Resource theory of asymmetric distinguishability

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Motivation

- **Distinguishability** plays a central role in all sciences
- Repeated trials of an experiment allow for increasing the distinguishability between two different hypotheses
- If the two different hypotheses are relatively distinguishable, then fewer trials are needed
- So distinguishability is a **resource** in this sense because it limits the amount of effort needed to make decisions
- Statistical and, more generally, quantum hypothesis testing provide a rigorous setting for studying distinguishability

Main message

- Distinguishability is a resource that can be quantified and interconverted (**resource theory of asymmetric distinguishability**) (see also [Mat10, Mat11] for earlier work)
- Fundamental unit is the **bit of asymmetric distinguishability**
- Objects to manipulate include state boxes, channel boxes, and quantum strategy (or comb) boxes, and basic tasks include distillation, dilution, and box transformations
- One-shot tasks give **operational meaning** to one-shot relative entropies, like non-smooth and smooth min- and max-relative entropy
- Key Result: **Q. relative entropy** is fundamental exchange rate
- Key Observation: Concepts underpin many other resource theories

FAQ

I don't see why the resource-theoretic viewpoint is useful. Is it simply because resource theories are currently in fashion?

No. We have made important progress on the sequential quantum Stein's lemma for quantum channels, and it is unclear whether this would have occurred without the resource-theoretic perspective.

Resource theory of asymmetric distinguishability

- Basic object to manipulate is a "**state box**," consisting of two quantum states ρ and σ:

$$(\rho, \sigma)$$

- Interpretation: quantum system prepared in an unknown state

Free operations

- What can you do with a state box?
- Any quantum channel $\mathcal{N}$ is allowed for free
- You can then convert one state box to another one as follows:

$$(\rho, \sigma) \quad \to \quad (\mathcal{N}(\rho), \mathcal{N}(\sigma))$$

- Some channels are **reversible**, i.e., isometric channels $\mathcal{U}$ or those that append a common quantum state τ:

$$(\rho, \sigma) \quad \leftrightarrow \quad (\mathcal{U}(\rho), \mathcal{U}(\sigma)) \quad \leftrightarrow \quad (\rho \otimes \tau, \sigma \otimes \tau)$$

Exact box transformation problem

Fundamental question of the resource theory

- Given state boxes (ρ, σ) and (τ, ω), is there a quantum channel $\mathcal{N}$ that takes the state box (ρ, σ) to the state box (τ, ω)?
- Equivalently, is there a quantum channel $\mathcal{N}$ such that

$$\mathcal{N}(\rho) = \tau, \qquad \mathcal{N}(\sigma) = \omega?$$

- This question has a long history both in classical and quantum information theory [Bla53, AU80, CJW04, MOA11, Bus12, HJRW12, BDS14, BaHN$^+$15, Ren16, BD16, Bus16, GJB$^+$18, Bus17, BG17]
- It can be solved by semi-definite programming (efficient algorithm)
- It is also known as **quantum relative majorization** [BG17] and some entropic characterizations are known

Approximate box transformation problem

- Performing exact transformations can be challenging in practice.
- Moreover, if the transformation were performed with small error, this would not be noticeable in practice
- Motivates a relaxation of the previous problem

More fundamental question of the resource theory

- Given state boxes (ρ, σ) and (τ, ω), how well can a quantum channel $\mathcal{N}$ take the state box (ρ, σ) to (τ, ω) approximately?
- Specifically, how small can the following error ε be for some quantum channel $\mathcal{N}$, such that

$$\mathcal{N}(\rho) \approx_{\varepsilon} \tau, \qquad \text{and} \qquad \mathcal{N}(\sigma) = \omega \quad ?$$

- Allowing error in conversion of first state but not in second state is why this is the resource theory of **asymmetric** distinguishability

Approximate box transformation problem (ctd.)

- We quantify error in terms of **normalized trace distance**, due to its strong operational meaning in terms of absolute deviation of observable probabilities in any quantum-physical experiment:

$$\zeta_1 \approx_\varepsilon \zeta_2 \qquad \Longleftrightarrow \qquad \frac{1}{2}\left\|\zeta_1 - \zeta_2\right\|_1 \leq \varepsilon$$

- Then approx. box transformation is the following optimization:

$$\varepsilon((\rho,\sigma) \to (\tau,\omega)) := \inf_{\mathcal{N}\in\mathrm{CPTP}} \{\varepsilon \in [0,1] : \mathcal{N}(\rho) \approx_\varepsilon \tau,\ \mathcal{N}(\sigma) = \omega\},$$

- This can be written as a semi-definite program:

$$\inf_{Y_B, J^{\mathcal{N}}_{RB} \geq 0} \left\{ \begin{array}{c} \mathrm{Tr}[Y_B] : Y_B \geq \tau_B - \mathrm{Tr}_R[\rho^T_R J^{\mathcal{N}}_{RB}], \\ \mathrm{Tr}_R[\sigma^T_R J^{\mathcal{N}}_{RB}] = \omega_B,\ \mathrm{Tr}_B[J^{\mathcal{N}}_{RB}] = I_R \end{array} \right\}$$

Asymptotic approximate box transformations

- Let's think like Claude Shannon and Charlie Bennett...
- (How 'bout that Shannon Award!!!)
- Let $n, m \in \mathbb{Z}^+$ and $\varepsilon \in [0, 1]$.
- An (n, m, ε) box transformation protocol for the boxes (ρ, σ) and (τ, ω) consists of a channel $\mathcal{N}^{(n)}$ such that

$$\mathcal{N}^{(n)}(\rho^{\otimes n}) \approx_\varepsilon \tau^{\otimes m}, \qquad \mathcal{N}^{(n)}(\sigma^{\otimes n}) = \omega^{\otimes m}.$$

- A rate R is *achievable* if for all $\varepsilon \in (0, 1]$, $\delta > 0$, and sufficiently large n, there exists an $(n, n[R - \delta], \varepsilon)$ box transformation protocol.
- **Optimal box transformation rate** $R((\rho, \sigma) \to (\tau, \omega))$ is equal to the supremum of all achievable rates.

