Universal fluctuations and scaling relations in interacting dimer models

Alessandro Giuliani, Univ. Roma Tre

Based on joint works with V. Mastropietro and F. Toninelli

BIRS, November 18, 2019

Outline

1. The non-interacting dimer model: exact solution and universality
2. Interacting dimers: weak universality and main results
3. Sketch of the proof

Non-interacting dimer model

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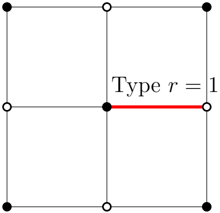

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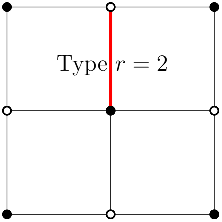

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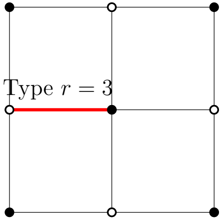

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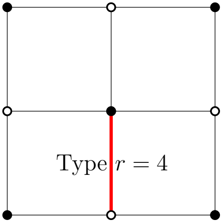

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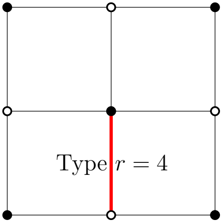

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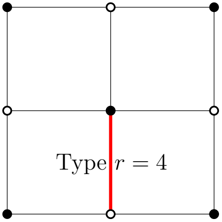

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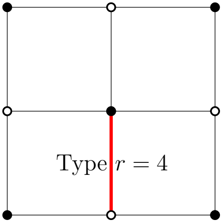

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The model is exactly solvable, e.g., $Z_L^0 = \det K(\underline{t})$, with $K(\underline{t})$ the Kasteleyn matrix. This implies

$$F(\underline{t}) = \lim_{L \to \infty} \frac{1}{L^2} \log Z_L^0 = \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \log |\mu(k)|,$$

with: $$\mu(k) = t_1 + it_2 e^{ik_1} - t_3 e^{ik_1 + ik_2} - ie^{ik_2}.$$

Non-interacting dimer correlations

Non-interacting dimer-dimer correlations can also be computed exactly. E.g.,

$$\langle \mathbb{1}_{(x,1)}; \mathbb{1}_{(y,1)} \rangle_0 = -t_1^2 \, K^{-1}(x-y) \, K^{-1}(y-x),$$

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Analyticity of $F(\underline{t})$ and decay of $\langle \mathbb{1}_{(x,r)}; \mathbb{1}_{(y,r')} \rangle_0$ can be read from the zeros of $\mu(k)$:

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Liquid phase: two non-degenerate zeros, in which case $K^{-1}(x)$ decays algebraically, as $(dist.)^{-1}$.

Asymptotics of dimer correlations

Let $p_\pm$ be the two non-degenerate zeros of $\mu(k)$, $\alpha_\pm = \partial_{k_1}\mu(p_\pm)$, $\beta_\pm = \partial_{k_2}\mu(p_\pm)$.

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Correspondingly,

$$\langle \mathbb{1}_{(x,r)}; \mathbb{1}_{(0,r')}\rangle_0 = \frac{1}{4\pi^2}\sum_{\omega=\pm}\frac{K_{\omega,r}K_{\omega,r'}}{(\beta_\omega x_1 - \alpha_\omega x_2)^2}$$
$$+\frac{1}{4\pi^2}\sum_{\omega=\pm}\frac{K_{-\omega,r}K_{\omega,r'}}{|\beta_\omega x_1 - \alpha_\omega x_2|^2}e^{i(p_\omega - p_{-\omega})\cdot x} + O(|x|^{-3})$$

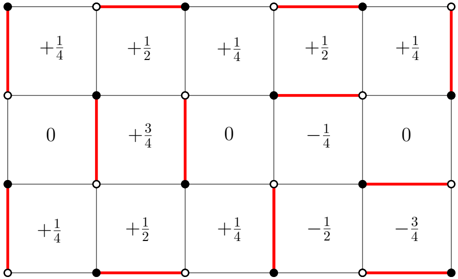
where: $K_{\omega,1} = t_1$, $K_{\omega,2} = it_2 e^{i(p_\omega)_1}$,
$K_{\omega,3} = -t_3 e^{i(p_\omega)_1 + i(p_\omega)_2}$, $K_{\omega,4} = -ie^{i(p_\omega)_2}$.

