First-order relativistic hydrodynamics is stable

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Banff, November 28, 2019

What is relativistic hydrodynamics?

If you don't know anything about relativistic hydrodynamics, you can try learning about it from classic textbooks.

The classics: Hydrodynamics is the dynamics of conserved densities, so the equations must include

$$\partial_\alpha T^{\alpha\beta} = 0\,, \qquad \partial_\alpha J^\alpha = 0$$

Question: What exactly are these $T^{\alpha\beta}, J^\alpha$?

That's where the classic textbooks will differ.

Open Landau-Lifshitz “Fluid mechanics”

$$\boxed{\begin{aligned} &T^{\mu\nu} = p\eta^{\mu\nu} + (\epsilon+p)u^\mu u^\nu + \tau^{\mu\nu}\,, \\ &J^\mu = nu^\mu + \nu^\mu\,, \end{aligned}}$$

$\tau^{\mu\nu}$ is transverse & traceless, contains the viscosities

ν^μ is transverse, contains charge conductivity

Open Weinberg* “Gravitation and cosmology”

$$\boxed{\begin{aligned} &T^{\mu\nu} = p\eta^{\mu\nu} + (\epsilon+p)u^\mu u^\nu + (q^\mu u^\nu + q^\nu u^\mu) + \tau^{\mu\nu}\,, \\ &J^\mu = nu^\mu\,, \end{aligned}}$$

$\tau^{\mu\nu}$ is transverse & traceless, contains the viscosities

q^μ is transverse, contains heat conductivity

*This formulation of hydrodynamics is due to Eckart (1940)

The equations look different, so what?

Let's shut up and calculate: solve for linear perturbations of the thermal equilibrium state. Easy!

Both Landau-Lifshitz' and Eckart's equations predict that:

a) thermal equilibrium does not exist

b) things propagate faster than light

Hiscock, Lindblom, 1984

Hiscock, Lindblom, 1987

What exactly is the problem?

Perturbations $e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}$, solve hydro equations: $\omega=\omega(\mathbf{k})$

Gapless modes: $\omega(\mathbf{k}\to 0)=0$, b/c of conserved charges. These correspond to normal hydrodynamics (sound etc).

But the equations also predict gapped modes $\omega(\mathbf{k}\to 0)\neq 0$, moreover with $\mathrm{Im}(\omega)>0$. These are unphysical modes.

These “fake” modes are outside of the validity regime of the low-energy hydro approximation. These are UV modes.

But if you want to actually solve the hydro equations in practice, these unphysical modes ruin predictability: cutoff-scale physics messes up the infrared behavior.

How is the problem fixed?

So the classic-textbook hydrodynamics is *not* what you solve in practice e.g. to study the quark-gluon plasma.

Most popular fix is the Israel-Stewart theory: the hydro equations are coupled to extra UV degrees of freedom, which in turn kill the unphysical UV modes.

These extra degrees of freedom are the dynamical stresses and heat fluxes, in addition to the dynamical T, u^α, μ.

The extra degrees of freedom of the Israel-Stewart theory play the role of a UV regulator. Note that in the non-relativistic Navier-Stokes eq-s, no UV regulator is needed.

Other regulators?

Can one find a regulator of hydrodynamics that does not involve introducing extra UV degrees of freedom?

E.g. in field theory, the Pauli-Villars regularization introduces extra UV degrees of freedom, but dimreg does not.

In a CFT, it is possible to have a sensible relativistic hydrodynamics whose only dynamical variables are T and u^α, and no extra UV degrees of freedom.

Bemfica, Disconzi, Noronha, arXiv:1708.06255

Claim: Regardless of CFT, there *is* a sensible relativistic hydrodynamics whose only variables are T, u^α, μ, and no extra UV d.o.f. You need to choose a suitable out-of-equilibrium definition of T, u^α, μ.