Solution of asymptotic box transformation problem

Result: Quantum relative entropy is the fundamental exchange rate

Given state boxes (ρ, σ) and (τ, ω), the optimal box transformation rate is equal to the ratio of quantum relative entropies:

$$R((\rho, \sigma) \to (\tau, \omega)) = \frac{D(\rho\|\sigma)}{D(\tau\|\omega)}$$

where $D(\rho\|\sigma) := \mathrm{Tr}[\rho[\log_2 \rho - \log_2 \sigma]]$ [Ume62].

- Highlights the fundamental role of quantum relative entropy in the resource theory of asymmetric distinguishability
- Observation: Resource theory is **asymptotically reversible**
- See also [BST19]

Solution of asymptotic box transformation problem (ctd.)

- How to prove this? Inspired by entanglement theory [BBPS96, BDSW96], break task into two: **distillation** and **dilution**
- For distillation, convert $(\rho^{\otimes n}, \sigma^{\otimes n})$ to fiducial currency (bits of asymmetric distinguishability), & for dilution, convert these to $(\tau^{\otimes m}, \omega^{\otimes m})$. This is the main idea behind the achievability part.
- For the (strong) converse part, use a **pseudo-continuity bound** for sandwiched Rényi relative entropy and data processing:

Pseudo-continuity bound

Let ρ_0, ρ_1, and σ be states such that $\operatorname{supp}(\rho_0) \subseteq \operatorname{supp}(\sigma)$. Fix $\alpha \in (1/2, 1)$ and $\beta \equiv \beta(\alpha) := \alpha / (2\alpha - 1) > 1$. Then

$$\widetilde{D}_\beta(\rho_0 \| \sigma) - \widetilde{D}_\alpha(\rho_1 \| \sigma) \geq \frac{\alpha}{1-\alpha} \log F(\rho_0, \rho_1).$$

Bits of asymmetric distinguishability

- We introduce the fundamental unit called "**bit of asymmetric distinguishability**":

$$(|0\rangle\langle 0|, \pi) \qquad \text{where } \pi = I/2$$

- m **bits of asymmetric distinguishability** are encoded in the box

$$(|0\rangle\langle 0|^{\otimes m}, \pi^{\otimes m})$$

- Common quantum channels lead to the following equivalence:

$$(|0\rangle\langle 0|^{\otimes m}, \pi^{\otimes m}) \qquad \leftrightarrow \qquad (|0\rangle\langle 0|, \pi_{2^m}),$$

where $\pi_{2^m} = \frac{1}{2^m}|0\rangle\langle 0| + \left(1 - \frac{1}{2^m}\right)|1\rangle\langle 1|$.

Bits of asymmetric distinguishability (ctd.)

More generally, $\log_2 M$ bits of asymmetric distinguishability are encoded in the following state box:

$$(|0\rangle\langle 0|, \pi_M)$$

where

$$\pi_M := \frac{1}{M}|0\rangle\langle 0| + \left(1 - \frac{1}{M}\right)|1\rangle\langle 1|.$$

Exact distinguishability distillation

- Goal: distill from state box (ρ, σ) as many exact bits of AD as possible
- That is, we want to perform the conversion:

$$(\rho, \sigma) \to (|0\rangle\langle 0|, \pi_M)$$

with M as large as possible.

- Formally, **one-shot exact distillable distinguishability** is given by

$$D_d^0(\rho, \sigma) := \log_2 \sup_{\mathcal{P} \in \text{CPTP}} \{M : \mathcal{P}(\rho) = |0\rangle\langle 0|,\ \mathcal{P}(\sigma) = \pi_M\}$$

- Key Result: It is equal to the **min-relative entropy** of [Dat09]:

$$D_d^0(\rho, \sigma) = D_{\min}(\rho \| \sigma)$$

where $D_{\min}(\rho\|\sigma) := -\log_2 \text{Tr}[\Pi_\rho \sigma]$

Exact distinguishability dilution

- Goal: prepare state box (ρ, σ) with as few exact bits of AD as possible
- That is, we want to perform the conversion:

$$(|0\rangle\langle 0|, \pi_M) \to (\rho, \sigma)$$

  with M as small as possible.
- Formally, **one-shot exact distinguishability cost** is given by

$$D_c^0(\rho, \sigma) := \log_2 \inf_{\mathcal{P} \in \mathrm{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) = \rho, \ \mathcal{P}(\pi_M) = \sigma\}$$

- Key Result: It is equal to the **max-relative entropy** of [Dat09]:

$$D_c^0(\rho, \sigma) = D_{\max}(\rho \| \sigma)$$

  where $D_{\max}(\rho\|\sigma) := \inf \left\{\lambda \geq 0 : \rho \leq 2^{\lambda}\sigma\right\}$

Approximate distinguishability distillation

- Goal: distill from state box (ρ, σ) as many approx. bits of AD as possible
- That is, we want to perform the conversion:

$$(\rho, \sigma) \to (\widetilde{0}_\varepsilon, \pi_M)$$

  with M as large as possible and $\widetilde{0}_\varepsilon \approx_\varepsilon |0\rangle\langle 0|$.
- Formally, **one-shot distillable distinguishability** is given by

$$D_d^\varepsilon(\rho, \sigma) := \log_2 \sup_{\mathcal{P} \in \text{CPTP}} \{M : \mathcal{P}(\rho) \approx_\varepsilon |0\rangle\langle 0|, \ \mathcal{P}(\sigma) = \pi_M\}$$

- Equal to **smooth min-relative entropy** of [BD10, BD11, WR12]:

$$D_d^\varepsilon(\rho, \sigma) = D_{\min}^\varepsilon(\rho \| \sigma)$$

  where $D_{\min}^\varepsilon(\rho\|\sigma) := -\log_2 \inf_{\Lambda \geq 0}\{\text{Tr}[\Lambda\sigma] : \Lambda \leq I, \text{Tr}[\Lambda\rho] \geq 1 - \varepsilon\}$

