On the global bifurcation diagram of mean field equations

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Physical, Geometrical and Analytical Aspects of Mean Field Systems of Liouville Type

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- **[B]** D. Bartolucci, "*Global bifurcation analysis of mean field equations and the Onsager microcanonical description of two-dimensional turbulence*", arXiv:1609.04139;
- **[BJLY1]** D.B., A. Jevnikar, Y. Lee, W. Yang "*Uniqueness of bubbling solutions of mean field equations*", arXiv:1704.02354, to appear on **J. M. P. A.**;
- **[BJLY2]** D. Bartolucci, A. Jevnikar, Y. Lee, W. Yang, "*Non degeneracy, Mean Field Equations and the Onsager theory of 2D turbulence* ", arXiv:1711.09970, to appear on **A.R.M.A.**;
- **[BJLY3]** D. Bartolucci, A. Jevnikar, Y. Lee, W. Yang, "*Local uniqueness of m-bubbling sequences for the Gelfand equation* ", Preprint (2017).

The Mean Field Equation

We are concerned with the global bifurcation diagram of solutions of the Mean Field Equation (M.F.E.),

$$\begin{cases} -\Delta u = \rho \dfrac{he^u}{\int\limits_\Omega he^u} & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (\mathbf{P}_{\rho,\Omega})$$

where,

- $\Omega \subset \mathbb{R}^2$ is any open, smooth and bounded domain;
- $h \in C^1(\overline{\Omega})$, $h \geq a > 0$ in $\overline{\Omega}$;
- $\rho \in [0, +\infty)$.

A useful notation,

$$< f >_\rho = \int\limits_\Omega \frac{he^{u_\rho} f}{\int\limits_\Omega he^{u_\rho}}.$$

The Mean Field Equation

$$\begin{cases} -\Delta u = \rho \left(\dfrac{he^u}{\int\limits_\Omega he^u} - \dfrac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma \\ \\ \int\limits_\Sigma u = 0 \end{cases} \qquad (\mathbf{P}_{\rho,\Sigma})$$

where,

- Σ is a compact surface without boundary;
- $h \in C^1(\Sigma)$, $h \geq a > 0$ in Σ;
- $\rho \in [0, +\infty)$.

Motivations

The analysis of these Liouville-type equations (J. Liouville, **J.M.P.A. (1853)**), possibly with h having power-type isolated zeroes, often arises in problems of pure and applied mathematics, such as:
- **the conformal geometry of surfaces**:J. L. Kazdan, F. W. Warner, **Ann. Math. (1974)**, M. Troyanov, **Trans. A.M.S. (1991)**;

- **quantum gravity in 2d, gauge-field vortices**: A.M. Polyakov, **Phys. Lett. (1981)**, G. Dunne, **L.N.P. (1995)**, Y. Yang, **S. M. M. (2001)**, G. Tarantello, **P.N.L.D.E. (2007)**;

- **the statistical mechanics description of 2d-turbulence**, [**CLMP**] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, **C. M. P. (1995)** and of **self-gravitating systems**, G. Wolansky, **A.R.M.A. (1992)**;

- **the chemotaxis dynamics of bacteria aggregation**, T. Suzuki **A.M.E.S. (2008)** and the **ignition models of combustion theory**, J. Bebernes, D. Eberly, **A.M.S. (1989)**;

- **the monodromy group of Fuchsian equations** C.L. Chai, C.S. Lin, C.L. Wang, **Jour. ec. pol. Math. (2017)**.

$(\mathbf{P}_{\rho,\Sigma})/(\mathbf{P}_{\rho,\Omega})$ as variational problems

Solutions of $(\mathbf{P}_{\rho,\Omega})$ (and similarly of $(\mathbf{P}_{\rho,\Sigma})$) can be found by as critical points of a suitable functional J_ρ, defined on $H_0^1(\Omega)$. As a consequence of the Moser-Trudinger inequality (J. Moser, **I.U.M.J. (1971)**), J_ρ is:

- coercive and bounded from below if $\rho < 8\pi$:
  $\rho < 8\pi$ **is subcritical, existence of solutions as minimizers of J_ρ is granted**;
- bounded from below but not coercive if $\rho = 8\pi$:
  $\rho = 8\pi$ **is critical, existence of solutions is not granted**;
- neither bounded from below nor coercive if $\rho > 8\pi$:
  $\rho > 8\pi$ **is supercritical, existence of solutions is not granted**.

