Adams' inequality with the exact growth

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Theorem [Nader Masmoudi, F.S. 2017]

Let m be a positive integer with $2 < m < N$. Then

$$\int_{\mathbb{R}^N} \frac{\exp_{\lceil \frac{N}{m} - 2 \rceil}\left\{ \beta_{N,m} |u|^{\frac{N}{N-m}} \right\}}{(1 + |u|)^{\frac{N}{N-m}}} dx \leqslant C_{N,m} \|u\|_{\frac{N}{m}}^{\frac{N}{m}} \quad \forall u \in W^{m,\frac{N}{m}}(\mathbb{R}^N),\ \|\nabla^m u\|_{\frac{N}{m}} \leqslant 1\,.$$

The above inequality fails if the power $\frac{N}{N-m}$ in the denominator is replaced with any $p < \frac{N}{N-m}$.

Here

- $\exp_k(t) := e^t - \sum_{j=0}^{k} \frac{t^j}{j!}\,, \quad k \in \mathbb{N};$
- $\lceil x \rceil$ denotes the smallest integer grater than or equal to $x \in \mathbb{R}$;
- $\beta_{N,m}$ is the sharp exponent of Adams' inequality on bounded domains;
- $\nabla^m u := (-\Delta)^{\frac{m}{2}} u$ if m is even, and $\nabla^m u := \nabla(-\Delta)^{\frac{m-1}{2}} u$ if m is odd.

Idea: It is possible to reach a **limiting sharp higher order** inequality exploiting **refined limiting and non-limiting second order** inequalities.

Let us consider first order Sobolev spaces in the **limiting case** of Sobolev embeddings

$$(W_0^{1,N}(\Omega), \|\nabla \cdot\|_N)$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a bounded domain. In this framework,

$$TM(\alpha) := \sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leqslant 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}}\, dx, \quad \alpha > 0$$

- **Sobolev embeddings**:
  - $W_0^{1,N}(\Omega) \subset L^q(\Omega) \quad \forall q \geqslant 1$
  - but $W_0^{1,N}(\Omega) \not\subseteq L^\infty(\Omega)$

  In particular, for any $q \geqslant 1$, $\displaystyle\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leqslant 1} \int_\Omega |u|^q\, dx < +\infty$

- **S. I. Pohozaev** (1965) and **N. S. Trudinger** (1967):
  - If $\gamma > \frac{N}{N-1}$ then there exists $u \in W_0^{1,N}(\Omega)$ with $\|\nabla u\|_N \leqslant 1$ such that

    $$\int_\Omega e^{\alpha |u|^\gamma}\, dx = +\infty, \quad \alpha > 0$$
  - there exists $\alpha = \alpha(N) > 0$ small such that $TM(\alpha) < +\infty$

- **J. Moser** (1970): $\alpha_N := N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} surface measure of $S^{N-1} \subset \mathbb{R}^N$

Trudinger-Moser inequality

J. Moser found the **sharp exponent** and proved the following result

Theorem [Moser 1970]

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain. There exists a constant $C_N > 0$ such that

$$\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leqslant 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}}\, dx \begin{cases} \leqslant C_N |\Omega| & \forall \alpha \leqslant \alpha_N \\ = +\infty & \forall \alpha > \alpha_N \end{cases}$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface measure of $S^{N-1} \subset \mathbb{R}^N$.

Key ideas of the proof of the **critical** inequality

$$\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leqslant 1} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx \leqslant C_N |\Omega|$$

1. Reduction of the problem to the **radial case**
2. Moser's change of variable

   $+$

   Moser's one-dimensional Lemma

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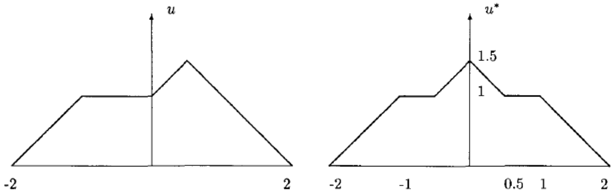
Reduction of the problem to the radial case

