# The late time expansion in linearized relativistic hydrodynamics

Ongoing work with Michal Heller, Michal Spaliński, Viktor Svensson and Ben Withers

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# Motivation

- In the highly symmetric cases where explicit computations can be performed, it has been found that the hydrodynamic expansion is a divergent asymptotic series [recall Michal's talk]
- Nevertheless, this is not the case for the dispersion relation of the hydrodynamic modes [Saso's talk]
- We would like to understand how the analyticity properties of the dispersion relation imprint themselves on the hydrodynamic expansion in real space.
- As a first step do address this question, in this talk I am going to discuss the late-time expansion of linearized relativistic hydrodynamics.

# Benchmark model

We will we working with conformal Mueller-Israel-Steward hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0,$$
$$(\tau_\Pi \, u^\alpha \mathcal{D}_\alpha + 1)\pi^{\mu\nu} = -\eta \, \sigma^{\mu\nu},$$

where the stress-energy tensor is given by

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + P(\mathcal{E})\Delta^{\mu\nu} + \pi^{\mu\nu}.$$

We will consider a static fluid in thermal equilibrium at temperature $T$ placed in four-dimensional Minkowski spacetime, and analyze infinitesimal fluctuations around it:

- Shear channel: $u^\mu = (1,0,0,0) + \epsilon\delta u^1(t,x^3), \pi^{1,3} = \epsilon\delta\pi^{1,3}(t,x^3)$.
- Sound channel: $u^\mu = (1,0,0,0) + \epsilon\delta u^3(t,x^3)$, $\pi^{i,i} = \epsilon\delta\pi^{i,i}(t,x^3)$, $\mathcal{E} = \mathcal{E}_0 + \epsilon\delta\mathcal{E}(t,x^3)$,

# The shear channel

The shear channel e.o.m are

$$(\mathcal{E}_{(0)} + P(\mathcal{E}_{(0)}))\partial_t \delta u^1 + \partial_x \delta\pi^{1,3} = 0,$$
$$\partial_t \delta\pi^{1,3} + \frac{\eta}{\tau_\Pi}\partial_x \delta u^1 = -\frac{1}{\tau_\Pi}\delta\pi^{1,3},$$

They are equivalent to the quasidydrodynamical description of diffusion-to-sound crossover [Grozdanov,Lucas & Poovuttikul].

They can be subsumed into a single equation for $\rho = (\mathcal{E}_{(0)} + P(\mathcal{E}_{(0)}))\delta u^1$: telegrapher's equation.

After a suitable coordinate redefinition

$$(\partial_t^2 + 2\partial_t - \partial_x^2)\rho(t,x) = 0.$$

The dispersion relation is $\omega^2 + 2i\omega - k^2 = 0$. There are two solutions

- Hydrodynamic mode: $\omega_H = -i + i\sqrt{1-k^2}$.
- Non-hydrodynamic mode: $\omega_{NH} = -i - i\sqrt{1-k^2}$.

Propagating modes only exist for $k > k_g = 1$, k-gap (see [Baggioli, Brazhkin, Trachenko, Vasin] for a survey of this phenomenon across many areas of physics).

The shear channel is special, in the sense that we can obtain closed form results for some physically relevant quantities.

This is very useful, since it allows us to test the general techniques that we will develop against explicit analytic results.

# The shear channel

Our objective now is computing the Green's function of the telegrapher's equation. This can be done explicitly:

$$G(t,x) = \Theta(t)\int_{\mathbb{R}} \frac{dk}{4\pi\sqrt{1-k^2}}\left(e^{-i(\omega_H(k)t-kx)} - e^{-i(\omega_{NH}(k)t-kx)}\right) =$$
$$= \frac{1}{2}e^{-t} I_0(\sqrt{t^2-x^2})\Theta(t-|x|).$$

For fixed $x$, $t\to\infty$, we recover the Green's function of the heat equation,

$$G(t,x) \to \frac{1}{2}\frac{e^{-\frac{1}{2}\frac{x^2}{t}}}{\sqrt{2\pi t}}.$$

This regime is controlled by the small $k$ behavior of $\omega_H(k) = -i/2k^2 + ....$

In the near light cone limit, $\tau = \sqrt{t^2-x^2} \to 0$,

$$G(t.x) \to \frac{1}{2}e^{-t}$$

This agrees with the Green's function of a system supporting exponentially dampled, massless excitations with dispersion relation $\omega_\pm = -i \pm |k|$: this regime is controlled by the large $k$ behavior of the dispersion relation.