Existence of solutions for $\rho > 8\pi$

However it is also well known that solutions exists for any $\rho \in (8\pi, +\infty) \setminus 8\pi\mathbb{N}$ on **compact surfaces and on domains with non trivial topology**:

- M. Struwe, G. Tarantello, **Boll. U.M.I. (1998)**;
- W. Ding, J. Jost, J. Li, G. Wang, **A.I.H.P. (1999)**;
- [**CL1-2**] C.C. Chen, C.S. Lin, **C.P.A.M. (2002)-(2003)**[**topological degree, existence for $\rho \in 8\pi\mathbb{N}$**];
- Z. Djadli, **C.C.M. (2008)**;
- A. Malchiodi, **Adv. Diff. Eq. (2008)** [**topological degree**] and **D.C.D.S. (2008)**;
- F. De Marchis, **J.F.A. (2010)**[**multiplicity**];
- D. B., F. De Marchis, **Jour. Math. Phys. (2012)**.

The existence for $\rho \in 8\pi\mathbb{N}$ is more subtle.

The lack of coercivity causes the so called **blow-up** phenomenon. The underlying concentration-compactness-quantization theory for Liouville-type equations has been developed in **[BM]** H. Brezis, F. Merle, **Comm. P.D.E. (1991)**, Y. Li, I. Shafrir, **I.U.M.J. (1994)** and **[L]** Y. Li, **Comm. Math. Phys. (1999)**, as refined in **[CL1]**.

Let $G(x,y)$ be the Green function,

$$\begin{cases} -\Delta G(x,y) = \delta_{x=y}, \quad x \in \ \Omega, \\ G(x,y) = 0, \quad x \in \ \partial\Omega. \end{cases}$$

m-bubbling sequences

As described by the concentration-compactness theory **[BM,L]**, **any sequence u_n of solutions of $(\mathbf{P}_{\rho,\Omega})$ (and of course of $(\mathbf{P}_{\rho,\Sigma})$) such that $\sup\limits_n \|u_n\|_\infty \to +\infty$, with $\rho = \rho_n \leq C$, along a subsequence blows up in the following sense:** as $n \to +\infty$ we have,

$$\rho_n \to 8\pi m, \quad \text{for some } m \in \mathbb{N},$$

$$\rho_n \frac{he^{u_n}}{\int\limits_\Omega he^{u_n}} \rightharpoonup 8\pi \sum_{i=1}^m \delta_{p_i},$$

where δ_p is the Dirac measure, $\{p_1, \cdots, p_m\} \subset \Omega$ are m distinct points **(blow up points)** and

$$u_n(x) \to 8\pi \sum_{i=1}^m G(x, p_i),$$

in $W_0^{1,q}(\Omega)$ for any $q \in [1, 2)$.

Existence of m-bubbling sequences

In particular $\underline{\mathbf{p}} = (p_1, \cdots, p_m)$ must be a critical point of the m-vortex Hamiltonian on Ω (and similarly on Σ):

$$H_m(x_1, \cdots, x_m) = \sum_{i=1}^{m} (R(x_i, x_i) + \frac{1}{4\pi} \log(h(x_i))) + \sum_{i,j=1; i \neq j}^{m} G(x_i, x_j),$$

where $R(x, y) = G(x, y) + \frac{1}{2\pi} \log |x - y|$ is the regular part of $G(x, y)$ (L. Ma, J.C. Wei, **C.M.H. (2001)**). We refer to a sequence of such solutions as an ***m*-point blow up sequence** or either as an ***m*-bubbling sequence**. It is well known that m-bubbling sequences exists (under suitable non degeneracy assumptions on the critical point $\underline{\mathbf{p}}$ of H_m):

- S. Baraket, F. Pacard, **Calc. Var. & P.D.E. (1998)**;
- **[CL1-2]**;
- P. Esposito, M. Grossi & A. Pistoia, **A.I.H.P. (2005)**;
- M. Kowalczyk, M. Musso & M. del Pino, **Calc. Var. & P.D.E. (2005)**.

