Reconstruction Methods for Inverse Problems

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The report as written surveys parts of the original submission to review the state of the art in the field. Then come the presentations and their abstracts, and then outcomes of the meeting where we put the talks in perspective.

1 Overview of the Field

Inverse problems require to determine the cause from a set of observations. Such problems are of importance in medical imaging, non destructive testing of materials, computerized tomography, source reconstructions in acoustics, computer vision, and geophysics, to mention but a few, and their mathematical solutions represent breakthroughs in applications. In many situations the mathematical modeling of these problems leads to the study of inverse boundary value problems for partial differential equations and systems that are highly nonlinear and ill-posed in the sense of Hadamard; small errors in the data may cause uncontrollable errors in the solution. It is precisely this feature that makes crucial the analysis of these instabilities and their regularization towards a successful computational reconstruction. The strategy of reconstruction is the following: Given a discrete set of (noisy) measurements, reconstruct an image of the unknown physical quantity inside the examined object. The natural approach is to reduce the problem to a minimization problem for a least-square constrained type functional. Due to the ill-posedness of the underlying inverse problems, all the functional reconstruction methods involve some form of regularization which enables stable reconstruction. These methods are called regularization techniques (see for instance [8]).

An illuminating example of ill-posed nonlinear inverse problem is the inverse conductivity problem modelling electrical impedance tomography (EIT), a nondestructive imaging technique with applications in medical imaging, geophysics and testing of materials, respectively. The problem was introduced the first time by Calderon in the early 80's motivated by an application in geophysical prospection. The goal is to detect the conductivity inside an object from boundary measurements encoded by the so-called Dirichlet to Neumann map. The conductivity problem is severely ill-posed as was proved in 1988 by Alessandrini [1]. In fact, despite of a-priori smoothness assumptions on the unknown conductivity, a conditional stability estimate of logarithmic type is the best possible. This has led to tackle the ill-posedness of the problem establishing regularization strategies for the effective determination of the solution to the problem. A recent trend is to restrict the set of admissible conductivities; for example assuming a smooth background conductivity containing a finite number of unknown small inclusions with a significantly different conductivity [9] or considering conductivities that are linear combinations of finitely many (known) profiles [2]. In fact, under these assumptions it is possible to prove Lipschitz stability estimates that imply local convergence of iterative methods, see for instance [7, 4].

We would like to mention several numerical approaches, that have been developed in the context of nonlinear tomographic problems, in particular EIT, inverse scattering, and inverse conductivity problems: these are for instance, level set methods (Santosa [17] based on Osher & Sethian [16]), shape derivatives (based on Sokolowski & Zolesio [18]), Statistical methods (Kaipio & Somersalo [11]), Dbar methods (Nachman [15]), Iterative regularization methods for nonlinear problems (Hanke [10]), Regularization by projection (Kaltenbacher [12]), Topological gradients (Masmoudi), variational regularization methods (Mueller & Siltanen [14]), PDE constraint optimization (Haber [3]).

The year 2016 marks the 110th birthday of the great Russian academician Andrey Nikoayevich Tikhonov (1906 - 1993). Tikhonov's work provides the mathematical foundations of regularization theory for solving inverse problems, which is a core topic of this workshop. Exciting experimental developments and the possibility of implementing regularization algorithms on computers made the mathematical results as prominent as they appear today. In 1979 Allan MacLeod Cormack (1924-1998) and Godfrey Hounsfield (1919-2004) won the Nobel Prize in "Physiology or Medicine" for the first development of a CT-scanner, which was based on inversion of the parallel beam transform, which is probably the most prominent inverse problem. Today new imaging concepts are the major driving force for discoveries in a variety of research areas, ranging from the nanoscale of single molecule imaging, via biomedical research, to macroscopic scales in Astrophysics.

This workshop tried to survey the zoo of regularization methods and to stimulate new research by productive interactions of the different computational and theoretical fields which were represented in the workshop.