Let us consider separately the contribution of hydro./non-hydro. modes to the Green's function

$$G(t,x) = G_H(t,x) - G_{NH}(t,x),$$
$$G_{H,NH}(t,x) = \frac{e^{-t}}{2\pi}\int_0^\infty \frac{dk}{\sqrt{1-k^2}} e^{\pm\sqrt{1-k^2}t}\cos(kx).$$

Again, these quantities can be explicitly computed

$$G_H = \frac{1}{2}e^{-t}I_0(\tau) - i\frac{e^{-t}K_0(\tau)}{2\pi},$$
$$G_{NH} = -i\frac{e^{-t}K_0(\tau)}{2\pi}.$$

Note that $G_H = \frac{1}{2}e^{-t}I_0(\tau) + G_{NH}$. This will be important later on.

# A general expression for the coefficients of the late-time expansion

We are working with integrals of the form

$$I(t) = \int_0^{k_{max}} dk f(k) e^{\varphi(k)t}.$$

We would like to have an algorithm which, given a general dispersion relation, allows us to obtain the coefficients of the late-time asymptotic expansion.

Such an algorithm exists.

Assume:

- $\varphi'(k=0) = 0$, $\varphi(k=0) = 0$.
- $\varphi(k)$ is monotonically decreasing.
- $\varphi(k)$ is even.

Then, $I(t)$ has a late-time expansion given by [Berry & Howls]

$$I \sim \sum_{r=0}^{\infty} \frac{\Gamma\left(r+\frac{1}{2}\right)}{t^{r+\frac{1}{2}}} c_r, \quad c_r = \frac{1}{4\pi i} \oint_{\gamma(0)} dk f(k) \left(\frac{k^2}{-\varphi(k)}\right)^{r+\frac{1}{2}} \frac{1}{k^{2r+1}}.$$

To compute the late-time expansion of $G_H(t, x=0)$, we take

$$\varphi(k) = \mathrm{Im}\ \omega_H(k) = -1 + \sqrt{1-k^2}, \quad f(k) = \frac{1}{2\pi\sqrt{1-k^2}}.$$

The integrand has two branch cuts on the real axis extending from $(-\infty, -1] \cup [1, \infty)$. The integration contour $\gamma(0)$ can be blown-up to go around both; there is no contribution from infinity.

Computing the cut integrals explicitly, we get

$$c_r = \frac{\Gamma\left(\frac{1}{2}+r\right)}{2^{r+\frac{3}{2}}\pi^{\frac{3}{2}}\Gamma(1+r)}.$$

This agrees with the late-time expansion of the known closed form result.

Since $c_r$ is not factorially suppressed, the late-time expansion is a divergent series.

# Borel resummation in the shear channel

Given the divergent, asymptotic series

$$f(t) \sim \sum_{r=0}^{\infty} \frac{d_r}{t^r},$$

the Borel transform

$$f_B(z) = \sum_{r=0}^{\infty} \frac{d_r}{r!} z^r.$$

returns a series with finite convergence radius.

The Laplace transform of $f_B(z)$ gives us a function $\mathcal{S}(t)$ analytic in some region in the complex $t$-plane whose late-time expansion agrees with the original asymptotic series

$$\mathcal{S}(t) = t \int_0^{\infty} dz e^{-tz} f_B(z).$$

In our case, the Borel transform of $\sqrt{t}G(t,0)$ can be obtained in closed form

$$f_B(z) = \sum_{r=0}^{\infty} \frac{c_r \Gamma\left(r+\frac{1}{2}\right)}{\Gamma(r+1)} z^r = \frac{1}{\sqrt{2}\pi^{\frac{3}{2}}} K\left(\frac{z}{2}\right).$$

$f_B$ has branch cut starting at $z = 2$ and running along the positive real axis: the asymptotic series is not Borel summable.

Consider lateral Borel resummations,

$$\mathcal{S}_\theta(t) = \sqrt{t} \int_0^{e^{i\theta}\infty} dz e^{-tz} f_B(z).$$

In particular, focus on $\mathcal{S}_\pm \equiv \mathcal{S}_{0^\pm}$. They can be explicitly computed

$$\mathcal{S}_\pm(t) = \frac{1}{2} e^{-t} I_0(t) \pm \frac{i e^{-t} K_0(t)}{2\pi}.$$

There is an imaginary ambiguity that depends on the choice of integration contour.

When promoting the original asymptotic series to a transseries, which takes into account the contribution of the non-hydrodynamical sector, it is possible to choose the relative weight between the two parts in such a way that the lateral Borel resummation of the whole transseries agrees with the true answer.