The subcritical case

The structure of the set of solutions of **($\mathbf{P}_{\rho,\Omega}$)/($\mathbf{P}_{\rho,\Sigma}$)** is heavily affected by the lack of compactness/blow up phenomenon.
For example, a well known consequence of the Pohožaev identity shows that **if Ω is strictly starshaped and (say) $h \equiv 1$, then there exists $\rho^*(\Omega) \geq 8\pi$ such that no solutions of ($\mathbf{P}_{\rho,\Omega}$) exist for $\rho \geq \rho^*(\Omega)$.**
In particular if Ω is simply connected and $h \in C^1(\overline{\Omega})$ is positive, then [CL2] the topological degree vanishes for $\rho > 8\pi$.

Example of concentration: $\Omega = B_1$, $h \equiv 1$.

Let $\Omega = B_1 := \{x \in \mathbb{R}^2 : |x| < 1\}$, $h \equiv 1$. It turns out that $\rho^*(B_1) = 8\pi$ and it is sharp. Indeed solutions on B_1 are radial and take the form

$$u_\rho(x) = 2\log\left(\frac{1+\gamma^2(\rho)}{1+\gamma^2(\rho)|x|^2}\right), \qquad \gamma^2(\rho) = \frac{\rho}{8\pi-\rho},\ \rho \in (0, 8\pi).$$

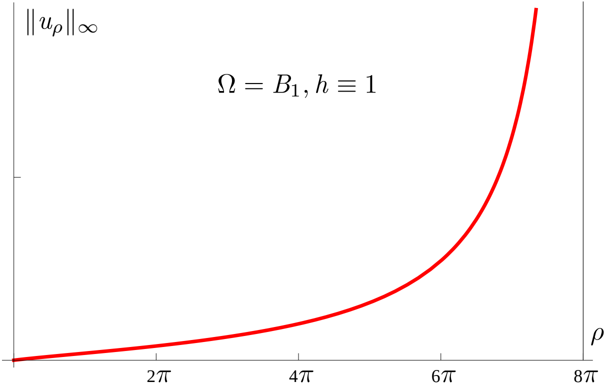
In particular it is easy to see that,

$$\rho\frac{e^{u_\rho}}{\int\limits_{B_1} e^{u_\rho}} \rightharpoonup 8\pi\delta_{x=0}, \quad \text{as} \quad \rho \to (8\pi)^-,$$

weakly in the sense of measures and

$$u_\rho(x) \to 8\pi G(x,0) = 4\log\left(\frac{1}{|x|}\right), \quad \text{as} \quad \rho \to (8\pi)^-.$$

The bifurcation diagram on $\Omega = B_1$, $h \equiv 1$.



Uniqueness of solutions for $\rho \leq 8\pi$, $\Delta \log h \geq 0$.

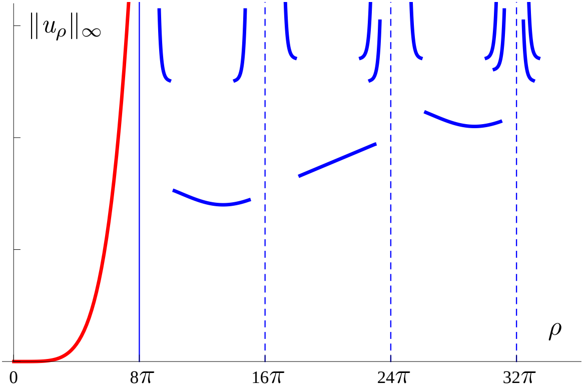
In particular, if $\rho \leq 8\pi$, then, whenever they exist, solutions for $(\mathbf{P}_{\rho,\Omega})$ are **unique** if $\Delta \log(h) \geq 0$ and **non degenerate** (if Ω is simply connected it is enough $\Delta \log(h) \geq 0$, otherwise it is enough $\Delta \log h = 0$):

- T. Suzuki, **A.I.H.P. (1992)**;
- **[CCL]** S.Y.A. Chang, C.C. Chen, C.S. Lin, **New Stud. Adv. Math. (2003)**;
- D. B., C.S. Lin, **C.P.D.E. (2009)**;
- **[BLin]** D. B., C.S. Lin, **Math. Ann. (2014)**;
- C. Gui, A. Moradifam, **Proc. A.M.S. (2018)**.