Dimers and height function

Dimer correlations ⇒ fluctuations of $h(f)$:

$$h(f') - h(f) = \sum_{b \in C_{f \to f'}} \sigma_b(\mathbb{1}_b - 1/4)$$

[$\sigma_b = \pm 1$ if b crossed with white on the right/left.]



Non-interacting height fluctuations

E.g., variance of the height difference:

$$\mathrm{Var}_0(h(f) - h(f')) = \sum_{b,b' \in C_{f \to f'}} \sigma_b \sigma_{b'} \langle \mathbb{1}_b ; \mathbb{1}_{b'} \rangle_0 .$$

Formula for $\langle \mathbb{1}_b ; \mathbb{1}_{b'} \rangle_0$ + path-indep. of $h(f) - h(f')$

$$\Rightarrow \mathrm{Var}_0(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'|$$

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NB: the pre-factor $\frac{1}{\pi^2}$ is independent of t_1, t_2, t_3.

Building upon this (Kenyon):

- height fluctuations converge to massless **GFF**
- scaling limit is **conformally covariant**

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Q: Does universality survives in the presence of perturbations breaking the determinant structure?

Outline

Interacting model:

$$Z_L^{\lambda} = \sum_{D \in \mathcal{D}_L} \Big(\prod_{b \in D} t_{r(b)} \Big) e^{\lambda \sum_{x \in \Lambda} f(\tau_x D)},$$

where: λ is small, f is a local function around the origin, τ_x translates by x.

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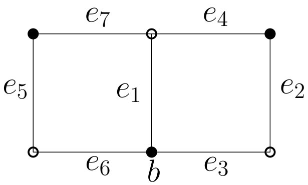
where: λ is small, f is a local function around the origin, τ_x translates by x. Two examples:

1. Dimers with plaquette interaction:

$$f_P(D) = \mathbb{1}_{e_1}\mathbb{1}_{e_2} + \mathbb{1}_{e_3}\mathbb{1}_{e_4} + \mathbb{1}_{e_1}\mathbb{1}_{e_5} + \mathbb{1}_{e_6}\mathbb{1}_{e_7}$$

2. The 6-vertex model: $f_{6V}(D) = \mathbb{1}_{e_1}\mathbb{1}_{e_2} + \mathbb{1}_{e_3}\mathbb{1}_{e_4}$

[Recall: $6V \leftrightarrow AT$ via discrete bosonization (Dubedat)]



Critical exponents and weak universality

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Therefore, if the model exhibits some form of universality, it cannot be in a naive way. Right notion: weak universality, proposed by Kadanoff:

all critical exponents can be deduced by one of them.

$$\text{E.g.,} \qquad X_c^{AT} X_e^{AT} = 1, \qquad X_p^{AT} = \frac{1}{4} X_e^{AT}.$$

Main results: interacting dimer-dimer correlation

Theorem [G.-Mastropietro-Toninelli (2015, 2017, 2019)]:

Let t_1, t_2, t_3 be s.t. $\mu(k)$ has two distinct non-degen. zeros, $p_\pm$ (non-degenerate $\Leftrightarrow \alpha_\omega = \partial_{k_1}\mu(p_\omega)$ and $\beta_\omega = \partial_{k_2}\mu(p_\omega)$ are not parallel).

