Asymptotics and the gradient expansion

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Universality near equilibrium

Consider the expectation value of the energy momentum tensor in some microscopic theory. Close to equilibrium we will find

$$\langle \hat{T}^{\mu\nu} \rangle = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E})(g^{\mu\nu} + u^\mu u^\nu) + \eta \sigma^{\mu\nu} + \ldots$$

What do we expect?

- Option 1: for gradients of fixed magnitude, adding more terms will give an increasingly more accurate answer
- Option 2: for a fixed number of terms, the answer will become more accurate as the magnitude of the gradients diminishes

The second possibility means that the series is asymptotic but not necessarily convergent.

At late times: universal asymptotic behaviour across many theories.

Some quantities such as dispersion relations in the linearised theory are represented by series with a finite radius of convergence [Withers 1803.18058; Grozdanov, Kovtun, Starinets, Tadic1904.01018, 1904.12862].

Divergent examples: late proper-time expansion of Bjorken flow in

- N=4 SYM via AdS/CFT
- Kinetic Theory (RTA)

The divergence of the gradient expansion:

- expresses the fact that subdominant contributions had been dropped
- explains why hydrodynamics works so well: “divergent series converge faster than convergent series” (G. Carrier)
- is connected with non-hydrodynamic modes in the microscopic theory (fast processes) so it should be seen as generic.

Relativistic Hydrodynamics

Conservation equation:

$$\nabla_\alpha T^{\alpha\beta} = 0$$

Constitutive relations as a gradient expansion

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu + \mathscr{P}(\mathscr{E})(g^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

The goal of hydrodynamics is match the (asymptotic) gradient expansion of any microscopic theory:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \dots$$

Why should the gradient expansion by divergent in hydrodynamics?

Causality and regulators

Known ways to do avoid acausal behaviour of Navier-Stokes theory:

- Mueller; Israel, Stewart (2nd order causal hydro)
- Bemfica, Disconzi, Noronha; Kovtun (1st order causal hydro)

Both these approaches introduce non-hydrodynamic modes which appear as a UV-regulator needed to maintain causality.

E.g. MIS (sound channel)

The relaxation time is the regulator parameter

$$\omega_{\mathrm{NH}} = -i\left(\frac{1}{\tau_\pi} - \frac{4}{3T}\frac{\eta}{s}k^2\right) + \dots$$

Group velocity

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\frac{\eta/s}{T\tau_\pi}} < 1 \qquad \Longleftrightarrow \qquad T\tau_\pi > 2\eta/s$$

Two options:

- use hydrodynamics in regimes where it is independent of the regulator
- match the non-hydrodynamic sector to some microscopic theory

Testing for regulator independence:

- Compare results between different MIS variants and BDNK
- Look at sensitivity to 2nd order TCs [Habich et al. 1512.05354]
- Check separation of hydro and non-hydro modes [MS 1607.06381]

Modeling the non-hydrodynamic sector:

- Leading QNM of N=4 SYM [Heller, Janik, MS, Witaszczyk 1409.5087]
- Quasihydrodynamics [Grozdanov, Lucas, Poovutticul 1810.10016]

Asymptotics of Bjorken flow in MIS

The equations of MIS hydrodynamics imply a first order ODE which determines the pressure anisotropy

$$C_{\tau_\pi}\left(1+\frac{\mathcal{A}}{12}\right)\mathcal{A}' + \left(\frac{C_{\tau_\pi}}{3w}+\frac{C_{\lambda_1}}{8C_\eta}\right)\mathcal{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w}-\mathcal{A}\right)$$

where

$$\mathcal{A} \equiv \frac{\mathcal{P}_T-\mathcal{P}_L}{\mathcal{P}}, \quad w \equiv \tau T$$

Asymptotic late-time solution (the gradient expansion):

$$\mathcal{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta(C_{\tau_\pi}-C_{\lambda_1})}{3w^2}}_{\text{2nd order}} + \ldots$$

Universal - no dependence on initial conditions.

