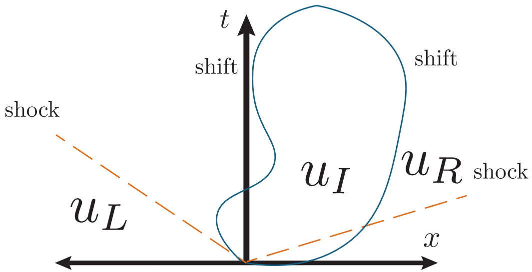


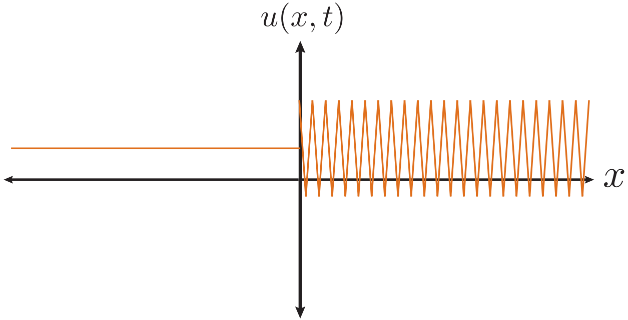
Further Positive Results

The theory of shifts within the context of the relative entropy method is now mature. On top of the L^2 Riemann stability, in other recent results,

- We can handle a non-local source term (Burgers–Hilbert equation) – arXiv:1904.09468 (K.-Vasseur '19).
- We get novel L^2-type control on the shift functions – arXiv:1904.09468 (K.-Vasseur '19) and arXiv:1904.09475 (K. '19).

The negative side: some thoughts on convex integration

Must entropic solutions have traces?



De Lellis-Otto-Westdickenberg (03'): YES for multi-D scalar.

However, not clear we can get traces for systems.

- Chiodaroli-De Lellis-Kreml ('15)
- Chiodaroli-Kreml ('14)
- Klingenberg-Markfelder ('18)
- Feireisl-Klingenberg-Markfelder ('19)

Let's consider one particular 1-D conservation law

$$(S) \qquad \begin{cases} u_t - a(v)_x = 0 \\ v_t - u_x = 0 \\ \eta(u,v)_t - q(u,v)_x \leq 0 \end{cases}$$

- $U := (u, v)$ and $U(x,t) \colon \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ is the unknown.
- $a \colon \mathbb{R} \to \mathbb{R}$ some given function
- $\eta(u,v) := \frac{1}{2}u^2 + F(v)$, where $F(v) = \int_0^v a(s)\, ds$.
- $q(u,v) := ua(v)$.

(see Müller-Šverák '03 and Kirchheim-Müller-Šverák '03)

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$$(S) \qquad \begin{cases} u_t - a(v)_x = 0 \\ v_t - u_x = 0 \\ \eta(u,v)_t - q(u,v)_x \leq 0 \end{cases}$$

Now, consider a stream function $\psi(x,t)\colon \mathbb{R}^2 \to \mathbb{R}^3$

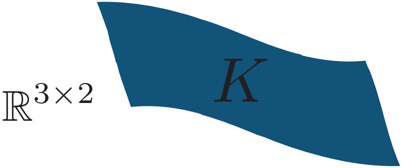
$$\begin{aligned} (u, -a(v)) &= (\psi^1_x, -\psi^1_t) \\ (v, -u) &= (\psi^2_x, -\psi^2_t) \\ (\eta, -q) &= (\psi^3_x, -\psi^3_t). \end{aligned}$$

Let's consider one particular 1-D conservation law

$$\nabla\psi \in K \subset \mathbb{R}^{3\times 2},$$

$$K := \left\{ \begin{pmatrix} u & a(v) \\ v & u \\ \eta(u,v) & q(u,v) \end{pmatrix} : u, v \in \mathbb{R} \right\}.$$

K is a 2 dimensional manifold.



Let's consider one particular 1-D conservation law

Assume

$$\{U^\epsilon\}_\epsilon$$

is a sequence of approximate solutions to (S), and

$$U^\epsilon \overset{*}{\rightharpoonup} U \quad \text{in } L^\infty.$$

$\nu_{x,t}$ – the Young measure associated with this weak* convergence.

Let's consider one particular 1-D conservation law

If we denote

$$P(u, v) := \begin{pmatrix} u & a(v) \\ v & u \\ \eta(u, v) & q(u, v) \end{pmatrix}.$$

Use P to push forward the Young measures onto K.

The probability measures which satisfy Jensen's inequality for polyconvex functions

$$\mathcal{M}^{\mathrm{pc}}(K) = \{\mu \in \mathcal{P}(K) : \int f(A)\, d\mu(A) \geq f(\bar{u}) \text{ for all } f : \mathbb{R}^{3\times 2} \to \mathbb{R} \text{ polyconvex}\}$$

By Div-Curl, the push forward of the Young measures is in $\mathcal{M}^{\mathrm{pc}}(K)$.

Theorem (Lorent-Peng '18)
Suppose $a \in C^2(\mathbb{R})$. Given $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbb{R}^2$, if $a'(\tilde{\alpha}_2) > 0$, then there exist non-trivial measures in $\mathcal{M}^{pc}(K \cap B_\delta(P(\tilde{\alpha})))$ for all $\delta > 0$. On the other hand, if $a'(\tilde{\alpha}_2) < 0$, then there exists $\delta_0 > 0$ depending on the function a and $\tilde{\alpha}_2$ such that $\mathcal{M}^{pc}(K \cap B_\delta(P(\tilde{\alpha})))$ is trivial for all $0 < \delta \leq \delta_0$.

Conclusion/Questions

Thank you!

The definition of strong traces

Fix $T > 0$. Let $u\colon \mathbb{R} \times [0,T) \to \mathbb{R}^n$ verify $u \in L^\infty(\mathbb{R} \times [0,T))$. We say u has the *strong trace property* if for every fixed Lipschitz continuous map $h\colon [0,T) \to \mathbb{R}$, there exists $u_+, u_-\colon [0,T) \to \mathbb{R}^n$ such that

$$\lim_{n\to\infty} \int_0^{t_0} \operatorname*{ess\,sup}_{y\in(0,\frac{1}{n})} \left| u(h(t)+y,t) - u_+(t) \right| \, dt$$

$$= \lim_{n\to\infty} \int_0^{t_0} \operatorname*{ess\,sup}_{y\in(-\frac{1}{n},0)} \left| u(h(t)+y,t) - u_-(t) \right| \, dt = 0$$

for all $t_0 \in (0,T)$.

(from Leger-Vasseur ’11)