Key ingredient: **Schwarz symmetrization**



$$u : \Omega \to \mathbb{R} \rightsquigarrow u^* : (0, |\Omega|] \to [0, +\infty)$$

$$\rightsquigarrow u^\sharp : \Omega^\sharp \to [0, +\infty), \; u^\sharp(x) := u^*\left(\frac{\omega_{N-1}}{N}|x|^N\right)$$

If $u \in W_0^{1,N}(\Omega)$ then $u^\sharp \in W_0^{1,N}(\Omega^\sharp)$ and

- $\displaystyle\int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx = \int_{\Omega^\sharp} e^{\alpha_N [u^\sharp]^{\frac{N}{N-1}}}\, dx$
- (Pólya-Szegö inequality) $\|\nabla u^\sharp\|_N \leqslant \|\nabla u\|_N$

Moser's change of variable and one-dimensional Lemma

Let $\Omega \subset \mathbb{R}^N$ and let $R > 0$ be such that $|B_R| = |\Omega|$, i.e. $\Omega^\sharp := B_R$.

1. Given any $u \in W_0^{1,N}(\Omega)$ with $\|\nabla u\|_N \leqslant 1$, we have

$$\int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}}}\, dx = \int_{B_R} e^{\alpha_N [u^\sharp]^{\frac{N}{N-1}}}\, dx \quad \text{and} \quad \|\nabla u^\sharp\|_N \leqslant 1$$

2. Performing the change of variable $r = |x| = Re^{-\frac{t}{N}}$ and setting $w(t) := \alpha_N^{\frac{N-1}{N}}\, u^\sharp(r)$,

$$\int_{B_R} e^{\alpha_N [u^\sharp]^{\frac{N}{N-1}}}\, dx = |B_R| \int_0^{+\infty} e^{w^{\frac{N}{N-1}} - t}\, dt \quad \text{and} \quad \|\nabla u^\sharp\|_N^N = \int_0^{+\infty} [w']^N\, dt$$

One-dimensional Lemma [Moser 1970]

There exists $c_N > 0$ such that for any non-negative measurable function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

$$\int_0^{+\infty} \phi^N(t)\, dt \leqslant 1$$

the following inequality holds

$$\int_0^{+\infty} \exp\left\{ \left(\int_0^t \phi(s)\, ds \right)^{\frac{N}{N-1}} - t \right\} dt \leqslant c_N,$$

Trudinger-Moser inequality

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where $\alpha_N := N \omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface measure of $S^{N-1} \subset \mathbb{R}^N$.

Related results can be found in several papers:
Adachi, Adams, Adimurthi, Bahouri, Carleson, Chang, Cianchi, de Figueiredo, do Ó, Dolbeault, Druet, Esteban, Flucher, Fontana, Ibrahim, Ishiwata, Kozono, Lam, Li, Lin, Lu, Majdoub, Malchiodi, Martinazzi, Masmoudi, Morpurgo, Nakanishi, Ogawa, Ozawa, Ruf, Strichartz, Struwe, Tanaka, Tarantello, Tintarev, Yang, . . .

Remark:

- $J_{N,\alpha}(u) := \int_{\mathbb{R}^N} \exp_N(\alpha |u|^{\frac{N}{N-1}})\, dx$
- $\exp_N(t) := e^t - \sum_{k=0}^{N-2} \frac{t^j}{j!}$

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The problem on the whole space $\mathbb{R}^N$ with $N \geqslant 2$

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$$J_{N,\alpha}(u) \leqslant C_{\alpha,N} \|u\|_N^N \quad \forall u \in W_0^{1,N}(\mathbb{R}^N) \text{ with } \|\nabla u\|_N \leqslant 1. \qquad \text{(AT)}$$

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Trudinger-Moser inequality on $\mathbb{R}^N$ with the exact growth condition, $N \geqslant 3$

Let $F : \mathbb{R} \to [0, +\infty)$ be continuous and let us consider $u \mapsto J(u) := \int_{\mathbb{R}^N} F(u)dx$.