Approximate distinguishability dilution

- Goal: prepare state box (ρ, σ) approximately using as few bits of AD as possible
- That is, we want to perform the conversion:

$$(|0\rangle\langle 0|, \pi_M) \to (\widetilde{\rho}, \sigma)$$

  with M as small as possible and $\widetilde{\rho} \approx_\varepsilon \rho$.
- Formally, **one-shot distinguishability cost** is given by

$$D_c^\varepsilon(\rho, \sigma) := \log_2 \inf_{\mathcal{P} \in \text{CPTP}} \{M : \mathcal{P}(|0\rangle\langle 0|) \approx_\varepsilon \rho,\ \mathcal{P}(\pi_M) = \sigma\}$$

- Key Result: It is equal to **smooth max-relative entropy** of [Dat09]:

$$D_c^\varepsilon(\rho, \sigma) = D_{\max}^\varepsilon(\rho \| \sigma)$$

  where $D_{\max}^\varepsilon(\rho \| \sigma) := \inf_{\widetilde{\rho} \approx_\varepsilon \rho} D_{\max}(\widetilde{\rho} \| \sigma)$.

Asymptotics

- **Asymptotic distillable distinguishability**:

$$D_d(\rho,\sigma) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}D_d^{\varepsilon}(\rho^{\otimes n},\sigma^{\otimes n}) = D(\rho\|\sigma)$$

  Last equality follows from quantum Stein's lemma [HP91] (refinements available in [ON00, Nag06, Hay07, TH13, Li14, MO15])

- **Asymptotic distinguishability cost**:

$$D_c(\rho,\sigma) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}D_c^{\varepsilon}(\rho^{\otimes n},\sigma^{\otimes n}) = D(\rho\|\sigma)$$

  Last equality follows from asymptotic equipartition property [TCR09] (refinements available in [TH13]). Open questions about error and strong converse exponents

- Observation: Resource theory is **asymptotically reversible**

Resource theory for channels

- We can generalize the resource theory of asymmetric distinguishability to **quantum channels** [WW19b]
- The basic object to manipulate is a **channel box**, consisting of two channels $\mathcal{N}$ and $\mathcal{M}$:

$$(\mathcal{N}, \mathcal{M})$$

- Quantum channel boxes have inputs and outputs, and so the ways that we can manipulate them are richer than for state boxes
- Tasks for the state theory have generalizations to the channel theory (distillation, dilution, channel box transformations)

Free operations are quantum superchannels

- Most general physical transformation of a quantum channel is a **superchannel** [CDP08], which accepts as input a quantum channel and outputs a quantum channel

- The superchannel $\Theta_{(A\to B)\to(C\to D)}$ takes as input a quantum channel $\mathcal{N}_{A\to B}$ and outputs a quantum channel $\mathcal{K}_{C\to D}$, which we denote by

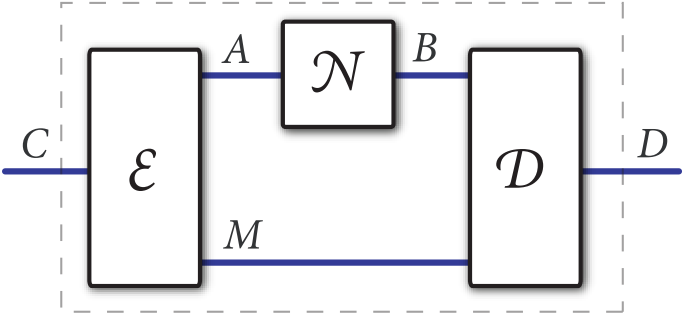
$$\Theta_{(A\to B)\to(C\to D)}(\mathcal{N}_{A\to B}) = \mathcal{K}_{C\to D}.$$

Physical realizations of quantum superchannels

- Superchannel has a **physical realization** in terms of pre- and post-processing quantum channels [CDP08] (see also [Gou18]):

$$\Theta_{(A\to B)\to(C\to D)}(\mathcal{N}_{A\to B}) = \mathcal{D}_{BM\to D} \circ \mathcal{N}_{A\to B} \circ \mathcal{E}_{C\to AM},$$

where $\mathcal{E}_{C\to AM}$ and $\mathcal{D}_{BM\to D}$ are pre- and post-processing channels



Exact channel box transformation problem

Fundamental question [Gou18]

- Given channel boxes $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{K}, \mathcal{L})$, is there a quantum superchannel Θ that takes the channel box $(\mathcal{N}, \mathcal{M})$ to the channel box $(\mathcal{K}, \mathcal{L})$?
- Specifically, is there a quantum superchannel Θ such that

$$\Theta(\mathcal{N}) = \mathcal{K}, \qquad \Theta(\mathcal{M}) = \mathcal{L}?$$

- This was called "comparison of channels" in [Gou18]
- [Gou18] showed that it can be solved by means of a semi-definite program and characterized by the extended conditional min-entropy

Approximate channel box transformation problem

Fundamental question of the resource theory [WW19b]

- Given channel boxes $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{K}, \mathcal{L})$, how well can a quantum superchannel Θ take the channel box $(\mathcal{N}, \mathcal{M})$ to the channel box $(\mathcal{K}, \mathcal{L})$ approximately?

- Specifically, how small can the following error ε be for some superchannel Θ such that

$$\Theta(\mathcal{N}) \approx_{\varepsilon} \mathcal{K}, \qquad \text{and} \qquad \Theta(\mathcal{M}) = \mathcal{L} \qquad ?$$

Approximate channel box transformation problem (ctd.)