PK, arXiv:1907.08191
Bemfica, Disconzi, Noronha, arXiv:1907.12695

Two pillars of classical hydrodynamics

Symmetry: $\partial_\alpha T^{\alpha\beta}[T, u^\lambda, \mu] = 0\,, \qquad \partial_\alpha J^\alpha[T, u^\lambda, \mu] = 0$

Note: $T^{\alpha\beta}, J^\alpha$ are always well-defined microscopically.
But: T, u^λ, μ are only well-defined in equilibrium.

There are many ways to define T, u^λ, μ out of equilibrium.
This is why Landau-Lifshitz and Eckart's eq-s are different.

Derivative expansion: Locality, as in any effective theory

$$T^{\alpha\beta}, J^\alpha = O(\partial^0) + O(\partial^1) + O(\partial^2) + \dots$$

Both Landau-Lifshitz and Eckart equations only keep $O(\partial^1)$ terms. This is the physics of viscosity and heat conduction.

To repeat:

In general, out of equilibrium, the notions of “local rest frame”, “local isotropy” etc are ambiguous, and are a matter of pure convention/taste.

How to write $T^{\alpha\beta}$ and J^{α} in terms of T, u^{λ}, μ

In rest frame: J^0 = charge density, J^i = charge current,

T^{00} = energy density, T^{ii} = pressure, T^{ij} = stress,

T^{0i} = momentum density/energy current

Write covariantly: $$J^{\mu} = \mathcal{N} u^{\mu} + \mathcal{J}^{\mu}$$

$$T^{\mu\nu} = \mathcal{E} u^{\mu} u^{\nu} + \mathcal{P} \Delta^{\mu\nu} + (\mathcal{Q}^{\mu} u^{\nu} + \mathcal{Q}^{\nu} u^{\mu}) + \mathcal{T}^{\mu\nu}$$

$\Delta^{\alpha\beta} = g^{\alpha\beta} + u^{\alpha} u^{\beta}$ spatial projector, $\mathcal{J} \cdot u = \mathcal{Q} \cdot u = \mathcal{T} \cdot u = 0$, $\mathcal{T}_{\alpha}{}^{\alpha} = 0$

Hydrodynamics: $\mathcal{E}, \mathcal{P}, \mathcal{Q}^{\alpha}, \mathcal{T}^{\alpha\beta}, \mathcal{N}, \mathcal{J}^{\alpha}$ must be written in terms of T, u^{λ}, μ

How to write $T^{\alpha\beta}$ and J^{α} in terms of T, u^{λ}, μ

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

derivative expansion in terms of
scalars made out of T, u^{α}, μ, ∂_{α}

derivative expansion in terms of
transverse vectors made out of T, u^{α}, μ, ∂_{α}

derivative expansion in terms of
transverse traceless tensors made out of T, u^{α}, μ, ∂_{α}

How to write $T^{\alpha\beta}$ and J^{α} in terms of T, u^{λ}, μ

O(∂) scalars: $\partial_\lambda u^\lambda \quad u^\lambda \partial_\lambda T \quad u^\lambda \partial_\lambda(\mu/T)$

O(∂) transverse vectors: $u^\lambda \partial_\lambda u^\mu \quad \Delta^{\mu\lambda}\partial_\lambda T \quad \Delta^{\mu\lambda}\partial_\lambda(\mu/T)$

O(∂) transverse traceless tensors: $\sigma^{\mu\nu}$

Such terms are of course not new. The new thing is to understand their implications for stability and causality.

Simple analogy: EFT

1) Identify the low-energy variables

2) Write down all the terms allowed by the symmetry,

3) Do this up to a given dimension, e.g.:

$$S = \int d^4x \left(a(\partial_\mu \varphi)^2 + b\varphi^2 + c\varphi^4 \right)$$

4) Constrain the coefficients a, b, c so that the physics is sensible, e.g. $c < 0$ for stability of the vacuum

Do the same in hydro

1) Identify the low-energy variables: $T,\ u^\alpha,\ \mu$

2) Write down all possible terms in the constitutive relations consistent with the symmetry

3) Do this up to a given order (say, 1-st order) in the derivative expansion

4) Constrain the coefficients so that the physics is sensible, e.g. demand stability of equilibrium