Boundedness [Ibrahim-Masmoudi-Nakanishi $N = 2$ and Masmoudi–S. $N \geqslant 3$]

The following conditions are equivalent:

1. $\lim_{|s|\to+\infty} \frac{|s|^{N/(N-1)}F(s)}{e^{\alpha_N |s|^{N/(N-1)}}} < +\infty$ and $\lim_{s\to 0} \frac{F(s)}{|s|^N} < +\infty$
2. There exists $C_{F,N} > 0$ such that

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Moreover

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2. For any sequence $\{u_k\}_k \subset W^{1,N}(\mathbb{R}^N)$ of radial functions satisfying $\|\nabla u_k\|_N \leqslant 1$ and weakly converging to some u in $W^{1,N}(\mathbb{R}^N)$, we have $J(u_k) \to J(u)$ as $k \to +\infty$.

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Theorem [Adams 1988]

Let m be an integer and let $\Omega \subset \mathbb{R}^N$ with $m < N$. There exists a constant $C_{m,N} > 0$ such that

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where $\overline{\beta} = \beta_{N,m}$ is explicitly known.

Here $\nabla^m u := (-\Delta)^{\frac{m}{2}} u$ if m is even, and $\nabla^m u := \nabla(-\Delta)^{\frac{m-1}{2}} u$ if m is odd.

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Adams' inequality: the particular case of $W_0^{2,2}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For Sobolev spaces of the form $W_0^{2,2}(\Omega)$, the Sobolev embedding theorem says that if $N > 4$ then

$$W_0^{2,2}(\Omega) \subset L^{\frac{2N}{N-4}}(\Omega)$$

and hence the limiting case is $\boxed{N = 4}$.

Theorem [Adams 1988]

Let $\Omega \subset \mathbb{R}^4$ be bounded. There exists a constant $C > 0$

$$\sup_{u \in W_0^{2,2}(\Omega),\, \|\Delta u\|_2 \leqslant 1} \int_\Omega e^{\alpha u^2}\, dx \begin{cases} \leqslant C|\Omega| & \forall \alpha \leqslant 32\pi^2 \,, \\ = +\infty & \forall \alpha > 32\pi^2 \,. \end{cases}$$

Main difficulty of the proof: how to reduce the problem to the radial case?

Problem: given $u \in W_0^{2,2}(\Omega)$, we do not know whether or not $u^\sharp \in W_0^{2,2}(\Omega^\sharp)$; even in the case $u^\sharp \in W_0^{2,2}(\Omega^\sharp)$, we would have to establish inequalities between $\|\Delta u\|_2$ and $\|\Delta u^\sharp\|_2$ and such estimates are unknown in general.

Adams' idea: $u \in \mathcal{C}_0^\infty(\Omega)$, $f := -\Delta u \Rightarrow u(x) = c I_2 * f(x) = c \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^2}\, dy$
$\Rightarrow u^*(t) \leqslant u^{**}(t) := \frac{1}{t}\int_0^t u^*(s)\, ds \leqslant t I_2^{**} f^{**} + \int_t^{+\infty} I_2^* f^*\, ds$

Adams inequality

Let $\nabla^m u := (-\Delta)^{\frac{m}{2}} u$ if m is even, and $\nabla^m u := \nabla(-\Delta)^{\frac{m-1}{2}} u$ if m is odd.

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Let m be an integer and let $\Omega \subset \mathbb{R}^N$ with $m < N$. There exists a constant $C_{m,N} > 0$ such that

$$\sup_{u \in W_0^{m,\frac{N}{m}}(\Omega),\, \|\nabla^m u\|_{\frac{N}{m}} \leqslant 1} \int_\Omega e^{\beta |u|^{\frac{N}{N-m}}} dx \begin{cases} \leqslant C_{m,N}|\Omega| & \forall \beta \leqslant \beta_{N,m}\,, \\ = +\infty & \forall \beta > \beta_{N,m}\,. \end{cases}$$

Theorem [Fontana, Morpurgo 2018]

Let m be a positive integer with $2 \leqslant m < n$. There exists a constant $C_{N,m} > 0$ such that

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The particular case of $W^{2,2}(\mathbb{R}^4)$

- $\|u\|^2_{W^{2,2}} := \|(-\Delta + I)u\|_2^2 := \|\Delta u\|_2^2 + 2\|\nabla u\|_2^2 + \|u\|_2^2$
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$$\sup_{u \in W^{2,2}(\mathbb{R}^4),\, \|u\|_{W^{2,2}} \leqslant 1} \int_\Omega (e^{\alpha u^2} - 1)\, dx \begin{cases} < C|\Omega| & \forall\, 0 < \alpha \leqslant 32\pi^2, \\ = +\infty & \forall \alpha > 32\pi^2. \end{cases}$$

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Adams' inequality with the exact growth condition on $\mathbb{R}^4$

Theorem [Masmoudi, S. 2014]

There exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^4} \frac{e^{32\pi^2 u^2} - 1}{(1 + |u|)^2}\, dx \leqslant C\|u\|_2^2 \quad \forall u \in W^{2,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2 \leqslant 1. \qquad (*)$$

Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$.