- Quantify error in terms of **normalized diamond distance** [Kit97], due to its strong operational meaning in terms of absolute deviation of observable probabilities in any quantum-physical experiment:

$$\mathcal{N}_1 \approx_\varepsilon \mathcal{N}_2 \qquad \Longleftrightarrow \qquad \frac{1}{2}\left\|\mathcal{N}_1 - \mathcal{N}_2\right\|_\diamond \leq \varepsilon$$

- Then approx. channel box transformation is the optimization

$$\varepsilon((\mathcal{N},\mathcal{M}) \to (\mathcal{K},\mathcal{L})) := \inf_{\Theta \in \mathsf{SC}} \left\{\varepsilon \in [0,1] : \Theta(\mathcal{N}) \approx_\varepsilon \mathcal{K},\ \Theta(\mathcal{M}) = \mathcal{L}\right\},$$

- This can be written as a semi-definite program:

$$\inf_{Z_{CD},\ \Gamma^{\Theta}_{CBAD} \geq 0} \left\|\mathrm{Tr}_D[Z_{CD}]\right\|_\infty, \text{ subject to}$$

$$Z_{CD} \geq \Gamma^{\mathcal{K}}_{CD} - \mathrm{Tr}_{AB}[(\Gamma^{\mathcal{N}}_{AB})^T \Gamma^{\Theta}_{CBAD}], \quad \Gamma^{\mathcal{L}}_{CD} = \mathrm{Tr}_{AB}[(\Gamma^{\mathcal{M}}_{AB})^T \Gamma^{\Theta}_{CBAD}],$$
$$\Gamma^{\Theta}_{CB} = I_{CB}, \quad \Gamma^{\Theta}_{CBA} = \Gamma^{\Theta}_{CA} \otimes \pi_B,$$

Asymptotic parallel channel box transformation

- Again think like Claude Shannon and Charlie Bennett...
- Let $n, m \in \mathbb{Z}^+$ and $\varepsilon \in [0,1]$.
- An (n, m, ε) **parallel channel box transformation protocol** for the channel boxes $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{K}, \mathcal{L})$ consists of a superchannel $\Theta^{(n)}$ such that

$$\Theta^{(n)}(\mathcal{N}^{\otimes n}) \approx_\varepsilon \mathcal{K}^{\otimes m}, \qquad \Theta^{(n)}(\mathcal{M}^{\otimes n}) = \mathcal{L}^{\otimes m}.$$

- A rate R is *achievable* if for all $\varepsilon \in (0,1]$, $\delta > 0$, and sufficiently large n, there exists an $(n, n[R-\delta], \varepsilon)$ parallel channel box transformation protocol.
- **Optimal parallel channel box transformation** rate $R^p((\mathcal{N}, \mathcal{M}) \to (\mathcal{K}, \mathcal{L}))$ is equal to supremum of all achievable rates.

Partial solution

Solution for classical–quantum and environment-seizable [BHKW18] channels in terms of **channel relative entropy** [CMW16, LKDW18]

Result: Quantum relative entropy is the fundamental exchange rate

Given classical–quantum or environment-seizable channel boxes $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{K}, \mathcal{L})$, the optimal parallel channel box transformation rate is equal to the ratio of channel relative entropies:

$$R^p((\mathcal{N}, \mathcal{M}) \to (\mathcal{K}, \mathcal{L})) = \frac{D(\mathcal{N}\|\mathcal{M})}{D(\mathcal{K}\|\mathcal{L})}$$

where $D(\mathcal{N}\|\mathcal{M}) := \sup_{\psi_{RA}} D(\mathcal{N}_{A\to B}(\psi_{RA})\|\mathcal{M}_{A\to B}(\psi_{RA}))$ [CMW16, LKDW18].

- (Parallel) resource theory asymptotically reversible for these channels

Partial solution (ctd.)

- How to prove this? Again break task into two: distillation and dilution
- For distillation, convert $(\mathcal{N}^{\otimes n}, \mathcal{M}^{\otimes n})$ to bits of asymmetric distinguishability, & for dilution, convert these to $(\mathcal{K}^{\otimes m}, \mathcal{L}^{\otimes m})$. This solves achievability part for special channels.
- For the (strong) converse part, use a pseudo-continuity bound for sandwiched Rényi relative entropy and data processing:

Pseudo-continuity bound

Let $\mathcal{N}^0_{A\to B}$, $\mathcal{N}^1_{A\to B}$, and $\mathcal{M}_{A\to B}$ be channels such that $D_{\max}(\mathcal{N}^0\|\mathcal{M}) < \infty$. Then for $\alpha \in (1/2, 1)$ and $\beta := \alpha/(2\alpha - 1) > 1$,

$$\widetilde{D}_\beta(\mathcal{N}^0\|\mathcal{M}) - \widetilde{D}_\alpha(\mathcal{N}^1\|\mathcal{M}) \geq \frac{\alpha}{1-\alpha} \log_2 F(\mathcal{N}^0, \mathcal{N}^1).$$

Bits of asymmetric distinguishability

- Identify $\log_2 M$ bits of asymmetric distinguishability as follows:

$$(|0\rangle\langle 0|, \pi_M) \quad \leftrightarrow \quad (\mathcal{R}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M})$$

where

$$\pi_M := \frac{1}{M}|0\rangle\langle 0| + \left(1 - \frac{1}{M}\right)|1\rangle\langle 1|$$

and $\mathcal{R}^{\sigma}(\rho) = \mathrm{Tr}[\rho]\sigma$ is a replacer channel that replaces the input state ρ with the state σ

Exact distinguishability distillation

- Goal: distill from box $(\mathcal{N}, \mathcal{M})$ as many exact bits of AD as possible
- That is, we want to perform the conversion:

$$(\mathcal{N}, \mathcal{M}) \to (\mathcal{R}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M})$$

  with M as large as possible.
- Formally, **one-shot exact distillable distinguishability** is given by

$$D_d^0(\mathcal{N}, \mathcal{M}) := \log_2 \sup_{\Theta \in \mathrm{SC}} \left\{ M : \Theta(\mathcal{N}) = \mathcal{R}^{|0\rangle\langle 0|},\ \Theta(\mathcal{M}) = \mathcal{R}^{\pi_M} \right\}$$

- Key Result: It is equal to the **channel min-relative entropy**:

$$D_d^0(\mathcal{N}, \mathcal{M}) = D_{\min}(\mathcal{N} \| \mathcal{M})$$

  where $D_{\min}(\mathcal{N}\|\mathcal{M}) := \sup_{\psi_{RA}} D_{\min}(\mathcal{N}_{A\to B}(\psi_{RA}) \| \mathcal{M}_{A\to B}(\psi_{RA}))$.