2 Recent Developments and Open Problems

The primary goal of the workshop has been to provide a forum on theoretical and numerical aspects related to stability in inverse problems. In particular we emphasize stability estimates for inverse problems, such as parameter estimation problems in wave equations, regularization methods in the discrete and continuous formulations, and multi-level techniques that make use of theoretical stability estimates and numerical algorithms. The workshop brought together fields which usually do not interact. Numerically oriented researchers usually do not make use of theoretical stability estimates and regularization researchers working on discrete and continuous formulations have focus on different aspects of numerics and analysis. The situation with interactions actually changed with the fundamental work of Alessandrini and Vessella [2], who derived stability estimates based on a continuous formulation for piecewise constant Ansatz functions. Their work bridged the gap between the discrete and continuous regularization world and also found its way to numerics recently. The recently very active topic of *uncertainty quantification* in continuous and discrete formulations is another example, considered by different communities, but solves the same problems (like the inverse aquifer problem). These communities use completely different computational approaches like Markov Chain Monte Carlo (MCMC), Kalman Filter (KF). The stochastic analysis can be considered the analog to the deterministic stability estimates. As with all workshops on inverse problems at BIRS, which were predominantly on theoretical aspects and concrete developments, we observed a broad and lively discussion of the theoretical developments, analytical and computational methodologies and new and existing applications. Moreover, we observed a growth in understanding of analysis, algorithms, and mathematical modeling. We aimed to bring mathematicians working on more abstract stability estimates in concrete examples as well as researchers working on more concrete computational algorithms.

3 Presentation Highlights

We start by giving a list of the titles and abstracts in chronological order:

Monday:

Peter Kuchment: Detecting presence of emission sources with low SNR. "Analysis" vs deep learning: The talk will discuss the homeland security problem of detecting presence of emission sources at high noise conditions. (Semi-)analytic and deep learning techniques will be compared. This is a joint work with W. Baines and J. Ragusa.

Luca Rondi: A multiscale approach for inverse problems: We extend the hierarchical decomposition of an image as a sum of constituents of different scales, introduced by Tadmor, Nezzar and Vese in 2004 [19], to a general setting. We develop a theory for multiscale decompositions which, besides extending the one of Tadmor, Nezzar and Vese to arbitrary L^2 functions, is applicable to nonlinear inverse problems, as well as to other imaging problems. As a significant example, we present applications to the inverse conductivity problem. This is a joint work with Klas Modin and Adrian Nachman.

Adrian Nachman: Two nonlinear harmonic analysis results: a Plancherel theorem for a nonlinear Fourier transform arising in the Inverse Conductivity Problem and multiscale decomposition of diffeomorphisms in Image Registration: The first part of this talk will be devoted to a well-studied nonlinear Fourier transform in two dimensions for which a proof of the Plancherel theorem had been a challenging open problem. I will describe the solution of this problem, as well as its application to reconstruction in the inverse boundary value problem of Calderon for a class of unbounded conductivities. This will include new estimates on classical fractional integrals and a new result on L^2 boundedness of pseudodifferential operators with non-smooth symbols. (Joint work with Idan Regev and Daniel Tataru).

The second part will be a continuation of Luca Rondi's lecture. It will be devoted to a multiscale decomposition of diffeomorphisms in image registration, inspired by the Tadmor Nezzar Vese hierarchical decomposition of images, with the sum replaced by composition of maps. (Joint work with Klas Modin and Luca Rondi).

Elisa Francini: Stable determination of polygonal and polyhedral interfaces from boundary measurements: We present some Lipschitz stability estimates for the Hausdorff distance of polygonal or polyhedral inclusions in terms of the Dirichlet-to-Neumann map based on a series of papers in collaboration with Elena Beretta (New York University AbuDhabi) and Sergio Vessella (Universit?? di Firenze).