# The sound channel

After the appropriate rescalings of $t$, $x$ the sound channel dispersion relation can be put in the form

$$P(\omega, k) = \omega^3 + i\omega^2 - (1+\gamma)\omega k^2 - ik^2 = 0, \quad \gamma = \frac{4\eta/s}{T\tau_\Pi}.$$

Having a group velocity such that $\lim_{k\to\infty} |v_g(k)| < 1$ demands that $\gamma < 2$.

There are three modes, two hydrodynamical

$$\omega_+(k) = k - \frac{i\gamma}{2}k^2 + \frac{(4-\gamma)\gamma}{8}k^3 + \dots,$$
$$\omega_-(k) = -k - \frac{i\gamma}{2}k^2 - \frac{(4-\gamma)\gamma}{8}k^3 + \dots,$$

and one non-hydrodynamical, which is purely imaginary

$$\omega_{NH}(k) = -i(1 - \gamma k^2 + (1-\gamma)\gamma k^4 + \dots).$$

In practice, these dispersion relations are determined numerically, by plugging $\omega(k) = \sum_{n=0}^{\infty} a_n k^n$ into $P(\omega(k), k) = 0$ and solving recursively order by order in a small $k$ expansion.

We will be considering integrals of the form

$$I_+(t,u) = \int_{\mathbb{R}} dk f(k) e^{\varphi_+(k)t}, \quad \varphi_+(k) = -i(\omega_+(k) - uk)t, \quad u = \frac{x}{t}$$

A particularly interesting case is to take $u = 1$ as $t \to \infty$ (this corresponds to a inertial observer moving at the speed of sound). For this choice of $u$, $I_+$ has a saddle point at $k = 0$, at which $\phi_+$ vanishes.

To procede further, we deform the original integration contour into a steepest descend path $k = k_{sd}(\xi)$,

$$k_{sd}(\xi) = \xi + ik_I(\xi),$$

along which $\mathrm{Im}\ \varphi_+(k_{sd}(\xi)) = \mathrm{Im}\ \varphi_+(k=0) = 0$.

Since the dispersion relation is only known as a series expansion, $k_I(\xi)$ has to be determined in the same way. The end result is

$$k_{sd}(\xi) = \xi + i\left(-\frac{4-\gamma}{8}\xi^2 - \frac{1}{512}(64 + 80\gamma - 4\gamma^2 - \gamma^3)\xi^4 + \dots\right),$$

$$\varphi_+(k_{sd}(\xi)) = -\frac{\gamma}{2}\xi^2 - \frac{\gamma(16 + 24\gamma + 5\gamma^2)}{128}\xi^4 + \dots$$

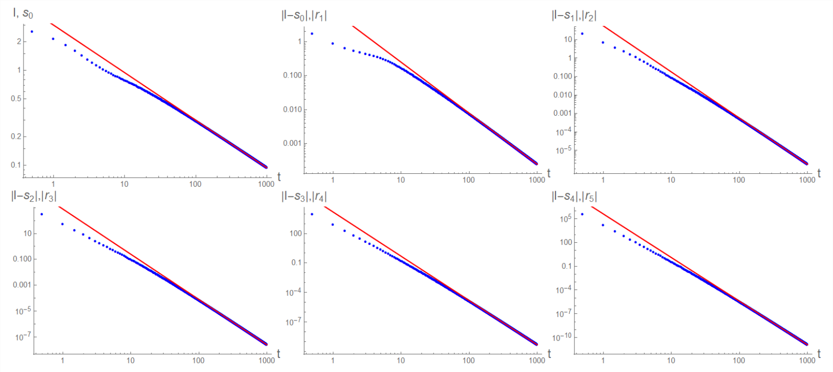
To arbitrary order, only even powers of $\xi$ appear in $\varphi_+(k_{sd}(\xi))$, always with a negative coefficient. As a consequence, the technical conditions spelled out before hold, and we can write

$$I_+ \sim \sum_{r=0}^{\infty} \frac{\Gamma\left(r+\frac{1}{2}\right)}{t^{r+\frac{1}{2}}} c_r, \quad c_r = \mathrm{Res}_{\xi=0}\left[\tilde{f}(\xi)\left(-\frac{\xi^2}{\phi_+(k_{sd}(\xi))}\right)^{r+\frac{1}{2}} \frac{1}{\xi^{2r+1}}\right],$$

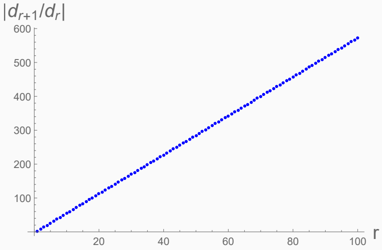
where $\tilde{f}$ is the even part of $k'_{sd}(\xi) f(k_{sd}(\xi))$.