Exact distinguishability dilution

- Goal: prepare channel box $(\mathcal{N}, \mathcal{M})$ with as few exact bits of AD as possible
- That is, we want to perform the conversion:

$$(\mathcal{R}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M}) \to (\mathcal{N}, \mathcal{M})$$

  with M as small as possible.
- Formally, **one-shot exact distinguishability cost** is given by

$$D_c^0(\mathcal{N}, \mathcal{M}) := \log_2 \inf_{\Theta \in \mathrm{SC}} \left\{ M : \mathcal{N} = \Theta(\mathcal{R}^{|0\rangle\langle 0|}),\ \mathcal{M} = \Theta(\mathcal{R}^{\pi_M}) \right\}$$

- $=$ **channel max-relative entropy** [CMW16, LKDW18, GFW$^+$18]

$$D_c^0(\mathcal{N}, \mathcal{M}) = D_{\max}(\mathcal{N} \| \mathcal{M})$$

  where $D_{\max}(\mathcal{N} \| \mathcal{M}) := \sup_{\psi_{RA}} D_{\max}(\mathcal{N}_{A\to B}(\psi_{RA}) \| \mathcal{M}_{A\to B}(\psi_{RA}))$

Approximate distinguishability distillation

- Goal: distill from box $(\mathcal{N}, \mathcal{M})$ as many approx. bits of AD as possible
- That is, we want to perform the conversion:

$$(\mathcal{N}, \mathcal{M}) \to (\widetilde{\mathcal{R}}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M})$$

  with M as large as possible and $\widetilde{\mathcal{R}}^{|0\rangle\langle 0|} \approx_\varepsilon \mathcal{R}^{|0\rangle\langle 0|}$.
- Formally, **one-shot distillable distinguishability** is given by

$$D_d^\varepsilon(\mathcal{N}, \mathcal{M}) := \log_2 \sup_{\Theta \in \mathrm{SC}} \left\{ M : \Theta(\mathcal{N}) \approx_\varepsilon \mathcal{R}^{|0\rangle\langle 0|},\ \Theta(\mathcal{M}) = \mathcal{R}^{\pi_M} \right\}$$

- Equal to **smooth channel min-relative entropy** of [CMW16]:

$$D_d^\varepsilon(\mathcal{N}, \mathcal{M}) = D_{\min}^\varepsilon(\mathcal{N} \| \mathcal{M})$$

  where $D_{\min}^\varepsilon(\mathcal{N} \| \mathcal{M}) := \sup_{\psi_{RA}} D_{\min}^\varepsilon(\mathcal{N}_{A\to B}(\psi_{RA}) \| \mathcal{M}_{A\to B}(\psi_{RA}))$

Approximate distinguishability dilution

- Goal: prepare box $(\mathcal{N}, \mathcal{M})$ approximately using as few bits of AD as possible
- That is, we want to perform the conversion:

$$(\mathcal{R}^{|0\rangle\langle 0|}, \mathcal{R}^{\pi_M}) \to (\widetilde{\mathcal{N}}, \mathcal{M})$$

with M as small as possible and $\widetilde{\mathcal{N}} \approx_\varepsilon \mathcal{N}$.

- Formally, **one-shot distinguishability cost** is given by

$$D_c^\varepsilon(\mathcal{N}, \mathcal{M}) := \log_2 \inf_{\Theta \in \mathrm{SC}} \left\{ M : \mathcal{N} \approx_\varepsilon \Theta(\mathcal{R}^{|0\rangle\langle 0|}),\ \mathcal{M} = \Theta(\mathcal{R}^{\pi_M}) \right\}$$

- Equal to **smooth channel max-relative entropy** of [GFW+18]:

$$D_c^\varepsilon(\mathcal{N}, \mathcal{M}) = D_{\max}^\varepsilon(\mathcal{N} \| \mathcal{M})$$

where $D_{\max}^\varepsilon(\mathcal{N} \| \mathcal{M}) := \inf_{\widetilde{\mathcal{N}} \approx_\varepsilon \mathcal{N}} D_{\max}(\widetilde{\mathcal{N}} \| \mathcal{M})$.

Asymptotics (parallel case)

- **Asymptotic parallel distillable distinguishability**:

$$D_d(\mathcal{N}, \mathcal{M}) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} D_d^{\varepsilon}(\mathcal{N}^{\otimes n}, \mathcal{M}^{\otimes n}) = \lim_{m \to \infty} \frac{1}{m} D(\mathcal{N} \| \mathcal{M})$$

  Follows essentially from quantum Stein's lemma [HP91] and converse bounds for $D_{\min}^{\varepsilon}$ [WR12, MW14, KW17]

- **Asymptotic parallel distinguishability cost**:

$$D_c(\rho, \sigma) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} D_c^{\varepsilon}(\mathcal{N}^{\otimes n}, \mathcal{M}^{\otimes n}) \geq \lim_{m \to \infty} \frac{1}{m} D(\mathcal{N} \| \mathcal{M})$$

  Last equality is operational (cost $\geq$ distillability). Whether equality holds is related to open question of [LW19]

- Resource theory is asymptotically reversible for classical–quantum and environment-seizable channel boxes

