CO-AXIAL MONODROMY

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We consider a surface S homeomorphic to the sphere and equipped with a Riemannian metric of constant curvature 1 with finitely many conic singularities with angles $\alpha_1, \ldots, \alpha_n$.

We measure angles in turns: 1 half$=$ 2π radians.

The question is: What angles are possible?

Necessary conditions:

$$\sum_{j=1}^{n} (\alpha_j - 1) + 2 > 0 \quad \text{(Gauss–Bonnet)},$$

$$d_1(\mathbf{Z}_o^n, \alpha - \mathbf{1}) \geq 1 \quad \text{(Closure condition)}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\mathbf{Z}_o^n$ is the set of integer lattice points with *odd* sums of coordinates, and d_1 is the ℓ_1 distance.

The standard metric (of area 1) on the sphere is

$$\rho_0(z)|dz| = \frac{|dz|}{\sqrt{\pi}(1+|z|^2)}.$$

Then our metric $\rho(z)|dz|$ has density $\exp(v/2)$ with respect to ρ_0, where

$$\Delta_{\rho_0} v + 2e^v - 8\pi = 4\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j}.$$

Our problem is to find out for which α_j this equation is solvable, *with some* a_j.

Necessity of the Closure condition is due to Mondello and Panov (2016). They also proved that the Gauss–Bonnet and the Closure condition with *strict inequality* are sufficient.

Developing map is a multi-valued function

$$S \backslash \{\text{singularities}\} \to \overline{\mathbf{C}},$$

where $\overline{\mathbf{C}}$ is the sphere equipped with the standard spherical metric (of curvature 1), and f is a local isometry away from the singularities. So f is analytic with respect to the conformal structure on S induced by the metric, and the monodromy group of f consists of rotations of $\overline{\mathbf{C}}$.

The monodromy is called *co-axial* if it is a subgroup of $SO(2)$.

Mondello and Panov proved that if the Closure condition holds with equality, then the monodromy must be co-axial.

α is called *admissible* if a co-axial metric with such angles exists.

Theorem 1. *Suppose that* $\alpha_1, \ldots \alpha_m$ *are not integers, while* $\alpha_{m+1}, \ldots, \alpha_n$ *are integers. For* α *to be admissible it is necessary that there exist* $\epsilon_j \in \{\pm 1\}$ *and integer* k' *such that:*

$$\sum_{j=1}^{m} \epsilon_j \alpha_j = k' \geq 0, \quad \textit{and the number}$$

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \textit{is non-negative and even.}$$

If the coordinates of the vector $\mathbf{c} := (\alpha_1, \ldots, \alpha_m, \underbrace{1, \ldots, 1}_{k'+k'' \text{ times}})$ *are incommensurable then these two conditions are also sufficient.*

If $\mathbf{c} = \eta\mathbf{b}$ *where coordinates of* $\mathbf{b}$ *are integers whose g.c.d. is* 1*, then there is an additional necessary condition*

$$2 \max_{m+1 \leq j \leq n} \alpha_j \leq \sum_{j=1}^{q} |b_j|, \quad q = m + k' + k'',$$

and all these three conditions together are sufficient.

This generalizes the previous results: for $n = 2$ (Troyanov, 1989), for $n = 3$ (Eremenko, 2004) and for $m = 2$ (Eremenko, Gabrielov, Tarasov, 2014), and completes the description of possible angles.

As the monodromy is co-axial, we have $df/f = Rdz$, where R is a rational function. The singularities are among zeros and poles (whose residues are not ± 1) of this function. One can show that for admissible α R can be always taken in the form

$$R(z) = \sum_{j=1}^{m} \frac{\epsilon_j \alpha_j}{z - a_j} - \sum_{j=1}^{k'} \frac{1}{z - b_j} + \sum_{j=k'+1}^{k'+k''} \frac{(-1)^j}{z - b_j},$$

the condition that k'' is even comes from the residue theorem. Notice that we can introduce any number of poles with residues ± 1; they are not singularities of the metric.

Zeros of R are singularities with integer angles: their multiplicities are $\alpha_j - 1$. Since all residues in this formula are determined by the angles, the question is whether one can construct such a function with prescribed residues and prescribed *multiplicities of zeros*.

We restate the problem: *For a given a vector $(c_1, \ldots, c_q)$ with $\sum_j c_j = 0$ and a given partition of $q-2 = \sum_{j=1}^s \ell_j$, does there exist a function*

$$R(z) = \sum_{j=1}^{q} \frac{c_j}{z - z_j}$$

with zeros of multiplicities ℓ_j ?