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Then, for λ small enough,

$$\langle \mathbb{1}_{(x,r)}; \mathbb{1}_{(0,r')}\rangle_\lambda = \frac{1}{4\pi^2}\sum_{\omega=\pm}\frac{K^\lambda_{\omega,r}K^\lambda_{\omega,r'}}{(\beta^\lambda_\omega x_1 - \alpha^\lambda_\omega x_2)^2}$$

$$+\frac{1}{4\pi^2}\sum_{\omega=\pm}\frac{H^\lambda_{-\omega,r}H^\lambda_{\omega,r'}}{|\beta^\lambda_\omega x_1 - \alpha^\lambda_\omega x_2|^{2\nu(\lambda)}}e^{-i(p^\lambda_\omega - p^\lambda_{-\omega})\cdot x} + O(|x|^{-3+O(\lambda)})$$

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where: $K^\lambda_{\omega,r}$, $H^\lambda_{\omega,r}$, α^λ_ω, β^λ_ω, p^λ_ω, $\nu(\lambda)$ are analytic in λ.

Moreover, $\nu(\lambda) = 1 + a\lambda + \cdots$ and, generically, $a \neq 0$.

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Use formula for $\langle \mathbb{1}_b; \mathbb{1}_{b'} \rangle_\lambda$ in that for height variance:

$$\mathrm{Var}_\lambda(h(f) - h(f')) = \sum_{b,b' \in C_{f \to f'}} \sigma_b \sigma_{b'} \langle \mathbb{1}_b; \mathbb{1}_{b'} \rangle_\lambda$$

it is not obvious that the growth is still logarithmic:
a priori, it may depend on the critical exp. $\nu(\lambda)$.

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- In general, $A(\lambda)$ depends on $\lambda, f, t_1, t_2, t_3$. Moreover,

$$\boxed{A(\lambda) = \nu(\lambda)}$$

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Analogue of $A = \nu$ previously proved in quantum 1D models (Haldane relation) (Benfatto-Mastropietro).

Our result is the first instance of such a 'non-local' scaling relation in a classical statmech model.

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1. The non-interacting dimer model:
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3. Sketch of the proof

Starting point: Grassmann representation of the non-interacting partition function:

$$Z_0 = \det(K) = \int \prod_x d\psi_x^+ d\psi_x^- e^{-(\psi^+, K\psi^-)}$$

$$= \int \mathcal{D}\psi \, e^{-\int \frac{dk}{(2\pi)^2} \hat{\psi}_k^+ \hat{\psi}_k^- \mu(k)}$$

where $\{\psi_x^\pm\}_{x\in\Lambda}$ are Grassmann variables.

Fermionic representation

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where $\{\psi_x^\pm\}_{x\in\Lambda}$ are Grassmann variables. Similarly,

$$K^{-1}(x,y) = \frac{1}{\det(K)} \int \mathcal{D}\psi \, e^{-\int \frac{dk}{(2\pi)^2} \hat{\psi}_k^+ \hat{\psi}_k^- \mu(k)} \psi_x^- \psi_y^+ .$$

Interacting dimers as interacting fermions

The partition function of the interacting model is

$$\frac{Z_\lambda}{Z_0} = \frac{1}{\det(K)} \int \mathcal{D}\psi \, e^{-\int \frac{dk}{(2\pi)^2} \hat{\psi}^+_k \hat{\psi}^-_k \mu(k) + V(\psi)},$$

where $V(\psi)$ is exp. decaying. E.g., if $f = f_P$,

$$V(\psi) = - \sum_{\gamma:\ |\gamma|>1} (1 - e^\lambda)^{|\gamma|-1} \prod_{e\in\gamma} (K_{r(e)} \psi^+_{b(e)} \psi^-_{w(e)}),$$