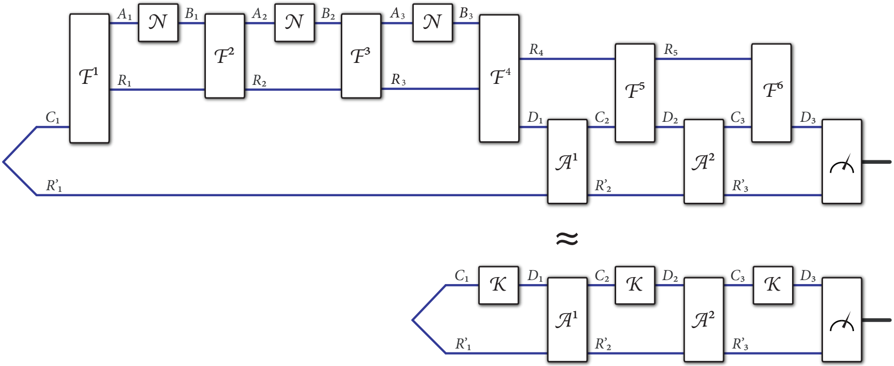
Sequential setting

(n, m, ε) sequential channel box transformation protocol

Goal is to convert n-round sequential channel box $(\mathcal{N}^{(n)}, \mathcal{M}^{(n)})$ to m-round sequential channel box $(\mathcal{K}^{(m)}, \mathcal{L}^{(m)})$ by means of a physical transformation $\Theta^{(n\to m)}$ (quantum strategy [GW07] or comb [CDP09]), such that

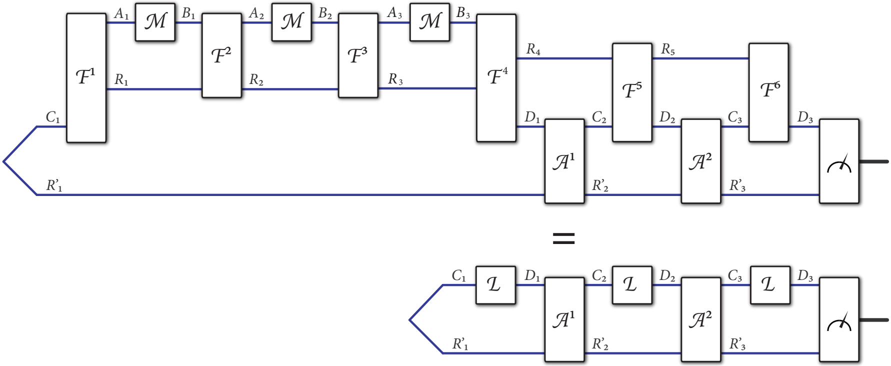
$$\Theta^{(n\to m)}(\mathcal{N}^{(n)}) \approx_{\varepsilon} \mathcal{K}^{(m)}, \qquad \Theta^{(n\to m)}(\mathcal{M}^{(n)}) = \mathcal{L}^{(m)}$$

Depiction of condition $\Theta^{(n\to m)}(\mathcal{N}^{(n)}) \approx_\varepsilon \mathcal{K}^{(m)}$



Observable probabilities between $\Theta^{(n\to m)}(\mathcal{N}^{(n)})$ and $\mathcal{K}^{(m)}$ deviate by no more than ε when paired up with an arbitrary co-strategy [GW07] or tester [CDP09] (operational definition of **strategy distance** [GW07, CDP09])

Depiction of condition $\Theta^{(n\to m)}(\mathcal{M}^{(n)}) = \mathcal{L}^{(m)}$



Observable probabilities between $\Theta^{(n\to m)}(\mathcal{N}^{(n)})$ and $\mathcal{K}^{(m)}$ do not deviate at all when paired up w/ an arbitrary co-strategy [GW07] or tester [CDP09] (equivalent to Choi states being equal [GW07, CDP09])

Exact sequential distinguishability dilution

- Goal: Prepare sequential channel box $(\mathcal{N}^{(n)}, \mathcal{M}^{(n)})$ with as few bits of AD as possible
- Formally, **exact distinguishability cost** is given by

$$D_c^0(\mathcal{N}^{(n)}, \mathcal{M}^{(n)}) := \inf_{\Theta^{(1\to n)}} \left\{ \begin{array}{c} \log_2 M : \mathcal{N}^{(n)} = \Theta^{(1\to n)}(\mathcal{R}^{|0\rangle\langle 0|}_{C\to D}), \\ \mathcal{M}^{(n)} = \Theta^{(1\to n)}(\mathcal{R}^{\pi_M}_{C\to D}) \end{array} \right\}.$$

- Key result: Using "bootstrapping" method of [GFW+18], normalized cost equal to channel max-relative entropy for all $n \geq 1$:

$$\frac{1}{n} D_c^0(\mathcal{N}^{(n)}, \mathcal{M}^{(n)}) = D_{\max}(\mathcal{N} \| \mathcal{M})$$

- Implies that asymptotic exact sequential cost is

$$D_c^0(\mathcal{N}, \mathcal{M}) := \lim_{n\to\infty} \frac{1}{n} D_c^0(\mathcal{N}^{(n)}, \mathcal{M}^{(n)}) = D_{\max}(\mathcal{N} \| \mathcal{M})$$

Approximate distinguishability distillation

- Goal: Distill from sequential channel box $(\mathcal{N}^{(n)}, \mathcal{M}^{(n)})$ as many approx. bits of AD as possible
- Formally, **approx. distillable distinguishability** is given by

$$D_d^{\varepsilon}(\mathcal{N}^{(n)}, \mathcal{M}^{(n)}) := \sup_{\Theta^{(n\to 1)}} \left\{ \begin{array}{c} \log_2 M : \Theta^{(n\to 1)}(\mathcal{N}^{(n)}) \approx_{\varepsilon} \mathcal{R}_{C\to D}^{|0\rangle\langle 0|}, \\ \Theta^{(n\to 1)}(\mathcal{M}^{(n)}) = \mathcal{R}_{C\to D}^{\pi_M} \end{array} \right\}.$$

- Key result: Using different "bootstrapping" method of [BHLS03, NGP15, GFW^{+}18], asymptotic sequential distillable distinguishability equals amortized channel relative entropy [BHKW18]:

$$D_d(\mathcal{N}, \mathcal{M}) := \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} D_d^{\varepsilon}(\mathcal{N}^{(n)}, \mathcal{M}^{(n)}) = D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M}), \text{ with}$$

$$D_{\mathcal{A}}(\mathcal{N}\|\mathcal{M}) := \sup_{\rho_{RA}, \sigma_{RA}} D(\mathcal{N}_{A\to B}(\rho_{RA})\|\mathcal{M}_{A\to B}(\sigma_{RA})) - D(\rho_{RA}\|\sigma_{RA})$$

- Can also be understood as solution of Stein's lemma for quantum channels in sequential setting

How to achieve amortized channel divergence?