Constitutive relations

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

Write down every possible term, with up to one derivative:

$$\mathcal{E} = \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda (\mu/T) + O(\partial^2)\,,$$

$$\mathcal{P} = p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda (\mu/T) + O(\partial^2)\,,$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \theta_2/T\, \Delta^{\mu\lambda} \partial_\lambda T + \theta_3 \Delta^{\mu\lambda} \partial_\lambda (\mu/T) + O(\partial^2)\,,$$

$$\mathcal{T}^{\mu\nu} = -\eta \sigma^{\mu\nu} + O(\partial^2)\,,$$

$$\mathcal{N} = n + \nu_1 \dot{T}/T + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda (\mu/T) + O(\partial^2)\,,$$

$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \gamma_2/T\, \Delta^{\mu\lambda} \partial_\lambda T + \gamma_3 \Delta^{\mu\lambda} \partial_\lambda (\mu/T) + O(\partial^2)\,,$$

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

Constitutive relations

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

Write down every possible term, with up to one derivative:

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

$$\begin{aligned}
\mathcal{E} &= \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,\\
\mathcal{P} &= p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,\\
\mathcal{Q}^\mu &= \theta_1 \dot{u}^\mu + \theta_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,\\
\mathcal{T}^{\mu\nu} &= -\eta\sigma^{\mu\nu} + O(\partial^2)\,,\\
\mathcal{N} &= n + \nu_1 \dot{T}/T + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,\\
\mathcal{J}^\mu &= \gamma_1 \dot{u}^\mu + \gamma_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,
\end{aligned}$$

Blue = perfect fluid, Red = one-derivative “transport coefficients”

Connection to the Landau frame

Redefine hydro fields by O(∂) corrections:

$$u_L^\mu \equiv u^\mu + \delta u^\mu\,, \quad T_L \equiv T + \delta T\,, \;\; \mu_L \equiv \mu + \delta\mu$$

Constitutive relations:

$$\left.\begin{aligned} \mathscr{E} &= \epsilon(T,\mu) + f_{\mathscr{E}}(\partial T, \partial\mu, \partial u) \\ \mathscr{N} &= n(T,\mu) + f_{\mathscr{N}}(\partial T, \partial\mu, \partial u) \\ \mathscr{Q}^\mu &= \mathscr{Q}^\mu(\partial T, \partial\mu, \partial u) \end{aligned}\right\}$$

Landau frame: choose $\delta T,\ \delta u,\ \delta\mu$ such that

$$\mathscr{E}_L = \epsilon(T_L, \mu_L)\,,$$
$$\mathscr{N}_L = n(T_L, \mu_L)\,,$$
$$\mathscr{Q}_L^\mu = 0$$

Connection to the Landau frame

Redefine hydro fields by O(∂) corrections:

$$u_L^\mu \equiv u^\mu + \delta u^\mu\,, \quad T_L \equiv T + \delta T\,, \;\; \mu_L \equiv \mu + \delta\mu$$

Landau frame amounts to choosing:

$$\delta u^\mu = \frac{\mathcal{Q}^\mu}{\epsilon + p}\,, \qquad \delta T = \frac{f_{\mathcal{E}}\,\frac{\partial n}{\partial \mu} - f_{\mathcal{N}}\,\frac{\partial \epsilon}{\partial \mu}}{\frac{\partial \epsilon}{\partial T}\frac{\partial n}{\partial \mu} - \frac{\partial \epsilon}{\partial \mu}\frac{\partial n}{\partial T}}\,, \qquad \delta\mu = \frac{-f_{\mathcal{E}}\,\frac{\partial n}{\partial T} + f_{\mathcal{N}}\,\frac{\partial \epsilon}{\partial T}}{\frac{\partial \epsilon}{\partial T}\frac{\partial n}{\partial \mu} - \frac{\partial \epsilon}{\partial \mu}\frac{\partial n}{\partial T}}$$

Connection to the Landau frame

$$u^\mu = u^\mu_L - \frac{1}{\epsilon + p}\left(\theta_1 \dot{u}^\mu + \theta_2 \frac{\Delta^{\mu\alpha}\partial_\alpha T}{T}\right) + O(\partial^2)$$

$$T = T_L - \frac{1}{\partial\epsilon/\partial T}\left(\varepsilon_1 \dot{T}/T + \varepsilon_2\, \partial\cdot u\right) + O(\partial^2)$$

Can loosely interpret θ_1, ε_1 as relaxation times to Landau-frame variables.