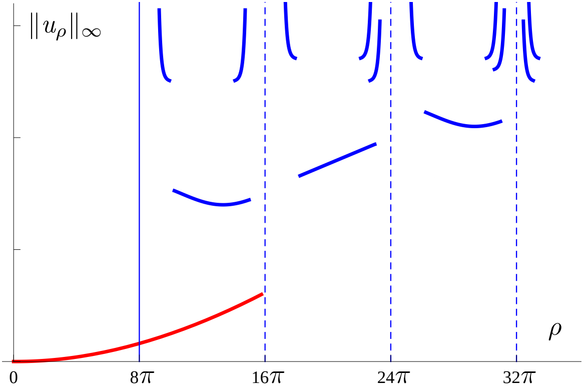
The uniqueness in the subcritical regime for $(\mathbf{P}_{\rho,\Sigma})$ is far more involved, see for example,

- C.S. Lin, **Calc. Var. & P.D.E. (2000)**;
- C.S. Lin, **A.R.M.A. (2000)**;
- C.S. Lin., M. Lucia, **J.D.E. (2006)**;
- C. Gui, A. Moradifam, **Invent. Math.** to appear;
- C. Gui, A. Moradifam, **Int. Math. Res. Not.** to appear;
- D.B., C. Gui, A. Jevnikar, A. Moradifam, Preprint (2018).

A temptative bifurcation diagram on a domain with "holes"



A temptative bifurcation diagram on a domain with "holes"



Some natural questions about the global bifurcation diagram

[Q1] Can we describe the full bifurcation diagram for domains which are not balls?

[Q2] Let u_n be an m-bubbling sequence with blow up points $\underline{\mathbf{p}} := (p_1, \cdots, p_m)$. Is it possible to evaluate the sign of $\rho_n - 8\pi m$?

[Q3] Let u_n be an m-bubbling sequence with blow up points $\underline{\mathbf{p}} := (p_1, \cdots, p_m)$. Is it true that u_n is unique for n large?

[Q4] Let u_n be an m-bubbling sequence with blow up points $\underline{\mathbf{p}} := (p_1, \cdots, p_m)$. Is it true that u_n is non degenerate for n large?

The sign of $\rho_n - 8\pi m$

A first answer to **[Q2]** has been provided in **[CL1]** where it was shown that if u_n is an m-bubbling sequence with blow up points "$\underline{\mathbf{p}}$", then for n large we have:

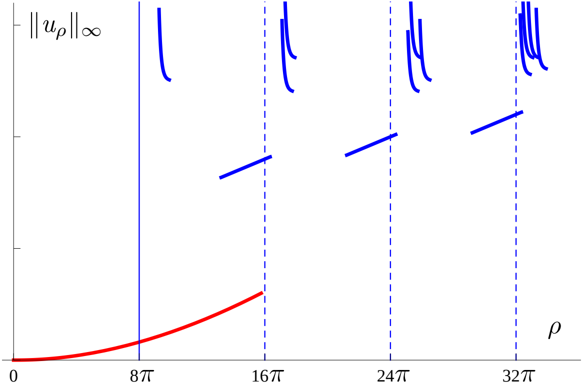
$$\rho_n - 8\pi m = c\Lambda_\Omega(\underline{\mathbf{p}})e^{-\lambda_n}\left(\lambda_n + \mathrm{O}_n(1)\right)$$

where c is a strictly positive constant, $\lambda_n = \max\limits_{B_r(p_1)} u_n - \log\left(\int\limits_\Omega he^{u_n}\right)$, (which satisfies $\lambda_n \to +\infty$) and

$$\Lambda_\Omega(\underline{\mathbf{p}}) = \sum_{i=1}^{m} \left(\Delta \log h(p_i)\right) h(p_i) e^{H^*_{i,m}(p_i)},$$

$$H^*_{i,m}(x) = 8\pi R(x,p_i) + 8\pi \sum_{j=1;j\neq i}^{m} G(x,p_j).$$

A refined bifurcation diagram on a domain with "holes" and $\Delta \log h > 0 (< 0)$



If $\Delta \log h > 0 (< 0)$ and the topology is non trivial, then for each $\rho \in 8\pi\mathbb{N}$ $(\mathbf{P}_{\rho,\Omega})$ admits at least one solution [CL2].