Idea: Use a block adaptive protocol

1. In a preliminary round, distill bits of AD at rate $D(\mathcal{N}_{A\to B}(\psi_{RA})\|\mathcal{M}_{A\to B}(\psi_{RA}))$ for some state ψ_{RA}
2. Then dilute these bits of AD to state box $(\rho_{RA}^{\otimes n}, \sigma_{RA}^{\otimes n})$
3. Now send states through channels to realize state box $([\mathcal{N}_{A\to B}(\rho_{RA})]^{\otimes n}, [\mathcal{M}_{A\to B}(\sigma_{RA})]^{\otimes n})$
4. Distill bits of AD at rate $D(\mathcal{N}_{A\to B}(\rho_{RA})\|\mathcal{M}_{A\to B}(\sigma_{RA}))$
5. Set aside fraction $D(\mathcal{N}_{A\to B}(\rho_{RA})\|\mathcal{M}_{A\to B}(\sigma_{RA})) - D(\rho_{RA}\|\sigma_{RA})$ and reinvest fraction $D(\rho_{RA}\|\sigma_{RA})$ for next round
6. Repeat 2-5 many times
7. Net rate of bits of AD produced is then $D(\mathcal{N}_{A\to B}(\rho_{RA})\|\mathcal{M}_{A\to B}(\sigma_{RA})) - D(\rho_{RA}\|\sigma_{RA})$

Conclusion and future directions

- Resource theory of asymmetric distinguishability developed for states [WW19a], channels [WW19b], and strategies/combs [WW19b]

- Strong links to other resource theories, as discussed in [WW19a]

- Many open questions about error and strong converse exponents, second-order expansions, etc.

- Interesting open question: Is there a channel box $(\mathcal{N}, \mathcal{M})$ such that

$$D_{\mathcal{A}}(\mathcal{N} \| \mathcal{M}) > \lim_{m \to \infty} \frac{1}{m} D(\mathcal{N}^{\otimes m} \| \mathcal{M}^{\otimes m}) \quad ?$$

  If so, the implication is that a sequential strategy can strictly outperform a parallel strategy in asymmetric quantum channel discrimination. Alternatively, is there equality above for all channels?

[AU80] P. M. Alberti and A. Uhlmann. A problem relating to positive linear maps on matrix algebras. *Reports on Mathematical Physics*, 18(2):163–176, October 1980.

[BaHN+15] Fernando G. S. L. Brandão, Michal Horodecki, Nelly Ng, Jonathan Oppenheim, and Stephanie Wehner. The second laws of quantum thermodynamics. *Proceedings of the National Academy of Sciences*, 112(11):3275–3279, March 2015. arXiv:1305.5278.

[BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Physical Review A*, 53(4):2046–2052, April 1996. arXiv:quant-ph/9511030.

[BD10] Francesco Buscemi and Nilanjana Datta. The quantum capacity of channels with arbitrarily correlated noise. *IEEE Transactions on Information Theory*, 56(3):1447–1460, March 2010. arXiv:0902.0158.

[BD11] Fernando G. S. L. Brandao and Nilanjana Datta. One-shot rates for entanglement manipulation under non-entangling maps. *IEEE Transactions on Information Theory*, 57(3):1754–1760, March 2011. arXiv:0905.2673.

References II

[BD16] Francesco Buscemi and Nilanjana Datta. Equivalence between divisibility and monotonic decrease of information in classical and quantum stochastic processes. *Physical Review A*, 93(1):012101, January 2016. arXiv:1408.7062.

[BDS14] Francesco Buscemi, Nilanjana Datta, and Sergii Strelchuk. Game-theoretic characterization of antidegradable channels. *Journal of Mathematical Physics*, 55(9):092202, September 2014. arXiv:1404.0277.

[BDSW96] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Physical Review A*, 54(5):3824–3851, November 1996. arXiv:quant-ph/9604024.

[BG17] Francesco Buscemi and Gilad Gour. Quantum relative Lorenz curves. *Physical Review A*, 95(1):012110, January 2017. arXiv:1607.05735.

[BHKW18] Mario Berta, Christoph Hirche, Eneet Kaur, and Mark M. Wilde. Amortized channel divergence for asymptotic quantum channel discrimination. August 2018. arXiv:1808.01498.

References III

[BHLS03] Charles H. Bennett, Aram W. Harrow, Debbie W. Leung, and John A. Smolin. On the capacities of bipartite Hamiltonians and unitary gates. *IEEE Transactions on Information Theory*, 49(8):1895–1911, August 2003. arXiv:quant-ph/0205057.

[Bla53] David Blackwell. Equivalent comparisons of experiments. *The Annals of Mathematical Statistics*, 24(2):265–272, June 1953.

[BST19] Francesco Buscemi, David Sutter, and Marco Tomamichel. An information-theoretic treatment of quantum dichotomies. July 2019. arXiv:1907.08539.

[Bus12] Francesco Buscemi. Comparison of quantum statistical models: Equivalent conditions for sufficiency. *Communications in Mathematical Physics*, 310(3):625–647, March 2012. arXiv:1004.3794.

[Bus16] Francesco Buscemi. Degradable channels, less noisy channels, and quantum statistical morphisms: An equivalence relation. *Problems of Information Transmission*, 52(3):201–213, July 2016. arXiv:1511.08893.

References IV

[Bus17] Francesco Buscemi. Comparison of noisy channels and reverse data-processing theorems. *2017 IEEE Information Theory Workshop*, pages 489–493, 2017. arXiv:1803.02945.

[CDP08] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Transforming quantum operations: Quantum supermaps. *Europhysics Letters*, 83(3):30004, August 2008. arXiv:0804.0180.

[CDP09] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80(2):022339, August 2009. arXiv:0904.4483.