Connection to the Landau frame

Conversely, if you happen to know u_L^μ, T_L from the exact $T^{\mu\nu}$ for uncharged fluids, then you find u^μ, T by:

$$u^\mu = u_L^\mu - \frac{1}{\epsilon + p}\left(\theta_1 \dot{u}_L^\mu + \theta_2 \frac{\Delta_L^{\mu\alpha}\partial_\alpha T_L}{T_L}\right) + O(\partial^2)$$

$$T = T_L - \frac{1}{\partial\epsilon/\partial T}\left(\varepsilon_1 \dot{T}_L/T_L + \varepsilon_2\, \partial\cdot u_L\right) + O(\partial^2)$$

Landau frame

Use field redefinitions and on-shell relations to push all red terms except π_2, η, γ_3 to $O(\partial^2)$.

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = \epsilon + \cancel{\varepsilon_1}\dot{T}/T + \cancel{\varepsilon_2}\partial_\lambda u^\lambda + \cancel{\varepsilon_3} u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{P} = p + \cancel{\pi_1}\dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \cancel{\pi_3} u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{Q}^\mu = \cancel{\theta_1}\dot{u}^\mu + \cancel{\theta_2}/T\,\Delta^{\mu\lambda}\partial_\lambda T + \cancel{\theta_3}\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + O(\partial^2)\,,$$

$$\mathcal{N} = n + \cancel{\nu_1}\dot{T}/T + \cancel{\nu_2}\partial_\lambda u^\lambda + \cancel{\nu_3} u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{J}^\mu = \cancel{\gamma_1}\dot{u}^\mu + \cancel{\gamma_2}/T\,\Delta^{\mu\lambda}\partial_\lambda T + \gamma_3\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

Eckart frame

Use field redefinitions and on-shell relations to push all red terms except π_2, η, $\theta_1=\theta_2$ to $O(\partial^2)$.

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = \epsilon + \cancel{\varepsilon_1}\dot{T}/T + \cancel{\varepsilon_2}\partial_\lambda u^\lambda + \cancel{\varepsilon_3} u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{P} = p + \cancel{\pi_1}\dot{T}/T + \pi_2\partial_\lambda u^\lambda + \cancel{\pi_3} u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \theta_2/T\,\Delta^{\mu\lambda}\partial_\lambda T + \cancel{\theta_3}\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + O(\partial^2)\,,$$

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$$\mathcal{J}^\mu = \cancel{\gamma_1}\dot{u}^\mu + \cancel{\gamma_2}/T\,\Delta^{\mu\lambda}\partial_\lambda T + \cancel{\gamma_3}\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

The fact that you *can* push most $O(\partial^1)$ terms to $O(\partial^2)$ doesn't mean that you *have to*. Let us keep all of them for now i.e. use the "<u>most general frame</u>".

Does it matter?

The choice of frame is not important from the point of view of the derivative expansion, or for classifying the transport coefficients. However, it is important from the point of view of the hydro equations themselves.

After all, the hydrodynamic equations (with the constitutive relations truncated at one-derivative order), when written in different frames, give rise to *different* differential equations.

The choice of frame may potentially affect such things as the well-posedness of the initial value problem for these partial differential equations, or lead to fictitious instabilities of the equilibrium state.