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The generating function for dimer correlations $W(A) = \langle \prod_e e^{A_e \mathbb{1}_e} \rangle_\lambda$ can be expressed similarly. E.g., if $f = f_P$, $V(\psi)$ is replaced by

$$V(\psi, A) = -\sum_{\gamma:\ |\gamma|>1} (1 - e^\lambda)^{|\gamma|-1} \prod_{e\in\gamma} (K_{r(e)} \psi^+_{b(e)} \psi^-_{w(e)} e^{A_e}),$$

which is lattice gauge invariant w.r.t.

$$\psi^\pm_x \to e^{i\alpha^\pm_x} \psi^\pm_x, \qquad A_e \to A_e - i\alpha^+_{b(e)} - i\alpha^-_{w(e)}$$

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$$p_\omega^\lambda = p_\omega + O(\lambda), \quad \alpha_\omega^\lambda = \alpha_\omega + O(\lambda), \quad \beta_\omega^\lambda = \beta_\omega + O(\lambda),$$

be the interacting ones, to be fixed a posteriori via a fixed point argument.

Counterterms

The interaction has the effect of moving the 'Fermi points' p_ω and the 'Fermi velocities' α_ω, β_ω. We let

$$p^\lambda_\omega = p_\omega + O(\lambda), \quad \alpha^\lambda_\omega = \alpha_\omega + O(\lambda), \quad \beta^\lambda_\omega = \beta_\omega + O(\lambda),$$

be the interacting ones, to be fixed a posteriori via a fixed point argument. Correspondingly, we write

$$\mu(k) = \mu_0(k) - n(k),$$

where, in the vicinity of p^λ_ω,

$$n(k) = \nu_{0,\omega} + a_{0,\omega}(k_1 - (p^\lambda_\omega)_1) + b_{0,\omega}(k_2 - (p^\lambda_\omega)_2)$$

Multiscale decomposition

We rewrite

$$\frac{Z_\lambda}{Z_0} = \frac{1}{\det(K)} \int \mathcal{D}\psi \, e^{-\int \frac{dk}{(2\pi)^2} \hat{\psi}^+_k \hat{\psi}^-_k \mu_0(k) + N(\psi) + V(\psi)} \equiv \left\langle e^{N(\psi)+V(\psi)} \right\rangle_0 .$$

where

$$N(\psi) = \int \frac{dk}{(2\pi)^2} \hat{\psi}^+_k \hat{\psi}^-_k n(k),$$

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and similarly for $W(A)$. Z_λ and $W(A)$ can be analyzed via a multiscale procedure (fermionic RG): we decompose

$$\int \frac{dk}{(2\pi)^2} \frac{e^{-ik\cdot(x-y)}}{\mu_0(k)} = \sum_{\omega=\pm} \sum_{h\le 0} e^{-ip^\lambda_\omega (x-y)} g^{(h)}_\omega (x-y),$$

where

$$g^{(h)}_\omega(x) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{-i(k-p^\lambda_\omega)x}}{\mu_0(k)} f_h(k - p^\lambda_\omega)$$

with $f_h(k)$ a smooth version of $\mathbb{1}(2^{h-1} \le |k| \le 2^h)$.

Multiscale integration

Correspondingly, we decompose the Grassmann field into quasi-particles and scales: $\psi^{\pm}_{x} = \sum_{\omega} e^{\pm i p_F^{\omega} x} \sum_{h \le 0} \psi^{(h)\pm}_{x,\omega}$ and integrate step by step $\psi^{(0)}, \psi^{(-1)}, \ldots$, thus getting for $h < 0$

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$$\begin{aligned} V^{(h)}(\psi) &= \sum_\omega \int \frac{dk}{(2\pi)^2} \hat\psi^+_{k,\omega} \hat\psi^-_{k,\omega} (2^h \nu_{h,\omega} + a_{h,\omega} k_1 + b_{h,\omega} k_2) \\ &+ \lambda_h \sum_x \psi^+_{x,+} \psi^-_{x,+} \psi^+_{x,-} \psi^-_{x,-} + \text{irrelevant terms.} \end{aligned}$$