Exponential corrections imply a transseries structure

$$\mathscr{A} = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\Phi_0(w)} + \sigma\, e^{-\frac{3}{2C_{\tau_\pi}} w} \underbrace{\left(w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau_\pi}}} \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n} \right)}_{\Phi_1(w)} + \dots$$

The form is determined by the non-hydrodynamic sector

$$\mathscr{A} = \sum_{n=0}^{\infty} \sigma^n\, e^{in\,\Omega w}\, \Phi_n(w), \qquad \Omega = i\frac{3}{2C_{\tau_\pi}} = -\,i\frac{3}{2}\mathrm{Im}(\omega)$$

- The hydro sector is universal: no memory of initial conditions
- The transseries parameter contains the initial data
- The transseries describes the dissipation of initial state information
- Resurgence: all coefficients can be recovered from the hydro ones!

The energy density of N=4 SYM as a transseries

$$\mathcal{E}\left(u,\boldsymbol{\sigma}\right)=\sum_{\boldsymbol{n}\in\mathbb{N}_0^{\infty}}\boldsymbol{\sigma}^{\boldsymbol{n}}\,\mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}\,u}\,\Phi_{\boldsymbol{n}}\left(u\right)\,,\quad u\equiv\tau^{2/3}$$

Vector of QNM frequencies:

$$\boldsymbol{A}=\left(A_1,\overline{A_1},A_2,\overline{A_2},\cdots\right)$$

Sectors labelled by

$$\boldsymbol{n}=\left(n_1,n_{\bar{1}},n_2,n_{\bar{2}},\cdots\right)=\sum_i n_i\boldsymbol{e}_i+\sum_i n_{\bar{i}}\overline{\boldsymbol{e}}_i$$

Transseries sectors

$$\Phi_{\boldsymbol{n}}\left(u\right)=u^{-\beta_{\boldsymbol{n}}}\sum_{k=0}^{+\infty}\varepsilon_k^{(\boldsymbol{n})}\,u^{-k}$$

Transseries parameters (integration constants):

$$\boldsymbol{\sigma}^{\boldsymbol{n}}\equiv\sigma_1^{n_1}\,\sigma_{\bar{1}}^{n_{\bar{1}}}\,\sigma_2^{n_2}\,\sigma_{\bar{2}}^{n_{\bar{2}}}\,\cdots$$

[Aniceto, Jankowski, Meiring, MS - 1810.07130]

- Hydro sector: $\boldsymbol{n} = \mathbf{0}$ (380 coeffs)
- Fundamental sectors corresponding to individual QNMs

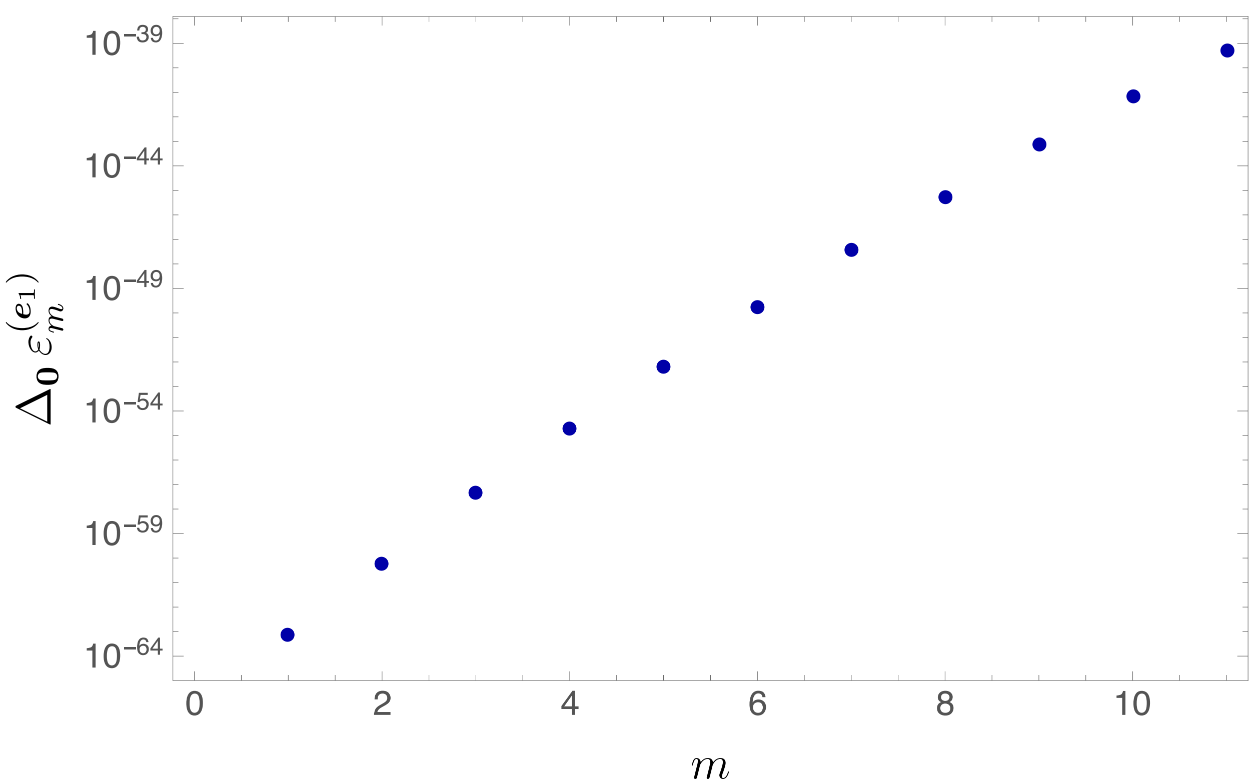
$$\boldsymbol{n} = \boldsymbol{e}_1 \text{ (250 coeffs)}$$