The sign of $\rho_n - 8\pi m$ when $\Delta \log h = 0$

However in many applications one faces the case where $\Delta \log h = 0$, as for example is the case for the problem itself which inspired the name M.F.E. **[CLMP]**, where $h = 1$. In this case the problem is **(far)** more subtle because **the sign of $\rho_n - 8\pi m$ depends by the geometry of the domain**. For $m = 1$ and Ω simply connected, the answer to **[Q2]** has been provided in **[CCL]** where it was shown that if u_n is a 1-bubbling sequence with blow up point p_1 and $\log h$ is harmonic in Ω, then for n large we have:

$$\rho_n - 8\pi = \overline{c}e^{-\lambda_n} \left(D_1(p_1) + \mathrm{o}_n(1) \right),$$

where $\overline{c}$ is a strictly positive constant, $\mathrm{o}_n(1) \to 0$ uniformly as $n \to +\infty$ and $D_1(p_1)$ is a geometric constant.

An improved estimate about the sign of $\rho_n - 8\pi m$

Theorem 1 [BJLY1]

Let u_n be an m-bubbling sequence of $(\mathbf{P}_{\rho,\Omega})$ (or of $(\mathbf{P}_{\rho,\Sigma})$) with blow up points "$\underline{\mathbf{p}}$". If $h \in C^{2,\sigma}(\overline{\Omega})$, then for any n large enough, we have:

$$\rho_n - 8\pi m = c_0 \Lambda_\Omega(\underline{\mathbf{p}}) e^{-\lambda_n} \left(\lambda_n + \log(a\delta^2) - 2\right) +$$

$$c_1 e^{-\lambda_n} (D_\Omega(\underline{\mathbf{p}}) + \mathrm{O}(\delta^\sigma)) + \mathrm{o}(e^{-\lambda_n}),$$

where $\delta > 0$ is a suitable positive fixed number.

$$D_\Omega(\underline{\mathbf{p}}) = \lim_{r\to 0} \sum_{i=1}^{m} h(p_i) e^{H^*_{i,m}(p_i)} \left[\int\limits_{\Omega_i / B_{r_i}(p_i)} e^{\Phi_i(x)}\, dx - \frac{\pi}{r_i^2} \right]$$

$$\Phi_i(x) = \sum_{j=1}^{m} 8\pi G(x, p_j) - H^*_{i,m}(p_i) + \log(h(x)) - \log(h(p_i)),$$

$$r_i = r\sqrt{8h(p_i)e^{H^*_{i,m}(p_i)}}, \quad a = 8\pi m h^2(p_1)e^{H^*_{1,m}(p_1)},$$

and

$$\bigcup_{i=1}^{m}\overline{\Omega_i} = \overline{\Omega},\ \Omega_i \cap \Omega_j = \emptyset\ \forall i \neq j,\ \text{and}\ p_i \in \Omega_i,\ i = 1, \cdots, m.$$

Clearly, $\overline{c}D_1(p_1) = c_1 D_\Omega(p_1)$ whenever $m = 1$, and $\log(h)$ is harmonic. The proof of Theorem 1 is based on a rather subtle evaluation based on the estimates in **[CL1]**.

For Ω convex let p be the unique critical point of the Robin function $R(x,x)(\equiv H_1(x)$, with $h \equiv 1)$. It is easy to see that $D_{B_1(p)}(p) = -\pi$. If Ω is a regular polygon then $D_\Omega(p) < 0$, **[CCL]**. On the other side it has been shown in **[BdM]** D.B., F. De Marchis, **A.R.M.A. (2015)** that there exists a universal constant $I_0 > 4\pi$ such that if Ω is a convex domain whose isoperimetric ratio I_Ω satisfies $I_\Omega > I_0$ then $D_\Omega(p) > 0$.