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Key point to be shown: if $\nu_{0,\omega}, a_{0,\omega}, b_{0,\omega}$ are properly fixed,

$$|\nu_{h,\omega}|, |a_{h,\omega}|, |b_{h.\omega}|, |\lambda_h - \lambda_{-\infty}| \le C|\lambda| 2^{h/2}, \text{ with } \lambda_{-\infty} = \lambda + O(\lambda^2).$$

The infrared reference model: Tomonaga-Luttinger

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In order to prove it, we compare the IR behaviour of our model with that of a reference, exactly solvable, model, playing the role of infrared fixed point theory, the TL model:

$$e^{W_N(J,\phi)} = \int P_Z^{[\leq N]}(d\psi) e^{\mathcal{V}(\sqrt{Z}\psi) + \sum_{j=1}^{2}(J^{(j)}, \rho^{(j)}) + Z(\psi,\phi)}.$$

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$\mathcal{V}$ is a non-local quartic interaction with kernel $\lambda_\infty v_0(x,y)$; $\rho^{(1)}_{x,\omega} = \psi^+_{x,\omega}\psi^-_{x,\omega}$ is the 'density', $\rho^{(2)}_{x,\omega} = \psi^+_{x,\omega}\psi^-_{x,-\omega}$ is the 'mass'.

Exact solution of the TL model

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and $\langle \rho^{(2)}_{x,\omega}; \rho^{(2)}_{0,\omega} \rangle_{TL} \sim (\text{const.}) |x|^{-2\nu_{TL}}$, with $\nu_{TL} = \frac{1-\tau}{1+\tau}$.

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Correspondingly, $\langle \hat{\psi}^-_{k+p^\lambda_\omega} \hat{\psi}^+_{k+p^\lambda_\omega} \rangle_\lambda \sim \langle \hat{\psi}^-_{k,\omega} \hat{\psi}^+_{k,\omega} \rangle_{TL}$, and

$$\langle \mathbb{1}_{(x,r)}; \mathbb{1}_{(y,r')} \rangle_\lambda = \sum_{\omega=\pm} \hat{K}_{\omega,r} \hat{K}_{\omega,r'} \langle \rho^{(1)}_{x,\omega}; \rho^{(1)}_{y,\omega} \rangle_{TL}$$

$$+ \sum_{\omega=\pm} e^{i(p^\lambda_\omega - p^\lambda_{-\omega})(x-y)} \hat{H}_{-\omega,r} \hat{H}_{\omega,r'} \langle \rho^{(2)}_{x,\omega}; \rho^{(2)}_{y,\omega} \rangle_{TL} + O(|x-y|^{-3+O(\lambda)}).$$

A similar relation, involving the same prefactors $\hat{K}_{\omega,r}, \hat{K}_{\omega,r}$, is valid for the vertex function.

Ward Identities for the dimer and TL models

Using the last relation between $\langle \mathbb{1}_{(x,r)}; \mathbb{1}_{(y,r')} \rangle_\lambda$ and $\langle \rho^{(j)}_{x,\omega}; \rho^{(j)}_{y,\omega} \rangle_{TL}$, with $j = 1, 2$, we obtain our main result on the asymptotics of the interacting dimer-dimer correlation, with

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If we now compare the vertex WI of the TL model with the lattice WI of the dimer model, associated with the local conservation law $\sum_{b \to x} \mathbb{1}_b = 1$, we find:

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Logarithmic growth and Kadanoff relation

We go back to $\mathrm{Var}_\lambda(h(f)-h(f')) = \sum_{b \in C_{f \to f'}} \sum_{b \in C'_{f \to f'}} \sigma_b \sigma_{b'} \langle \mathbb{1}_b ; \mathbb{1}_{b'} \rangle_\lambda .$

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- Related results, via similar methods, for: Ashkin-Teller, 8V, 6V, XXZ, non-planar Ising.

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Thank you!