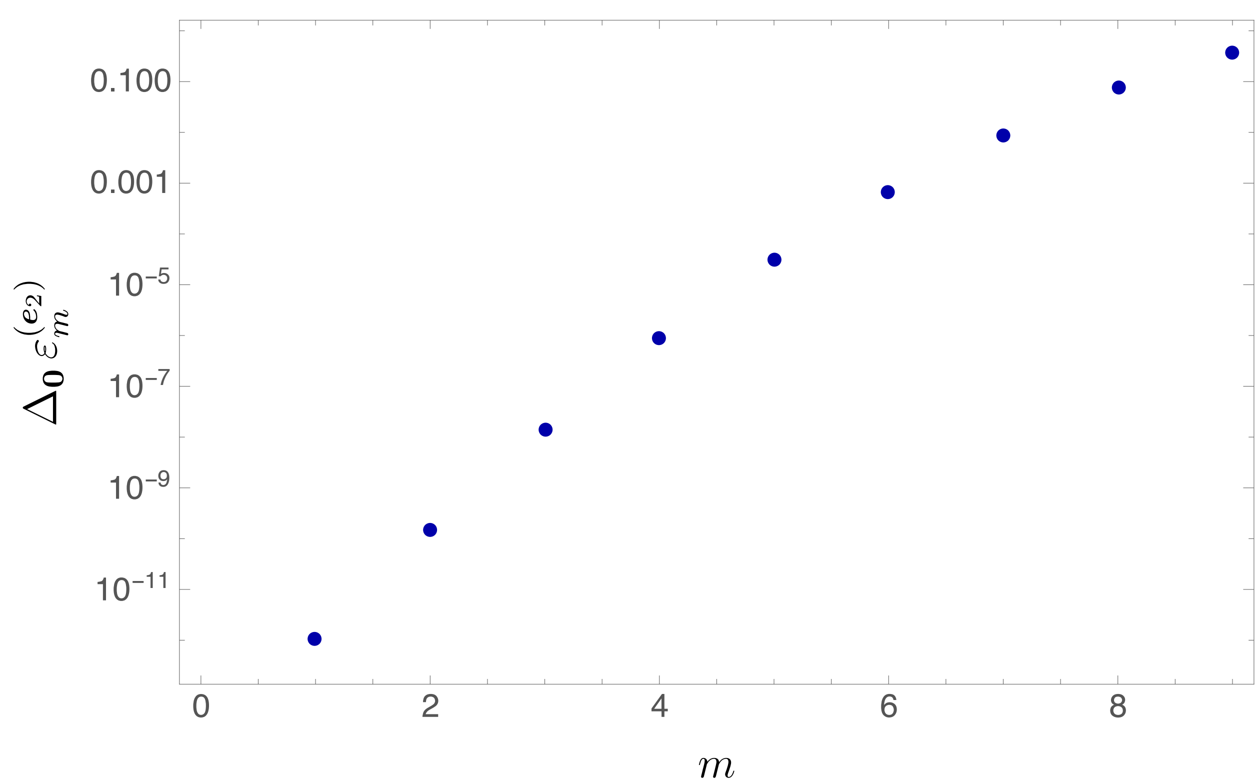
$$\boldsymbol{n} = \boldsymbol{e}_2 \text{ (200 coeffs)}$$