Now let's talk about the constraints on the 1-derivative transport coefficients

Constraints: extensivity in equilibrium

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

$$\mathcal{P} = p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \theta_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + O(\partial^2)\,,$$

$$\mathcal{N} = n + \nu_1 \dot{T}/T + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \gamma_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

Extensivity at $O(\partial^0)$: $\epsilon = -p + T\partial p/\partial T + \mu\,\partial p/\partial\mu, \;\; n = \partial p/\partial\mu$

Extensivity at $O(\partial^1)$: $\theta_1 = \theta_2, \;\; \gamma_1 = \gamma_2$

Constraints: positive viscosity and conductivity

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{P} = p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \theta_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + O(\partial^2)\,,$$

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$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \gamma_2/T\, \Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + O(\partial^2)\,,$$

$$\dot{T} \equiv u^\lambda \partial_\lambda T$$

$$\dot{u}^\mu \equiv u^\lambda \partial_\lambda u^\mu$$

Genuine O(∂) transport coefficients:

- Shear viscosity: η
- Bulk viscosity: combination of $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\nu_{1,2,3}$
- Charge/heat conductivity: combination of $\gamma_{1,3}$, $\theta_{1,3}$

Now comes the most important slide

Constraints: stability and causality

In the space of $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\theta_{1,2,3}$, $\nu_{1,2,3}$, $\gamma_{1,2,3}$, η there is a sub-space where hydro is stable and causal. It is necessary to keep ε_1, π_1, θ_1, ν_1, γ_1 non-zero, positive, and bounded from below.

PK, arXiv:1907.08191
Bemfica, Disconzi, Noronha, arXiv:1907.12695

Minimal stable and causal uncharged hydro

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}$$

$$\mathcal{E} = \epsilon + \varepsilon_1 \dot{T}/T + O(\partial^2)$$

$$\mathcal{P} = p + \pi_1 \dot{T}/T + \left(-\zeta + v_s^2(\pi_1 - v_s^2 \varepsilon_1)\right) \partial \cdot u + O(\partial^2)$$

$$\mathcal{Q}^\mu = \theta \left(\dot{u}^\mu + \frac{1}{T} \Delta^{\mu\lambda} \partial_\lambda T \right) + O(\partial^2)$$

$$\mathcal{T}^{\mu\nu} = -\eta \sigma^{\mu\nu} + O(\partial^2)$$

Three parameters $\varepsilon_1(T), \pi_1(T), \theta(T)$ besides $\eta(T), \zeta(T)$

Linear perturbations, uniform moving fluid

Stability: $$\mathrm{Im}\,\omega(\mathbf{k}) \leqslant 0$$

Causality: $$\lim_{k\to\infty} \frac{\mathrm{Re}\,\omega(\mathbf{k})}{k} < 1$$

Lorentz covariance gives no simple relations between $\omega(\mathbf{k})$ at $\mathbf{v}=0$ and $\omega'(\mathbf{k}')$ at $\mathbf{v}\neq 0$ unless $\omega(\mathbf{k})$ is linear.

If causality is not satisfied for the fluid at rest, then the uniformly moving fluid will have unstable modes.

Example: shear waves in a moving fluid, small **k**

$$\omega(\mathbf{k}) = \mathbf{k}\cdot\mathbf{v} - \frac{i\eta}{\epsilon + p}\sqrt{1-\mathbf{v}^2}\,\left(\mathbf{k}^2 - \left(\mathbf{k}\cdot\mathbf{v}\right)^2\right) + O(\mathbf{k}^3)$$

gapless, stable

$$\omega(\mathbf{k}) = \frac{i(\epsilon+p)\sqrt{1-\mathbf{v}^2}}{\eta\mathbf{v}^2 - \theta} + O(\mathbf{k}\cdot\mathbf{v})$$

$\theta \equiv \theta_1 = \theta_2$

gapped, stable for θ>η only!