Matteo Santacesaria: Infinite-dimensional inverse problems with finite measurements: In this talk I will discuss how ideas from applied harmonic analysis, in particular sampling theory and compressed sensing, may be applied to inverse problems for partial differential equations. The focus will be on inverse boundary value problems for the conductivity and the Schrodinger equations, but the approach is very general and allows to handle many other classes of inverse problems. I will discuss uniqueness, stability and reconstruction, both in the linearized and in the nonlinear case. This is joint work with Giovanni S. Alberti.

Ekaterina Sherina: Quantitative PAT-OCT Elastography for Biomechanical Parameter Imaging: Diseases like cancer or arteriosclerosis often cause changes of tissue stiffness in the micrometer scale. Our work aims at developing a non-invasive method to quantitatively image these biomechanical changes and study the potential of the method for medical diagnostics. We focus on quantitative elastography combined with photoacoustic (PAT) and optical coherence tomography (OCT). The problem we deal with consists in estimating elastic material parameters from internal displacement data, which are evaluated from OCT-PAT recordered successive images of a sample.

Tuesday:

Gabriele Steidl: Regularization of Inverse Problems via Time Discrete Geodesics in Image Spaces: This talk addresses the solution of inverse problems in imaging given an additional reference image. We combine a modification of the discrete geodesic path model of Berkels, Effland and Rumpf [5] with a variational model, actually the $L^2 - TV$ model, for image restoration. We prove that the space continuous model has a minimizer and propose a minimization procedure which alternates over the involved sequences of deformations and images. The minimization with respect to the image sequence exploits recent algorithms from convex analysis to minimize the L 2 -T V functional. For the numerical computation we apply a finite difference approach on staggered grids together with a multilevel strategy. We present proof-of-the-concept numerical results for sparse and limited angle computerized tomography as well as for superresolution demonstrating the power of the method. Further we apply the morphing approach for image colorization. This is joint work with Sebastian Neumayer and Johannes Persch (TU Kaiserslautern).

Uri Ascher: Discrete processes and their continuous limits: The possibility that a discrete process can be closely approximated by a continuous one, with the latter involving a differential system, is fascinating. Important theoretical insights, as well as significant computational efficiency gains may lie in store. A great success story in this regard are the Navier-Stokes equations, which model many phenomena in fluid flow rather well. Recent years saw many attempts to formulate more such continuous limits, and thus harvest theoretical and practical advantages, in diverse areas including mathematical biology, image processing, game theory, computational optimization, and machine learning. Caution must be applied as well, however. In fact, it is often the case that the given discrete process is richer in possibilities than its continuous differential system limit, and that a further study of the discrete process is practically rewarding. I will show two simple examples of this. Furthermore, there are situations where the continuous limit process may provide important qualitative, but not quantitative, information about the actual discrete process. I will demonstrate this as well and discuss consequences.

Markus Grasmair: Total variation based Lavrentiev regularisation: In this talk we will discuss a non-linear variant of Lavrentiev regularisation, where the sub-differential of the total variation replaces the identity operator as regularisation term. The advantage of this approach over Tikhonov based total variation regularisation is that it avoids the evaluation of the adjoint operator on the data. As a consequence, it can be used, for instance, for the solution of Volterra integral equations of the first kind, where the adjoint would require an integration forward in time, without the need of accessing future data points. We will discuss first the theoretical properties of this method, and then propose a taut-string based numerical method for the solution of one-dimensional problems.

Andrea Aspri: Analysis of a model of elastic dislocation in geophysics: In this talk we will discuss a model for elastic dislocations describing faults in the Earth???s crust. We will show how to get the well-posedness of the direct problem which consists in solving a boundary-value/transmission value problem in a half-space for isotropic, inhomogeneous linear elasticity with Lipschitz Lam?? parameters. Mostly we will focus the attention on the uniqueness result for the non-linear inverse problem, which consists in determining the fault and the slip vector from displacement measurements made on the boundary of the half-space. Uniqueness for the inverse problem follows by means of the unique continuation result for systems and under some geometrical constrains on the fault. This is a joint work with Elena Beretta (Politecnico di Milano & NYU ??? Abu Dhabi), Anna Mazzucato (Penn State University) and Maarten de Hoop (Rice University).