Theorem 2. *If the c_j are incommensurable, such an R exists. If $c_j = \eta_j b_j$ with mutually prime integers b_j, then the necessary and sufficient condition for existence of R is*

$$2\left(1 + \max_{1 \le j \le s} \ell_j\right) \le \sum_{j=1}^{q} |b_j|.$$

Commensurable case. $R = \eta g$,

$$g(z) = \sum_{j=1}^{q} \frac{b_k}{z - a_k}, \quad b_j \quad \text{are mutually prime integers.}$$

Then $g = h'/h$, h is rational, and we are looking for a rational function with prescribed multiplicities of zeros, poles and critical points other than zeros and poles. We have $\deg h = (1/2) \sum_j |b_j|$ and the necessary condition $\ell_j + 1 \leq \deg h$ is evident. Song and Yu (2016) proved that this is also sufficient.

This is a special case of the Hurwitz problem: *when there exist a rational function with given number of critical values and prescribed multiplicities of their preimages*. There is no simple general criterion, but the special case that we need is known.

General case. Consider the real projective space $\mathbf{RP}^{q-2}$ consisting of q-tuples $\mathbf{c} = (c_1, \ldots, c_q)$ with zero sum, modulo proportionality. Let Z be the union of the coordinate hyperplanes $c_j = 0$. Let P be a partition of $q - 2$. We say that a point $c \in \mathbf{RP}^{q-2}$ is *P-admissible* if there exists $g(z)$ with residues $\mathbf{c}$ and multiplicities of zeros P. Otherwise $\mathbf{c}$ is *P-exceptional*. A point $\mathbf{c}$ is called *rational* if its coordinates are commensurable.

Proposition. *For every q and P, the set of rational P-exceptional points in $\mathbf{RP}^{q-2}$ is finite.*

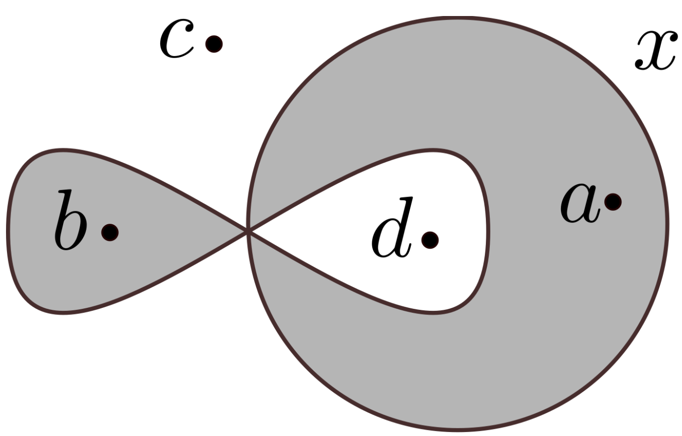
Indeed, they satisfy $\sum_{j=1}^{q} |b_j| \leq 2(\max \ell_j + 1)$, and b_j are integers.

Now we try to construct a rational function $R = f'/f$ with prescribed residues and multiplicities of zeros. Consider a *flat* metric ρ on $S^* = S\backslash\{$singularities$\}$ with developing map $\log f$. The metric space (S^*, ρ) breaks into flat cylinders by the critical level lines of $u = \log|f|$. Semi-infinite cylinders are neighborhoods of the punctures, and the cylinder surrounding a puncture z_j has "waist" $2\pi c_j$. There are also finite cylinders, and all cylinders are pasted together along certain boundary arcs. To construct such a surface, one chooses a scheme of the boundary identifications, and prescribes waists to all cylinders, and the lengths of the boundary arcs which are to be identified. Once such a flat surface is constructed, f is recovered by the uniformization theorem.

Example. $q = 4$. The residues are $a, b, -c, -d$ and we want a single critical point of multiplicity 3. The pattern in the figure consists of 4 infinite cylinders whose waists are known. One only need to determine the length of x. We have to find a positive solution to

$$a = x + d, \quad c = x + b.$$

Such an x exists iff $a - d + b - d = 0$ and $x > 0$ if $a > d$ and $c > b$.



The possibility of the construction that we outlined depends on the ability to choose the waists of all cylinders and the lengths of the arcs to be pasted together. The waists of the semi-infinite cylinders are prescribed. This leads to a set of equations and inequalities of the form

$$A_j(c_1, \ldots, c_q) = 0, \quad B_j(c_1, \ldots, c_q) > 0$$

with some *linear* functions A_j, B_j with *integer coefficients*. We conclude that the set of P-exceptional points $\mathbf{c}$ is a *rational polyhedron* in $\mathbf{RP}^{q-2}$. But we know from the consideration of the commensurable case that this rational polyhedron contains only finitely many rational points.

A rational polyhedron containing finitely many rational points must be finite and must consist of only rational points!

This completes the proof in the general case.

References

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