E.g. the Landau-Lifshitz frame sets θ=0, predicts instability

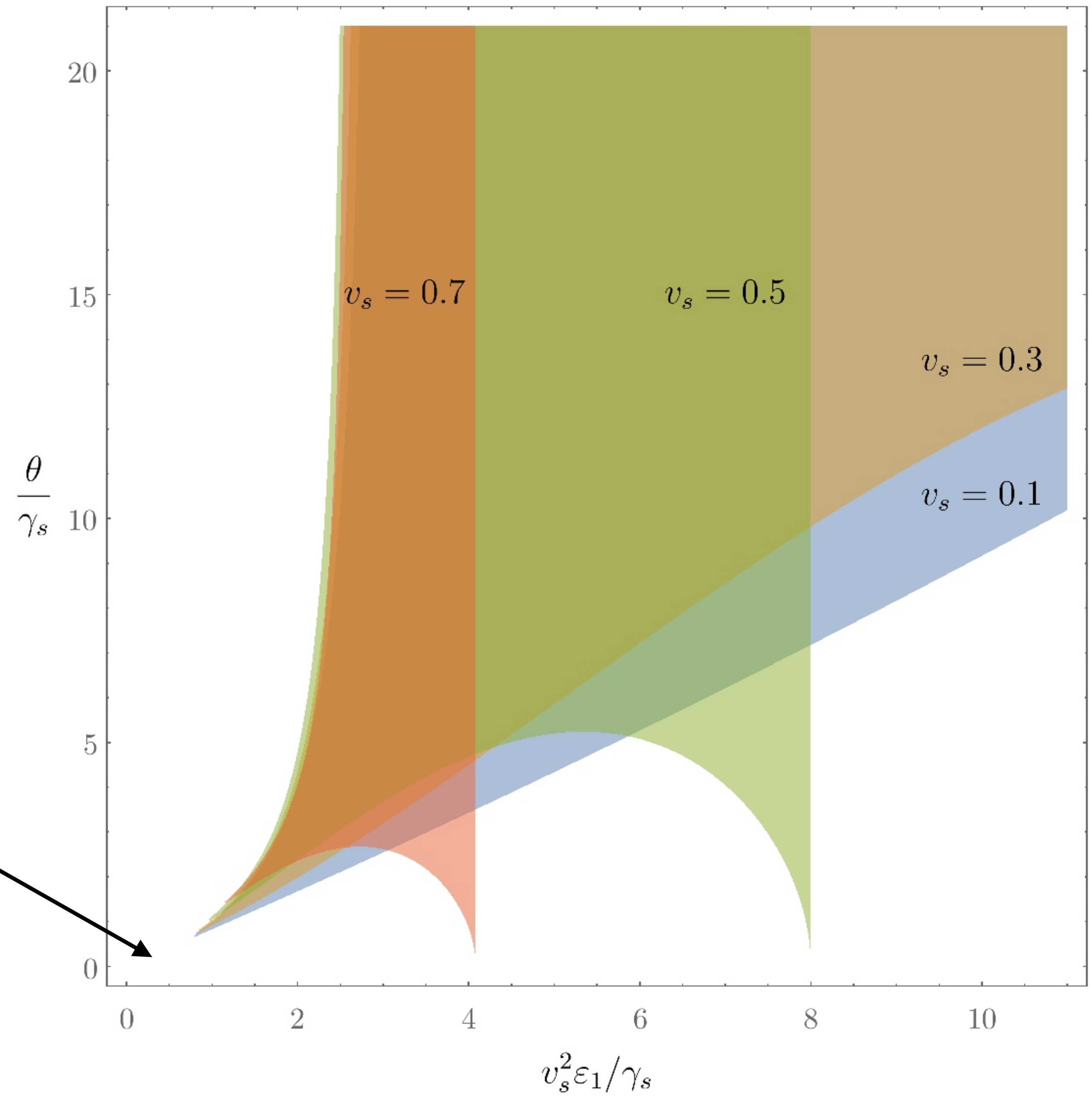
Example: shear waves in a static fluid, large **k**

$$\omega(\mathbf{k}) = \pm\,(\eta/\theta)^{1/2}\,|\,\mathbf{k}\,|$$

causal for θ>η only!