Barbara Kaltenbacher: Regularization of backwards diffusion by fractional time derivatives: The backwards heat equation is one of the classical inverse problems, related to a wide range of applications and exponentially ill-posed. One of the first and maybe most intuitive approaches to its stable numerical solution was that of quasireversibility, whereby the parabolic operator is replaced by a differential operator for which the backwards problem in time is well posed. After a short overview of approaches in this vein, we will dwell on a new one that relies on replacement of the first time derivative in the PDE by a fractional differential operator, which, due to the asymptotic properties of the Mittag-Leffler function as compared to the exponential function, leads to an only moderately ill-posed problem. Thus the order alpha of (fractional) differentiation acts as a regularization parameter and convergence takes place in the limit as alpha tends to one. We study the regularizing properties of this approach and a regularization parameter choice by the discrepancy principle. Additionally, a substantial numerical improvement can be achieved by exploiting the linearity of the problem by breaking the inversion into distinct frequency bands and using a different fractional order for each. This is joint work with William Rundell.

Bernd Hofmann: The impact of conditional stability estimates on variational regularization and the distinguished case of oversmoothing penalties: Conditional stability estimates require additional regularization for obtaining stable approximate solutions if the validity area of such estimates is not completely known. The focus of this talk is on the Tikhonov regularization under conditional stability estimates for non-linear ill-posed problems in Hilbert scales, where the case that the penalty is oversmoothing plays a prominent role. This oversmoothing problem has been studied early for linear forward operators, most notably in the seminal paper by Natterer 1984. The a priori parameter choice used there, just providing order optimal convergence rates, has in the oversmoothing case the unexpected property that the quotient of the noise level square and the regularization parameter tends to infinity when the noise level tends to zero. We provide in this talk some new convergence rate results for nonlinear problems and moreover case studies that enlighten the interplay of conditional stability and regularization. In particular, there occur pitfalls for oversmoothing penalties, because convergence can completely fail and the stabilizing effect of conditional stability may be lost.

Antonio Leitao: A convex analysis approach to iterative regularization methods: We address two well known iterative regularization methods for ill-posed problems (Landweber and iterated-Tikhonov methods) and discuss how to improve the performance of these classical methods by using convex analysis tools. The talk is based on two recent articles:

Range-relaxed criteria for choosing the Lagrange multipliers in nonstationary iterated Tikhonov method (with R.Boiger, B.F.Svaiter [6]), and On a family of gradient type projection methods for nonlinear ill-posed problems [13]

Lars Ruthotto: Deep Neural Networks motivated by PDEs: One of the most promising areas in artificial intelligence is deep learning, a form of machine learning that uses neural networks containing many hidden layers. Recent success has led to breakthroughs in applications such as speech and image recognition. However, more theoretical insight is needed to create a rigorous scientific basis for designing and training deep neural networks, increasing their scalability, and providing insight into their reasoning.

This talk bridges the gap between partial differential equations (PDEs) and neural networks and presents a new mathematical paradigm that simplifies designing, training, and analyzing deep neural networks. It shows that training deep neural networks can be cast as a dynamic optimal control problem similar to path-planning and optimal mass transport. The talk outlines how this interpretation can improve the effectiveness of deep neural networks. First, the talk introduces new types of neural networks inspired by to parabolic, hyperbolic, and reaction-diffusion PDEs. Second, the talk outlines how to accelerate training by exploiting multi-scale structures or reversibility properties of the underlying PDEs. Finally, recent advances on efficient parametrizations and derivative-free training algorithms will be presented.