Remark:

- It is easy to see that inequality $(*)$ implies that for any $\alpha \in (0, 32\pi^2)$ there exists $C_\alpha > 0$ such that

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Main difficulty: how to reduce the problem to the radial case?

- **Schwarz symmetrization**

  Problem: given $u \in W^{2,2}(\mathbb{R}^4)$, we do not know whether or not $u^\sharp \in W^{2,2}(\mathbb{R}^4)$; even in the case $u^\sharp \in W^{2,2}(\mathbb{R}^4)$, we would have to establish inequalities between $\|\Delta u\|_2$ and $\|\Delta u^\sharp\|_2$ and such estimates are unknown in general

- **Comparison principle**

Theorem [Talenti 1976]

Let $B_R \subset \mathbb{R}^4$ be the ball of radius $R > 0$ centered at the origin. Let u, v be weak solutions of

$$(P) \begin{cases} -\Delta u = f & \text{in } B_R \\ u \in W_0^{1,2}(B_R) \end{cases} \qquad (P^\sharp) \begin{cases} -\Delta v = f^\sharp & \text{in } B_R \\ v \in W_0^{1,2}(B_R) \end{cases}$$

then $u^\sharp \leqslant v$.

This comparison principle is a suitable tool if one works with the Dirichlet norm, in fact $\|\Delta u\|_2 = \|\Delta v\|_2$.

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Reduction of the problem to the radial case: a key tool

Given $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$, we denote by

$$f := -\Delta u \quad \text{in } \mathbb{R}^4 .$$

Let

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(t)\, dt.$$

Talenti's inequality

If $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ then

$$u^\sharp(r_1) - u^\sharp(r_2) \leqslant \frac{\sqrt{2}}{16\pi} \int_{|B_{r_1}|}^{|B_{r_2}|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\, d\xi \quad \text{for } 0 < r_1 \leqslant r_2. \tag{T}$$

Remark: the key ingredients in the proof of (T) are

- coarea formula
- isoperimetric inequality

Consequently the constant $\frac{\sqrt{2}}{16\pi}$ is sharp!

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Optimal descending growth condition

Talenti's inequality enables us to obtain the following result

Theorem [Masmoudi, S. 2014]

Let $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ and let $R > 0$. If $u^\sharp(R) > 1$ and $f := -\Delta u$ in $\mathbb{R}^4$ satisfies

$$\int_{|B_R|}^{+\infty} [f^{**}(s)]^2 \, ds \leqslant 4K$$

for some $K > 0$, then we have

$$\frac{\exp\left(\frac{32\pi^2}{K}[u^\sharp(R)]^2\right)}{[u^\sharp(R)]^2} R^4 \leqslant \frac{C}{K^2} \|u^\sharp\|^2_{L^2(\mathbb{R}^4 \setminus B_R)},$$

where C is a universal constant independent of u, R and K.

Here

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(t) \, dt$$

and

$$\int_0^{+\infty} [f^{**}(s)]^2 \, ds \leqslant 4 \int_0^{+\infty} [f^*(s)]^2 \, ds$$

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Adams' inequality with the exact growth condition: sketch of the proof

Key ingredients of the proof:

1. Optimal descending growth condition
2. Talenti's inequality + Moser's change of variable and one-dimensional Lemma

One-dimensional Lemma [Moser 1970]

There exists $c_0 > 0$ such that for any non-negative measurable function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

$$\int_0^{+\infty} \phi^2(t)\, dt \leqslant 1$$

the following inequality holds

$$\int_0^{+\infty} e^{F(t)}\, dt \leqslant c_0,$$

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$$F(t) := \Big(\int_0^t \phi(s)\, ds\Big)^2 - t.$$

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Adams' inequality with the exact growth condition: the second order Sobolev case

Remark: The key ingredients of the proof of Adams' inequality with the exact growth condition in $W^{2,2}(\mathbb{R}^4)$ are closely related to the properties of the Laplacian operator but they are not confined to the 4-dimensional case!