[CJW04] Anthony Chefles, Richard Jozsa, and Andreas Winter. On the existence of physical transformations between sets of quantum states. *International Journal of Quantum Information*, 02(01):11–21, March 2004. arXiv:quant-ph/0307227.

References V

[CMW16] Tom Cooney, Milán Mosonyi, and Mark M. Wilde. Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication. *Communications in Mathematical Physics*, 344(3):797–829, June 2016. arXiv:1408.3373.

[Dat09] Nilanjana Datta. Min- and max-relative entropies and a new entanglement monotone. *IEEE Transactions on Information Theory*, 55(6):2816–2826, June 2009. arXiv:0803.2770.

[GFW$^+$18] María García Díaz, Kun Fang, Xin Wang, Matteo Rosati, Michalis Skotiniotis, John Calsamiglia, and Andreas Winter. Using and reusing coherence to realize quantum processes. *Quantum*, 2:100, October 2018. arXiv:1805.04045.

[GJB$^+$18] Gilad Gour, David Jennings, Francesco Buscemi, Runyao Duan, and Iman Marvian. Quantum majorization and a complete set of entropic conditions for quantum thermodynamics. *Nature Communications*, 9(1):5352, December 2018. arXiv:1708.04302.

[Gou18] Gilad Gour. Comparison of quantum channels with superchannels. August 2018. arXiv:1808.02607.

References VI

[GW07] Gus Gutoski and John Watrous. Toward a general theory of quantum games. *Proceedings of the thirty-ninth annual ACM symposium on theory of computing*, pages 565–574, 2007. arXiv:quant-ph/0611234.

[Hay07] Masahito Hayashi. Error exponent in asymmetric quantum hypothesis testing and its application to classical-quantum channel coding. *Physical Review A*, 76(6):062301, December 2007. arXiv:quant-ph/0611013.

[HJRW12] Teiko Heinosaari, Maria A. Jivulescu, David Reeb, and Michael M. Wolf. Extending quantum operations. *Journal of Mathematical Physics*, 53(10):102208, October 2012. arXiv:1205.0641.

[HP91] Fumio Hiai and Dénes Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143(1):99–114, December 1991.

[Kit97] A Yu Kitaev. Quantum computations: algorithms and error correction. *Russian Mathematical Surveys*, 52(6):1191–1249, December 1997.

References VII

[KW17] Eneet Kaur and Mark M. Wilde. Upper bounds on secret key agreement over lossy thermal bosonic channels. *Physical Review A*, 96(6):062318, December 2017. arXiv:1706.04590.

[Li14] Ke Li. Second order asymptotics for quantum hypothesis testing. *Annals of Statistics*, 42(1):171–189, February 2014. arXiv:1208.1400.

[LKDW18] Felix Leditzky, Eneet Kaur, Nilanjana Datta, and Mark M. Wilde. Approaches for approximate additivity of the Holevo information of quantum channels. *Physical Review A*, 97(1):012332, January 2018. arXiv:1709.01111.

[LW19] Zi-Wen Liu and Andreas Winter. Resource theories of quantum channels and the universal role of resource erasure. April 2019. arXiv:1904.04201v1.

[Mat10] Keiji Matsumoto. Reverse test and characterization of quantum relative entropy. October 2010. arXiv:1010.1030.

References VIII

[Mat11] Keiji Matsumoto. Reverse test and characterization of quantum relative entropy. In *The Second Nagoya Winter Workshop on Quantum Information, Measurement, and Foundations*, 2011. Slides available at `https://sites.google.com/site/nww2011/home/talks-slides/matsumoto.pdf`.

[MO15] Milán Mosonyi and Tomohiro Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. *Communications in Mathematical Physics*, 334(3):1617–1648, March 2015. arXiv:1309.3228.

[MOA11] Albert W. Marshall, Ingram Olkin, and Barry C. Arnold. *Inequalities: Theory of Majorization and Its Applications*. Springer, second edition, 2011.

[MW14] William Matthews and Stephanie Wehner. Finite blocklength converse bounds for quantum channels. *IEEE Transactions on Information Theory*, 60(11):7317–7329, November 2014. arXiv:1210.4722.

[Nag06] Hiroshi Nagaoka. The converse part of the theorem for quantum Hoeffding bound. November 2006. arXiv:quant-ph/0611289.

References IX

[NGP15] Miguel Navascués and Luis Pedro García-Pintos. Nonthermal quantum channels as a thermodynamical resource. *Physical Review Letters*, 115(1):010405, July 2015. arXiv:1501.02597.

[ON00] Tomohiro Ogawa and Hiroshi Nagaoka. Strong converse and Stein's lemma in quantum hypothesis testing. *IEEE Transactions on Information Theory*, 46(7):2428–2433, November 2000. arXiv:quant-ph/9906090.

[Ren16] Joseph M. Renes. Relative submajorization and its use in quantum resource theories. *Journal of Mathematical Physics*, 57(12):122202, December 2016. arXiv:1510.03695.

[TCR09] Marco Tomamichel, Roger Colbeck, and Renato Renner. A fully quantum asymptotic equipartition property. *IEEE Transactions on Information Theory*, 55(12):5840–5847, December 2009. arXiv:0811.1221.

[TH13] Marco Tomamichel and Masahito Hayashi. A hierarchy of information quantities for finite block length analysis of quantum tasks. *IEEE Transactions on Information Theory*, 59(11):7693–7710, August 2013. arXiv:1208.1478.

References X

[Ume62] Hisaharu Umegaki. Conditional expectations in an operator algebra IV (entropy and information). *Kodai Mathematical Seminar Reports*, 14(2):59–85, 1962.

[WR12] Ligong Wang and Renato Renner. One-shot classical-quantum capacity and hypothesis testing. *Physical Review Letters*, 108(20):200501, May 2012. arXiv:1007.5456.

[WW19a] Xin Wang and Mark M. Wilde. Resource theory of asymmetric distinguishability. May 2019. arXiv:1905.11629.

[WW19b] Xin Wang and Mark M. Wilde. Resource theory of asymmetric distinguishability for quantum channels. July 2019. arXiv:1907.06306.