Wednesday:

Simon Arridge Combining learned and model based approaches for inverse problems: Deep Learning (DL) has become a pervasive approach in many machine learning tasks and in particular in image processing problems such as denoising, deblurring, inpainting and segmentation. The application of DL within inverse problems is less well explored because it is not trivial to include Physics based knowledge of the forward operator into what is usually a purely data-driven framework. In addition some inverse problems are at a scale much larger than image or video processing applications and may not have access to sufficiently large training sets. In this talk I will present some of our approaches for i) developing iterative algorithms combining data and knowledge driven methods with applications in medical image reconstruction ii) developing a learned PDE architecture for forward and inverse models of non-linear image flow. Joint work with : Marta Betcke, Andreas Hauptmann, Felix Lucka

Giovanni Alberti "Combining the Runge approximation and the Whitney embedding theorem in hybrid imaging":

Abstract The reconstruction in quantitative coupled physics imaging often requires that the solutions of certain PDEs satisfy some non-zero constraints, such as the absence of critical points or nodal points. After a brief review of several methods used to construct such solutions, I will focus on a recent approach that combines the Runge approximation and the Whitney embedding theorem.

Eldad Haber "Conservative architectures for deep neural networks": In this talk we discuss architectures for deep neural networks that preserve the energy of the propagated signal. We show that such networks can have significant computational advantages for some key problems in computer vision

Robert Plato "New results on a variational inequality formulation of Lavrentiev regularization for nonlinear monotone ill-posed problems": We consider nonlinear ill-posed equations $Fu = f$ in Hilbert spaces $\mathcal{H}$, where $F : \mathcal{H} \to \mathcal{H}$ is monotone on a closed convex subset $\mathcal{M} \subset \mathcal{H}$. For given data $f^\delta \in \mathcal{H}$, $\|f^\delta - f\| \le \delta$, a standard approach is Lavrentiev regularization $Fv_\alpha^\delta + \alpha v_\alpha^\delta = f^\delta$, with $v_\alpha^\delta \in \mathcal{M}$ and $\alpha > 0$ small. Since existence of a solution $v_\alpha^\delta \in \mathcal{M}$ may only be guaranteed for special cases, e.g., $\mathcal{M} = \mathcal{H}$ or $\mathcal{M} =$ ball, we replace this equation by a regularized variational inequality, i.e., we consider $u_\alpha^\delta \in \mathcal{M}$ satisfying

$$\langle Fu_\alpha^\delta + \alpha u_\alpha^\delta - f^\delta, w - u_\alpha^\delta \rangle \ge 0 \quad \text{for each } w \in \mathcal{M}.$$

We present new estimates of the error $u_\alpha^\delta - u_*$ for suitable choices of $\alpha = \alpha(\delta)$, if the solution $u_* \in \mathcal{M}$ of $Fu = f$ admits an adjoint source representation. Examples like parameter estimation problems or the autoconvolution equation are considered, and numerical illustrations are also given.

This is joint work with B. Hofmann (TU Chemnitz, Germany), to appear in JOTA.

Thursday:

Erkki Somersalo "A stable Bayesian layer stripping algorithm for electrical impedance tomography": In electrical impedance tomography (EIT) the goal is to estimate an unknown conductivity distribution inside a body based on current-voltage measurements on the boundary of the body. Mathematically, the problem is tantamount to recovering a coefficient function of an elliptic PDE from the knowledge of complete Cauchy data at the boundary. Layer stripping method is based on the idea that the Dirichlet-to-Neumann map of the elliptic PDE can in principle be propagated into the body using an operator-valued backwards Riccati equation, while simultaneously estimating the unknown coefficient around the inwards moving artificial boundary. The ill-posedness of the inverse problem manifests itself as instability of the approach: among other things, the backwards Riccati equation may blow up in finite time. In this talk, the layer stripping algorithm is revisited in a Bayesian framework, and using novel ideas from particle filtering and sequential Monte Carlo methods, a stable computational scheme is proposed and tested numerically.

