A critical equation with Hardy potential

(jointly with N. Ghoussoub, A. Pistoia and G. Vaira)

Pierpaolo Esposito,
Department of Mathematics and Physics,
University of Roma Tre

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$$U_\mu(x) = \frac{\alpha_N \mu^\Gamma}{|x|^{\beta^-}(\mu^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}}, \quad \mu > 0$$

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- F. Catrina, Z.Q. Wang, CPAM 53 (2000)
- S. Terracini, Adv. Differential Equations 2 (1996)

On bounded domains

On Ω bdd domain with $0 \in \Omega$ set

$$S_\gamma(\Omega) = \inf\{\int_\Omega [|\nabla u|^2 - \gamma \frac{u^2}{|x|^2}] : \ u \in H_0^1(\Omega) \text{ s.t. } \int_\Omega |u|^{\frac{2N}{N-2}} = 1\}$$

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with E-L equation

$$-\Delta u - \gamma \frac{u}{|x|^2} = |u|^{\frac{4}{N-2}} u + \lambda u \text{ in } \Omega \setminus \{0\}, \quad u = 0 \text{ on } \partial\Omega$$

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Remark: no soln's in general for $\lambda \leq 0$ and no positive soln's for $\lambda \geq \lambda_1$

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- N. Ghoussoub, F. Robert, Calc. Var., to appear
- E. Jannelli, JDE 156 (1999)
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See also the survey

- N. Ghoussoub, F. Robert, Bull. Math. Sci. 6 (2016)

Sign-changing solutions

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- D. Cao, P. Han, JDE 205 (2004)
- D. Cao, S. Peng, JDE 193 (2003)
- D. Cao, S. Yan, Calc. Var. 38 (2010)
- Z. Chen, W. Zou, JDE 252 (2012)
- A. Ferrero, F. Gazzola, JDE 177 (2001)

Some comments when $\lambda \to 0^+$

For $\gamma = 0$

- H. Brezis, L. Nirenberg, CPAM 36 (1983) (no ground states in Ω and no positive soln's in B when $N = 3$)
- Adimurthi, S.L. Yadava, Nonlinear Anal. 14 (1990) & F.V. Atkinson, H. Brezis, L. Peletier, JDE 85 (1990) (no radial sign-changing soln's in B when $N = 3, 4, 5, 6$)

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Q1: What about positive soln's for $\gamma < 0$?

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Q1: What about positive soln's for $\gamma < 0$?

Q2: What about sign-changing soln's for $\gamma < 0$ or $\gamma \geq \frac{(N-2)^2}{4} - 4$?

Main results

Attack existence issues by a perturbative approach for λ small:

Theorem 1

i) $\gamma \leq \frac{(N-2)^2}{4} - 1 \Rightarrow$ *positive solution* u_λ *developing a bubble at* 0
ii) $\gamma < \frac{(N-2)^2}{4} - 4 \Rightarrow$ *sign-changing solution* u_λ *shaped as a tower of* k *alternating bubbles centered at* 0

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By a fine asymptotic analysis:

Theorem 2

$\gamma \geq \frac{(N-2)^2}{4} - 4 \Rightarrow$ *no radial sign-chang. soln's in* B *for* $\lambda > 0$ *small*

A more general result

Set $\sigma_j = \frac{\Gamma}{2(\Gamma-1)}(\frac{\Gamma}{\Gamma-2})^{j-1} - \frac{1}{2}$ with $\Gamma = \sqrt{\frac{(N-2)^2}{4} - \gamma}$. Let u_n be solutions in B with $\lambda_n \to 0^+$. Theorem 2 follows by

Theorem 3

i) $u_n > 0$ *then* $\gamma \leq \frac{(N-2)^2}{4} - 1$ *and* $u_n \sim U_{\mu_n^1}$ *on the scale* $\mu_1^n \sim \lambda_n^{\sigma_1}$

ii) u_n *sign-changing, then* $\gamma < \frac{(N-2)^2}{4} - 4$

iii) u_n *have* $k-1$ *shrinking nodes* $M_k^n < \cdots < M_2^n \to 0$ *then* $u_n \sim U_{\mu_j^n}$ *at scale* $\mu_j^n \sim \lambda_n^{\sigma_j}$, $M_1^n = 1$ *and* $M_j^n \sim (\mu_{j-1}^n \mu_j^n)^{\frac{2\Gamma}{(N-2)^2}}$