Example: $\gamma = 0.7$, $f(k) = (1 + k^2)^{-2}$. We compute $I_+(t, 1)$ numerically, and compare the result against the late-time expansion.

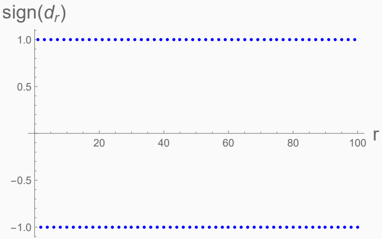
$$s_N = \sum_{p=0}^{N} \frac{\Gamma(p+1/2)}{t^{p+1/2}} c_p, \quad r_{N+1} = s_{N+1} - s_N,$$



What about the large order behavior? The series is factorially divergent:




However, it is also alternating, so it is Borel resummable:



Let us focus on the Green's function now. We have that

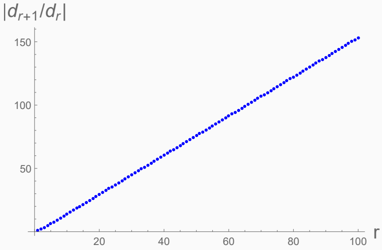
$$G(t,x) = \int_{\mathbb{R}} dk G(t,k) e^{ikx} \equiv I_+ + I_- + I_{NH},$$

$$I_+ = \frac{1}{2\pi} \int_{\mathbb{R}} dk \frac{1}{(\omega_+(k) - \omega_-(k))(\omega_+(k) - \omega_{NH}(k))} e^{-i(\omega_+(k)t - kx)},$$
$$I_- = \frac{1}{2\pi} \int_{\mathbb{R}} dk \frac{1}{(\omega_-(k) - \omega_+(k))(\omega_-(k) - \omega_{NH}(k))} e^{-i(\omega_-(k)t - kx)},$$
$$I_{NH} = \frac{1}{2\pi} \int_{\mathbb{R}} dk \frac{1}{(\omega_{NH}(k) - \omega_+(k))(\omega_{NH}(k) - \omega_-(k))} e^{-i(\omega_{NH}(k)t - kx)}.$$

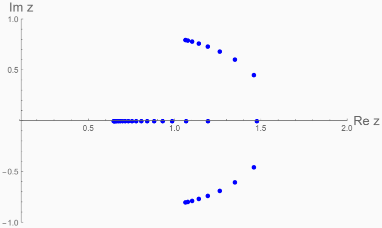
We are going to analyze the integral $I_{+,reg}$, defined as

$$I_{+,reg} = \frac{1}{2\pi} \int_{\mathbb{R}} dk \left( \frac{1}{(\omega_+(k) - \omega_-(k))(\omega_+(k) - \omega_{NH}(k))} - \frac{1}{2ik} \right) e^{-i(\omega_+(k)t - kx)}.$$

Our algorithm returns again a divergent late-time expansion, however, it is not alternating: not Borel resummable.




Singularity structure in complex Borel plane: three (discretized) branch cuts, one real and two complex conjugated.



Each branch cut corresponds to a particular imaginary ambiguity.

We expect that these ambiguities are cancelled out by non-perturbative sectors.

Can we find the corresponding non-perturbative saddle points? Yes!

- The real cut starts at $z_R \approx 0.6466$. There is a non-perturbative saddle point of $\varphi_+$ at $k = -0.775436i$, where $\phi_+ = -0.646093$
- The complex conjugated cuts start at $z_\pm \approx 1.0625 \pm 0.8000i$. There are two non-perturbative saddle points of $\varphi_{NH}$ at $k = \pm 0.612085 + 0.387718i$, where $\varphi_{NH} = -1.059306 \pm 0.801289i$.

Main novelty: viewing the dispersion relation as a function defined on a Riemann surface, these non-perturbative saddles might live on different sheets.

# Conclusions

- We have analyzed the late-time expansion of different physical quantities both in the shear and in the sound channels.
- We have found that these late-time expansions might be divergent asymptotic series.
- In some cases, we have found out that the singularity structure of the analytically continued Borel transform of the divergent series reveals the existence of the non-hydrodynamical saddles.

Many thanks for your attention!