Daniela Calvetti "Reconstruction via Bayesian hierarchical models: convexity, sparsity and model reduction": The reconstruction of sparse signals from indirect, noisy data is a challenging inverse problem. In the Bayesian framework, the sparsity belief can be encoded via hierarchical prior models. In this talk we discuss the convexity - or lack thereof - of the functional associated to different models, and we show that Krylov subspace methods for the computation of the MAP solution implicitly perform an effective and efficient model reduction.

Jari Kaipio "Born approximation for inverse scattering with high contrast media": Born approximation is widely used for inverse scattering problems with low contrast media. With high constrast media, the single scattering approximation is not a feasible one and the respective reconstructions are often rendered useless. In this talk, we consider the inverse scattering problem in the Bayesian framework for inverse problems. We show that with approximative marginalization, one may be able to use the Born approximation and, furthermore, compute statistically meaningful error estimates for the index of refraction.

Claudia Schillings "On the Analysis of the Ensemble Kalman Filter for Inverse Problems and the Incorporation of Constraints": The Ensemble Kalman filter (EnKF) has had enormous impact on the applied sciences since its introduction in the 1990s by Evensen and coworkers. It is used for both data assimilation problems, where the objective is to estimate a partially observed time-evolving system, and inverse problems, where the objective is to estimate a (typically distributed) parameter appearing in a differential equation. In this talk we will focus on the identification of parameters through observations of the response of the system - the inverse problem. The EnKF can be adapted to this setting by introducing artificial dynamics. Despite documented success as a solver for such inverse problems, there is very little analysis of the algorithm. In this talk, we will discuss well-posedness and convergence results of the EnKF based on the continuous time scaling limits, which allow to derive estimates on the long-time behavior of the EnKF and, hence, provide insights into the convergence properties of the algorithm. This is joint work with Dirk Bloemker (U Augsburg), Andrew M. Stuart (Caltech), Philipp Wacker (FAU Erlangen-Nuernberg) and Simon Weissmann (U Mannheim).

Noemie Debroux "A joint reconstruction, super-resolution and registration model for motion-compensated MRI": This work addresses a central topic in Magnetic Resonance Imaging (MRI) which is the motion-correction problem in a joint reconstruction, super-resolution and registration framework. From a set of multiple MR acquisitions corrupted by motion, we aim at -jointly- reconstructing a super-resolved single motion-free corrected image and retrieving the physiological dynamics through the deformation maps. To this purpose, we propose a novel variational model relying on hyperelasticity and compressed sensing principles. We demonstrate through numerical results that this combination creates synergies in our complex variational approach resulting in higher quality reconstructions. This is a joint work with A. Aviles-Rivero, V. Corona, M. Graves, C. Le Guyader, C. Sch??nlieb, G. Williams.

Shari Moskow "Reduced order models for spectral domain inversion: Embedding into the continuous problem and generation of internal data": We generate reduced order Galerkin models for inversion of the Schrödinger equation given boundary data in the spectral domain for one and two dimensional problems. We show that in one dimension, after Lanczos orthogonalization, the Galerkin system is precisely the same as the three point staggered finite difference system on the corresponding spectrally matched grid. The orthogonalized basis functions depend only very weakly on the medium, and thus by embedding into the continuous problem, the reduced order model yields highly accurate internal solutions. In higher dimensions, the orthogonalized basis functions play the role of the grid steps, and highly accurate internal solutions are still obtained. We present inversion experiments based on the internal solutions in one and two dimensions. This is joint with: L. Borcea, V. Druskin, A. Mamonov, M. Zaslavsky.