The sign of $\rho - 8\pi$, $h \equiv 1$.

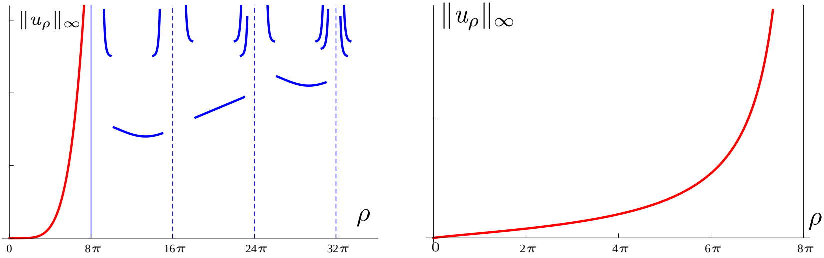
[BLin]: if $\Omega = B_1(0) \setminus \overline{B_a(0)}$, then $D_\Omega(p_1) > 0$ for any maximum point of H_1, while if $\Omega = B_1(0) \setminus \overline{B_r(x_0)}$, $x_0 \neq 0$, then, if r is small enough, H_1 has only one maximum point p_1 and $D_\Omega(p_1) < 0$.

Actually, it has been shown in **[CCL], [BLin]** that, for the branch of unique solutions of $(\mathbf{P}_{\rho,\Omega})$ with $\rho < 8\pi$, say

$$\Gamma_{8\pi} = \{(\rho, u_\rho), \rho \in (0, 8\pi)\},$$

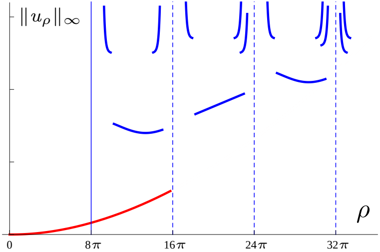
we have only two possibilities.

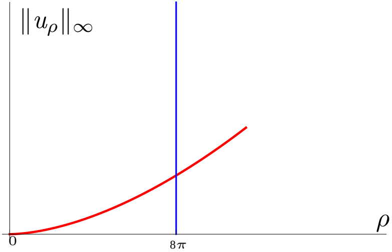
(**I**) either $\mathbf{\Gamma}_{8\pi}$ **blows up from the left at $\rho = 8\pi$ and $(\mathbf{P}_{\rho,\Omega})\big|_{\rho=8\pi}$ has no solution** (which happens if and only if H_1 has a unique and nondegenerate maximum point p_1 and $D_\Omega(p_1) \leq 0$). In this case **Ω is said to be a domain of type I/of first kind**.



The sign of $\rho - 8\pi$, $h \equiv 1$.

(**II**) or $\mathbf{\Gamma}_{8\pi}$ **can be continued to a smooth branch** $\mathbf{\Gamma}_{\mu}$**,** $\mu > 8\pi$**, which crosses the line** $\rho = 8\pi$ **and the solutions of** $(\mathbf{P}_{\rho,\Omega})$ **are nondegenerate for any** $\rho < \mu$ (which happens if and only if there is a maximum point of H_1, say p_1, such that $D_\Omega(p_1) > 0$). In this case Ω **is said to be a domain of type II/of second kind**.



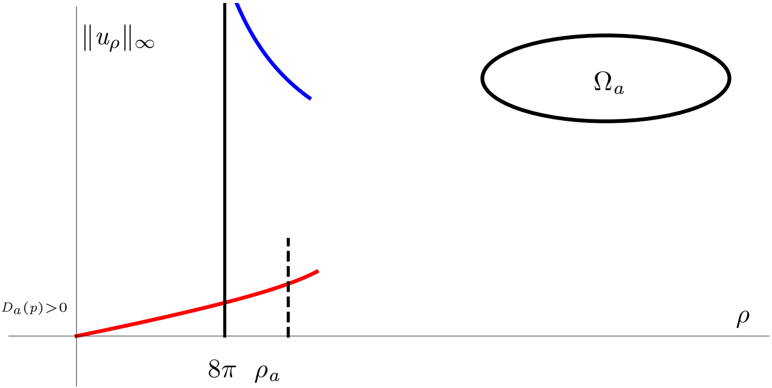


Ω simply connected: [CL2]
The topological degree vanishes if $\rho > 8\pi$!