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Change of variables: $v(r) \sim r^{\frac{(N-2)\beta_-}{2\Gamma}} u(r^{\frac{N-2}{2\Gamma}})$, $\alpha = \frac{2\beta_-}{\Gamma}$, satisfies

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i) $u_n > 0$ *then* $\gamma \leq \frac{(N-2)^2}{4} - 1$ *and* $u_n \sim U_{\mu_n^1}$ *on the scale* $\mu_1^n \sim \lambda_n^{\sigma_1}$
ii) u_n *sign-changing, then* $\gamma < \frac{(N-2)^2}{4} - 4$
iii) u_n *have* $k-1$ *shrinking nodes* $M_k^n < \cdots < M_2^n \to 0$ *then* $u_n \sim U_{\mu_j^n}$ *at scale* $\mu_j^n \sim \lambda_n^{\sigma_j}$, $M_1^n = 1$ *and* $M_j^n \sim (\mu_{j-1}^n \mu_j^n)^{\frac{2\Gamma}{(N-2)^2}}$

Change of variables: $v(r) \sim r^{\frac{(N-2)\beta_-}{2\Gamma}} u(r^{\frac{N-2}{2\Gamma}})$, $\alpha = \frac{2\beta_-}{\Gamma}$, satisfies

$$-\Delta v = |v|^{\frac{4}{N-2}} v + \lambda |x|^\alpha v \text{ in } B \setminus \{0\}, \; v = 0 \text{ on } \partial B \qquad (P)_\lambda$$

We recover the non-existence results for $\gamma = 0$, $3 \leq N \leq 6$ and $\gamma > \frac{(N-2)^2}{4} - 1$ & the asymptotics for $\gamma = 0$, $N \geq 7$ due to

- A. Iacopetti, Ann. Mat. Pura Appl. 194 (2015)

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$$V_j = (-1)^{k-j+1} \delta_j^{\frac{N-2}{2}} v(\delta_j x) \to V, \quad V = \Big(\frac{1}{1 + \frac{|x|^2}{N(N-2)}}\Big)^{\frac{N-2}{2}}$$

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- Bubbles of same sign don't superimpose by Pohozaev identity:

$$\int_{\partial A} \left[|x|(v')^2 + (N-2)vv' + \frac{N-2}{N}|x||v|^{\frac{2N}{N-2}} + \lambda |x|^{\alpha+1} v^2 \right]$$
$$= (\alpha+2)\lambda \int_A |x|^\alpha v^2 > 0$$

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Crucial estimate: as a by-product $|v| \leq CV_{\delta_j}$ in $[R_{j-1}, R_j]$

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- $\delta_j = \delta_j(\lambda)$ follows by Pohozaev identity if $R_j \sim \sqrt{\delta_j \delta_{j+1}}$

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$$\Rightarrow \frac{1}{\delta_{j+1}^{\frac{N-2}{2}}} = \int_{R_j}^{r_{j+1}} |v|' \sim \frac{\delta_j^{\frac{N-2}{2}}}{R_j^{N-2}} + o(\frac{1}{\delta_{j+1}^{\frac{N-2}{2}}})$$

A perturbative approach

Ansatz: $u = \sum_{j=1}^{k} (-1)^j PU_{\mu_j} + \phi$, where $P : H^1(\Omega) \to H_0^1(\Omega)$ projection, $\phi \in H_0^1(\Omega)$ small, $\mu_k << \cdots << \mu_1$

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Energy expansion: As $\lambda \to 0^+$

$$E = (B_N |m_{\gamma,0}(\Omega)| \mu_1^{2\Gamma} - \lambda \mu_1^2 f(\mu_1)) + \sum_{j=2}^{k} [C_N (\frac{\mu_j}{\mu_{j-1}})^\Gamma - D_N \lambda \mu_j^2] + \text{h.o.t}$$

where $f(\mu_1) = \log \frac{1}{\mu_1} / 1$ if $\Gamma = 1 / > 1$

The choice of the parameters

Need to require:

$$\mu_1^{2\Gamma} \sim \lambda \mu_1^2 f(\mu_1), \quad (\frac{\mu_j}{\mu_{j-1}})^\Gamma \sim \lambda \mu_j^2 \quad \Rightarrow \quad \begin{cases} \Gamma \geq 1 & \text{if } k = 1 \\ \Gamma > 2 & \text{if } k \geq 2 \end{cases}$$

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Thanks for your attention