Weihong Guo "PCM-TV-TFV: A Novel Two-Stage Framework for Image Reconstruction from Fourier Data": We propose in this paper a novel two-stage projection correction modeling (PCM) framework for image reconstruction from (nonuniform) Fourier measurements. PCM consists of a projection stage (P-stage) motivated by the multiscale Galerkin method and a correction stage (C-stage) with an edge guided regularity fusing together the advantages of total variation and total fractional variation. The P-stage allows for continuous modeling of the underlying image of interest. The given measurements are projected onto a space in which the image is well represented. We then enhance the reconstruction result at the C-stage that minimizes an energy functional consisting of a delity in the transformed domain and a novel edge guided regularity. We further develop ecient proximal algorithms to solve the corresponding optimization problem. Various numerical results in both one-dimensional signals and two-dimensional images have also been presented to demonstrate the superior performance of the proposed two-stage method to other classical one-stage methods. This is a joint work with Yue Zhang (now at Siemens Corporate Research) and Guohui Song (Clarkson University).

Friday:

Fioralba Cakoni "Inverse Scattering Problems for the Time Dependent Wave Equation": In this presentation we will discuss recent progress in non-iterative methods in the time domain. The use of time dependent data is a remedy for the large spacial aperture that these method need to obtain a reasonable reconstructions. Fist we consider the linear sampling method for solving inverse scattering problem for inhomogeneous media. A fundamental tool for the justification of this method is the solvability of the time domain interior transmission problem that relies on understanding the location on the complex plane of transmission eigenvalues. We present our latest result on the solvability of this problem. As opposed to the frequency domain case, in the time domain there are no known qualitative methods with a complete mathematical justification, such as e.g. the factorization method.This is still a challenging open problem and the second part of the talk addresses this issue. In particular, we discuss the factorization method to obtain explicit characterization of a (possibly non-convex) Dirichlet scattering object from measurements of time-dependent causal scattered waves in the far field regime. In particular, we prove that far fields of solutions to the wave equation due to particularly modified incident waves, characterize the obstacle by a range criterion involving the square root of the time derivative of the corresponding far field operator. Our analysis makes essential use of a coercivity property of the solution of the Dirichlet initial boundary value problem for the wave equation in the Laplace domain that forces us to consider this particular modification of the far field operator. The latter in fact, can be chosen arbitrarily close to the true far field operator given in terms of physical measurements. Finally we discuss some related open questions.

Marco Verani "Detection of conductivity inclusions in a semilinear elliptic problem via a phase field approach": We tackle the reconstruction of discontinuous coefficients in a semilinear elliptic equation from the knowledge of the solution on the boundary of the planar bounded domain. The problem is motivated by an application in cardiac electrophysiology. We formulate a constraint minimization problem involving a quadratic mismatch functional enhanced with a phase field term which penalizes the perimeter. After computing the optimality conditions of the phase-field optimization problem and introducing a discrete finite element formulation, we propose an iterative algorithm and prove convergence properties. Several numerical results are reported, assessing the effectiveness and the robustness of the algorithm in identifying arbitrarily-shaped inclusions. (Joint work with E. Beretta and L. Ratti)

Peter Elbau "About using dynamical systems as regularisation methods and their optimal convergence rates": A regularisation method for a linear, ill-posed problem may be seen as a family of bounded approximate inverse operators; for example, this family could be given as the solution of a dynamical system whose stationary limit corresponds to the exact inverse. Showalter's method, where the dynamical system is the gradient flow for the squared norm of the residuum, is a classical example of this sort of regularisation. Recently, second order dynamical systems have been used for this construction (despite their oscillating behaviour), and this setting allowed for a continuous formulation of Nesterov's algorithm which gave an explanation of its fast rate of convergence. In this talk, we want to restrict ourselves to the case where the inverse problem enters the dynamical system only via the gradient of the squared norm of the residuum so that we can apply spectral theory to solve the dynamical system explicitly, which allows us to characterise the convergence rate of the regularisation method uniquely by, for example, variational source conditions.

4 Outcome of the Meeting and Scientific Progress Made

The meeting took place in a friendly environment with a lot of interactions and many stimulating discussions. In addition to the scientific talks also a poster presentation was held, which yielded to lively interactions of the participants.