E.g. the Landau-Lifshitz frame sets θ=0, predicts acausality

Stable and causal frames for uncharged fluids



origin is excluded

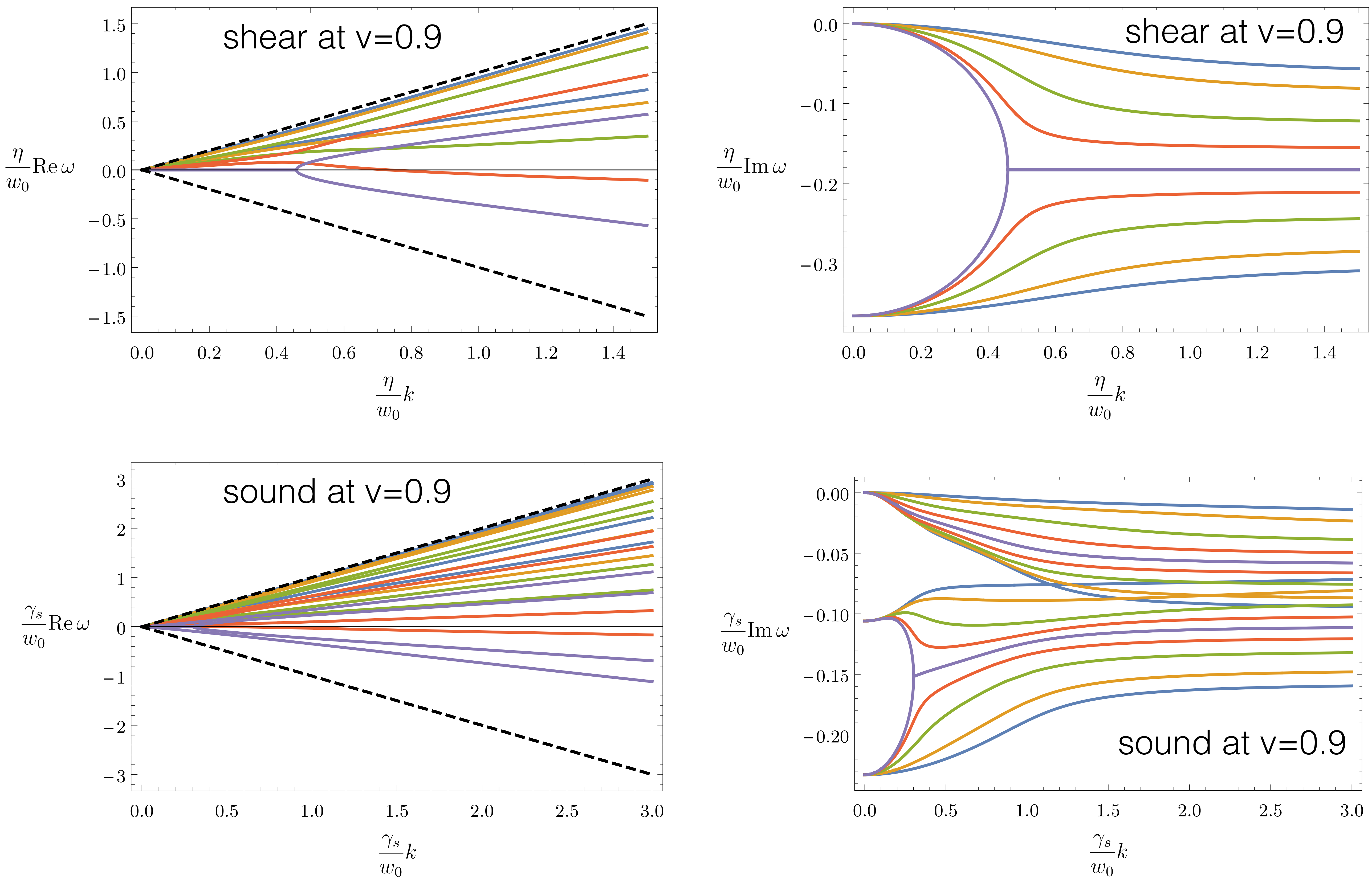
Plot for

$\pi_1/\gamma_s = 3/v_s^2$

$$\gamma_s \equiv \frac{4}{3}\eta + \zeta$$

PK, arXiv:1907.08191

Typical plots of ω(k) in the stable region



Entropy current

$$TS^{\mu}_{\text{canon}} = pu^{\mu} - T^{\mu\nu}u_{\nu}\,, \qquad \partial_{\mu}S^{\mu}_{\text{canon}} \overset{\text{on-shell}}{=} -\,T^{\mu\nu}_{(1)}\,\partial_{\mu}\left(\frac{u_{\nu}}{T}\right)$$