- Mixed sectors corresponding to QNM coupling

$$\boldsymbol{n} = 2\boldsymbol{e}_1 \text{ (100 coeffs)}$$

$$\boldsymbol{n} = \boldsymbol{e}_1 + \bar{\boldsymbol{e}}_1 \text{ (100 coeffs)}$$





Resurgence: all you need is the (hydrodynamic) gradient expansion.

Late times linearised MIS

The shear channel dispersion relation is (+ is the hydro mode)

$$\omega_\pm = \frac{1}{2\tau_R}\left(-i \pm \sqrt{4D\tau_R k^2 - 1}\right) , \quad D \equiv \frac{\eta}{Ts}$$

The Green's function of the corresponding linear problem solves

$$(\tau_R \partial_t^2 + \partial_t - D\partial_x^2)G(t,x) = \delta(t)\delta(x)$$

It can be calculated exactly and satisfies causality constraints.

$$G(t,x) = \theta(t - |x|)\Big(I_+(t,x) + I_-(t,x)\Big) \equiv \theta(t - |x|)\tilde{G}(t,x)$$

$$I_\pm(t,x) = \pm \frac{1}{2\pi}\int_0^\infty dk \frac{e^{i(kx - \omega_\pm t)}}{\omega_+ - \omega_-}$$

[Heller, Serantes, MS, Svensson, Withers, TBA]

Using standard asymptotic methods one finds

$$G_H \equiv \tilde{G}(t,0) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} \sum_{k=0}^{\infty} a_k t^{-k}$$

$$a_k = \frac{(-1)^k \Gamma(1/2 + k)}{2^{k+1} \Gamma(1/2 - k) \Gamma(1 + k)}$$

This series is factorially divergent:

$$\frac{a_{k+1}}{a_k} \sim \frac{k}{2t}$$

The Borel transform can be done analytically

$$\mathscr{B}[\sqrt{t} G_H](\xi) = \sum_{n \geq 0} \frac{a_n}{\Gamma[n+1]} \xi^n = \frac{1}{\sqrt{2}\pi^{3/2}} K(\xi/2) \equiv B(\xi)$$

The elliptic function K has a cut on the real axis; the location of the branch point of the Borel transform is set by the relaxation time.

Correspondingly, the Borel sum exhibits a complex ambiguity

$$\mathcal{S}_{\pm} G_H = \sqrt{t} \int_{C_\pm} d\xi \, e^{-w\xi} \, B(\xi) = \frac{1}{2} e^{-t} \left(I_0(t) \pm \frac{i}{\pi} K_0(t) \right)$$

Thus, the answer is given up to an exponentially damped contribution

$$\mathcal{S} G_H = \mathcal{S}_- G_H + \sigma \left(\mathcal{S}_+ G_H - \mathcal{S}_- G_H \right)$$

To get a real result the transseries parameter must be

$$\sigma = \frac{1}{2} + c, \quad c \in R$$

The non-hydrodynamic mode contribution cancels the ambiguity.

This matches the exact result for the Green function.

Summary

- Gradient expansions appear both in microscopic theories and in hydrodynamic models and are asymptotic (and often divergent)
- The divergence is connected with the presence of non-hydrodynamic (gapped) modes which act as a UV-regulator necessary for causality
- The realm of applicability of hydrodynamics can be understood as the region of regulator-independence
- It may be interesting and useful to formulate theories of hydrodynamics with specific non-hydrodynamic sectors