The workshop combined different aspects and techniques for the solution of inverse problems and image reconstruction, like machine learning, neural networks, stability, regularization methods in deterministic and stochastic settings and uncertainty quantification. The participants could identify some common view points. For instance, dynamical methods for solving inverse problems in a deterministic and a stochastic setting differ by the consideration of noise. Indeed, while in uncertainty quantification a time dependent noise process is considered, in the deterministic setting noise is static. However, both approaches have the same goal: to establish and analyze new methods for solving applied inverse problems. The similarities became evident in the talks of Daniela Calvetti, Claudia Schilling and Peter Elbau, respectively, and allow for a synergetic point of view. Dynamical approaches for registration and general imaging problems where discussed in the talks of Gabriele Steidl, Adrian Nachman and Luca Rondi. Here the considered analytical methods are the calculus of variations.

When discretizing inverse problems, stability becomes the dominant question, and the analysis of such was a common topic along many presenters, like Elisa Francini, Matteo Santacesaria and Bernd Hoffman. Stability could be investigated in a deterministic and a stochastic setting as well. Interestingly merging of the discrete setting with the continuous world is still not fully understood and a series of open questions needs to be solved. In particular, appropriate discretization spaces are still limited to piecewise constant and simple finite element spaces, while adaptive and advanced spaces like reduced basis spaces have not been investigated. The need of appropriate discretization in electrical impedance tomography has been documented in the talk by Matteo Santacesaria as an open question. Erkki Somersalo presented a stable layer stripping method for solving the problem of EIT in a stochastic setting. Similar as EIT also inverse scattering problems have been a driving source for inverse problems in general and regularization theory in particular. Fioralba Cakoni has reported on scattering problems for the wave equation. Jari Kaipio presented Born approximation methods for solving inverse scattering problems, which then were solved by a Bayesian regularization method.

Uniqueness of inverse problems was the main issue in the talk of Giovanni Alberti, who showed uniqueness results for hybrid inverse problems. While on the other hand Shari Moskow showed how to generate highly accurate internal data using reduced order models that are used in hybrid inverse problems.

Machine learning has become a major research topic in inverse problems: the expanding area is yet not well structured scientifically and it is indeed necessary to provide mathematically well defined problem formulations. The talks of Lars Ruhotto and Eldad Haber were highlights in this perspective: by presenting a class of neural networks with connections between layers at distance greater than one, which is the standard setting, they introduced a links to dynamical systems and differential equations; this approach relates the usual problem of weight determination in machine learning to parameter identification in partial differential equations. In the reverse direction, deep Learning and methods from artificial intelligence have been identified as new tools for solving inverse problems. Although the mathematical theory of machine learning is still at a premature stage there are already a series of well-established connections, such as to constrained optimization and to parameter identification in partial differential equations. Therefore this workshop can be considered as one of the first in which continuous limits of machine learning algorithms (layer to infinity) were shown. A careful investigation of continuous limits of discrete dynamical systems was presented in Uri Ascher's talk - he also showed how this algorithms can be used to solve inverse problems. Very intriguing was the talk of Peter Kuchment who explained his point of view of machine learning in the context of highly ill–posed problems in security applications with very little information on the object. In this case data driven models might outperform model driven approaches.

The field of regularization was covered in a wide generality: Novel aspects of infinite dimensional regularization theory in a deterministic setting were discussed in the talks of Markus Grasmair, Robert Plato, Barbara Kaltenbacher and Bernd Hofmann. Numerical methods for solving convex optimizations problems of regularized inverse problems were discussed in several talks, such as in particular in Antonio Leitao's talk.

Imaging problems, in particular with magnetic resonance data has been considered by Noemie Debroux, Weihong Guo and Simon Arridge. Here one could learn about (dynamical) total variation denoising and filtering. Simon Arridge combined and replaced filtering techniques by learning methods.

The speakers were chosen from all levels of the academic career: recent Ph.D.s (e.g., Noemie Debroux) were given the possibility to present their work alongside the more senior researchers in the field of inverse problems.

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