one-derivative contribution

$$T\partial_{\mu}S^{\mu}_{\text{canon}} \overset{\text{on-shell}}{=} -\,\varepsilon_1\left(\frac{\dot{T}}{T}\right)^2 - \pi_2\left(\partial_{\lambda}u^{\lambda}\right)^2 - \theta_2\frac{(\Delta^{\mu\alpha}\partial_{\alpha}T)(\Delta_{\mu\beta}\partial^{\beta}T)}{T^2} - \theta_1\dot{u}^{\mu}\dot{u}_{\mu} + \frac{\eta}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$$

$$-\,(\varepsilon_2+\pi_1)\frac{\dot{T}}{T}\partial_{\lambda}u^{\lambda} - (\theta_1+\theta_2)\frac{\dot{u}^{\mu}\Delta_{\mu\nu}\partial^{\nu}T}{T} + O(\partial^3)$$

Positive only if

$$\eta > 0\,,\quad \theta_1 = \theta_2 \leqslant 0\,,\quad \varepsilon_1 \leqslant 0\,,\quad \pi_2 \leqslant 0\,,\quad 4\varepsilon_1\pi_2 - (\varepsilon_2+\pi_1)^2 \geqslant 0$$

However recall that we are only computing $\partial_\mu S^\mu$ on-shell, and up to $O(\partial^2)$ only. On-shell we have:

$$\frac{\dot{T}}{T} = -v_s^2\, \partial_\lambda u^\lambda + O(\partial^2)\,, \qquad \frac{\Delta_{\mu\lambda}\partial^\lambda T}{T} = -\dot{u}_\mu + O(\partial^2)\,,$$

Entropy current

$$TS^{\mu}_{\text{canon}} = pu^{\mu} - T^{\mu\nu}u_{\nu}\,, \qquad \partial_{\mu}S^{\mu}_{\text{canon}} \overset{\text{on-shell}}{=} -\underbrace{T^{\mu\nu}_{(1)}}_{\text{one-derivative contribution}}\partial_{\mu}\left(\frac{u_{\nu}}{T}\right)$$

$$T\partial_{\mu}S^{\mu}_{\text{canon}} \overset{\text{on-shell}}{=} \underbrace{\left(-\pi_2 + v_s^2(\varepsilon_2+\pi_1) - v_s^4\varepsilon_1\right)}_{\text{this combination is }\zeta}\left(\partial_{\lambda}u^{\lambda}\right)^2 + \frac{\eta}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} + O(\partial^3)$$

Positive only if $\eta > 0\,,\ \ \zeta > 0$ 🙂

It had to be like this b/c on-shell and up to O(∂²) the theory is just the standard first-order hydro

Conclusions

Do what we've always been taught to do in field theory: write down every term allowed by the symmetry, then you will find a 1-st order relativistic hydro that is stable and causal, and only uses the same variables as the non-relativistic Navier-Stokes equations.

I only talked about linear stability and causality. One can show that the *non-linear* hydro equations in the general frame are causal, well-posed, and can be coupled to Einstein's equations

Bemfica, Disconzi, Noronha, arXiv:1907.12695

What's next for the stable-frame hydro?

Viable numerical schemes?

Heavy-ion applications?

How does it compare with the Israel-Stewart hydro?

Compare the new hydro to AdS/CFT evolution of $T^{\mu\nu}$?

What happens at $O(\partial^2)$? See also David's talk yesterday.

Thank you!