Supercritical solutions when the degree vanishes

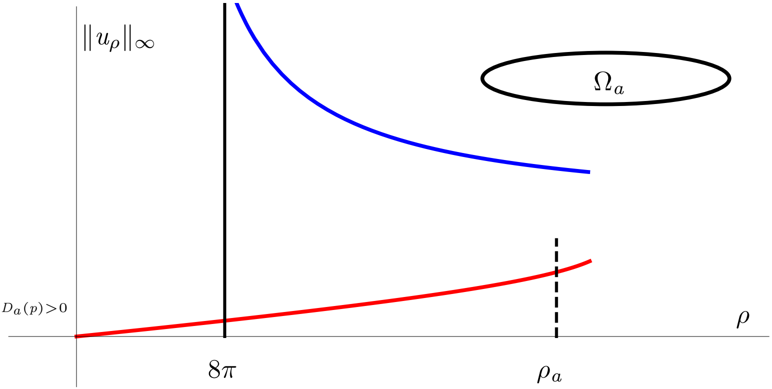
Let Ω_a be a canonical ellipse with semiaxis lenght 1 and a and put $h \equiv 1$. The degree for $\rho > 8\pi$ vanishes, but **[BdM]** as $a \to 0^+$, then Ω_a is of type II and we can find $\rho_a \to +\infty$ such that the following holds.

Supercritical solutions when the degree vanishes



Lower branch = local minimizers of J_ρ.
Upper "branch" = mountain pass solutions.

Supercritical solutions when the degree vanishes



Lower branch = local minimizers of J_ρ.
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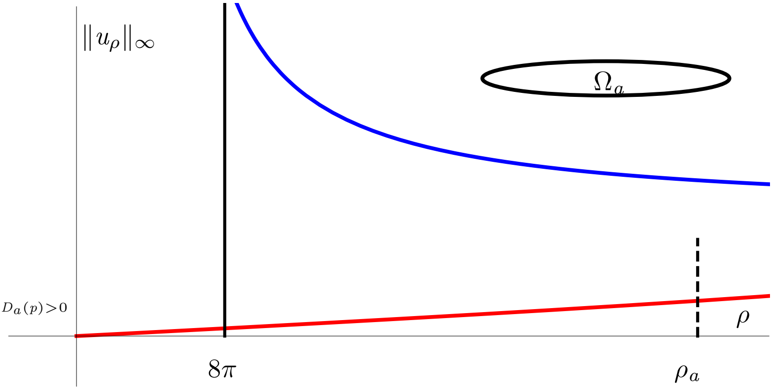
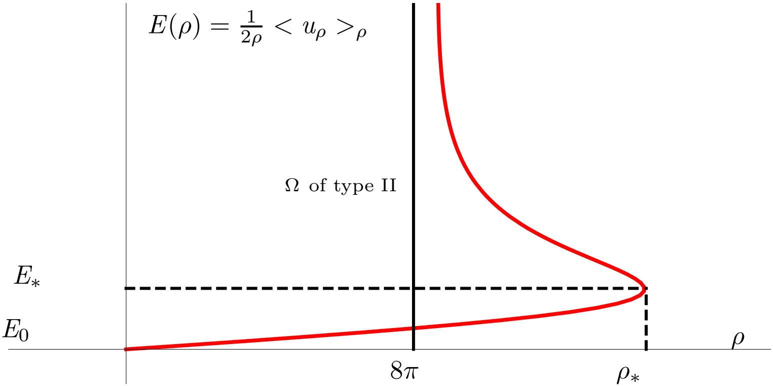


Lower branch = local minimizers of J_ρ.
Upper "branch" = mountain pass solutions.