Indeed, for any $N \geqslant 4$, using the same arguments one can prove the existence of a constant $C_N > 0$ such that

$$\int_{\mathbb{R}^N} \frac{\exp_{\lceil \frac{N}{2}-2\rceil}\{\beta_{N,2}|u|^{\frac{N}{N-2}}\}}{(1+|u|)^{\frac{N}{N-2}}}\,dx \leqslant C_N \|u\|_{\frac{N}{2}}^{\frac{N}{2}} \quad \forall u \in W^{2,\frac{N}{2}}(\mathbb{R}^N) \text{ with } \|\Delta u\|_{\frac{N}{2}} \leqslant 1$$

where

- $\exp_k(t) := e^t - \sum_{j=0}^{k} \frac{t^j}{j!}, \quad k \in \mathbb{N}$
- $\lceil x \rceil$ denotes the smallest integer grater than or equal to $x \in \mathbb{R}$
- $\beta_{N,2}$ is the sharp exponent of Adams' inequality on bounded domains

(See Lu-Tang-Zhu for the case $N \geqslant 3$ – In particular, the case $N = 3$ reveals some non-trivial technical difficulties!)

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Adams' inequality: the second order Lorentz-Sobolev case

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Adams' inequality with the exact growth condition: the higher order Sobolev case

Let

$$\nabla^m u := \begin{cases} (-\Delta)^{\frac{m}{2}} u & \text{if } m \text{ is even,} \\ \nabla(-\Delta)^{\frac{m-1}{2}} u & \text{if } m \text{ is odd.} \end{cases}$$

Theorem [Masmoudi, S. 2017]

Let m be a positive integer with $2 < m < N$. Then

$$\int_{\mathbb{R}^N} \frac{\exp_{\lceil \frac{N}{m} - 2 \rceil}\left\{ \beta_{N,m} |u|^{\frac{N}{N-m}} \right\}}{\left(1 + |u| \right)^{\frac{N}{N-m}}} \, dx \leqslant C_{N,m} \|u\|_{\frac{N}{m}}^{\frac{N}{m}} \quad \forall u \in W^{m,\frac{N}{m}}(\mathbb{R}^N), \ \|\nabla^m u\|_{\frac{N}{m}} \leqslant 1 \, .$$

The above inequality fails if the power $\frac{N}{N-m}$ in the denominator is replaced with any $p < \frac{N}{N-m}$.

Idea: It is possible to reach a **limiting sharp higher order** inequality exploiting **refined limiting and non-limiting second order** inequalities.

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Non-limiting sharp embeddings for Lorentz-Sobolev spaces

Theorem [Alvino 1977]

Assume $1 \leqslant p < N$ and $1 \leqslant q \leqslant p$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain,

$$\text{if } p^* := \frac{Np}{N-p} \quad \Rightarrow \quad \|u\|_{p^*,q} \leqslant \frac{p}{N-p} \left(\frac{N}{\omega_{N-1}} \right)^{\frac{1}{n}} \|\nabla u\|_{p,q} \quad \forall u \in W_0^1 L^{p,q}(\Omega) .$$

Theorem [Tarsi 2012]

Assume $N > 2$, $1 < p < N/2$ and $q > 1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let

$$\text{if } p^* := \frac{Np}{N-2p} \quad \Rightarrow \quad \|u\|_{p^*,q} \leqslant l_{N,p} \|\Delta u\|_{p,q} \quad \forall u \in W^2 L^{p,q}(\Omega) \cap W_0^1 L^{p,q}(\Omega) ,$$

where $l_{N,p}$ is explicitly known.

As a by-product of the argument introduced by Tarsi: if $2 < m < N$, we have

$$\|\Delta u\|_{\frac{N}{2},\frac{N}{m}} \leqslant \alpha_{N,m} \|\nabla^m u\|_{\frac{N}{m}} \quad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$$

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Summarizing, if $2 < m < N$

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Thank you!