Supercritical solutions when the degree vanishes

It is shown in **[B]**, under rather natural assumptions based on the dual **microcanonical variational principle** (**entropy maximization at fixed energy** **[CLMP]**), that if Ω is a strictly starshaped domain of type II ($h \equiv 1$), then the branch of solutions crossing 8π (say Γ_∞), takes the form:



Supercritical solutions when the degree vanishes

The underlying idea in **[B]** is to parametrize solutions not with ρ, but in terms of the natural thermodynamic independent variable, which is the **energy**,

$$E(\rho) = \frac{1}{2\rho} < u_\rho >_\rho = \frac{1}{2} \left(\int\limits_\Omega \int\limits_\Omega \frac{e^{u_\rho(x)}}{\int\limits_\Omega e^{u_\rho}} G(x,y) \frac{e^{u_\rho(y)}}{\int\limits_\Omega e^{u_\rho}} dxdy \right),$$

where $\rho = \rho(E)$ becomes the Lagrange multiplier relative to the energy constraint. Following this physical point of view, it turns out that the natural spectral theory is not the standard one, but rather the one whose eigenvalue ($\sigma = \sigma(\rho)$) equation is:

$$-\Delta\phi + \rho \frac{e^{u_\rho(x)}}{\int\limits_\Omega e^{u_\rho}} (\phi - < \phi >_\rho) = \sigma(\rho) \frac{e^{u_\rho(x)}}{\int\limits_\Omega e^{u_\rho}} (\phi - < \phi >_\rho),$$

with $\phi \in H_0^1(\Omega)$.

Supercritical solutions when the degree vanishes

In this modified spectral setting, it can be shown that if along a branch of solutions **the first eigenvalue $\sigma_1(\rho)$ is strictly positive, then $E(\rho)$ is strictly increasing**. This monotonicity property plays a crucial role in showing that the branch Γ_∞ is the branch of entropy maximizers (thermodynamic equilibrium states).

However in **[B]** also some **spectral assumptions** are made to prevent bifurcation and to guarantee monotonicity in the **high energy regime where $\sigma_1 \leq 0$**, which are hard to verify in general. It turns out that all the assumptions in **[B]** can be removed, at least for high values of E, by a series of results of independent interest **[BJLY1-2]** which answer to **[Q3], [Q4]**.

Uniqueness of m-bubbling sequences

Theorem 2 [BJLY1]

Let $u_n^{(1)}$, $u_n^{(2)}$ be a pair of m-bubbling sequences of $(\mathbf{P}_{\rho,\Omega})$ (or of $(\mathbf{P}_{\rho,\Sigma})$) with blow up points "$\underline{\mathbf{p}}$" and $\rho_n^{(1)} = \rho_n^{(2)}$. Suppose that:
- $\det(D^2 H_m(\underline{\mathbf{p}})) \neq 0$;
- either $\Lambda_\Omega(\underline{\mathbf{p}}) \neq 0$ or $D_\Omega(\underline{\mathbf{p}}) \neq 0$.

Then, for any n large enough, $u_n^{(1)} \equiv u_n^{(2)}$.

Non degeneracy of m-bubbling sequences

Theorem 3 [BJLY2]

Let u_n be an m-bubbling sequence of $(\mathbf{P}_{\rho,\Omega})$ (or of $(\mathbf{P}_{\rho,\Sigma})$) with blow up points "$\underline{\mathbf{p}}$". Suppose that:
- $\det(D^2 H_m(\underline{\mathbf{p}})) \neq 0$;
- either $\Lambda_\Omega(\underline{\mathbf{p}}) \neq 0$ or $D_\Omega(\underline{\mathbf{p}}) \neq 0$.

Then, for n large enough, the linearized equation at u_n admits only the trivial solution.

The proofs of Theorems 2 and 3 are based on the estimates in **[CL1]**, Theorem 1 and on the analysis of some subtle Pohozaev-type identities, as recently introduced in:
[LY] C.S. Lin, S. Yan, *On the Chern-Simons-Higgs equation: Part II, local uniqueness and exact number of solutions.*

Monotonicity of $\rho = \rho(E)$ for large E on convex domains of type II

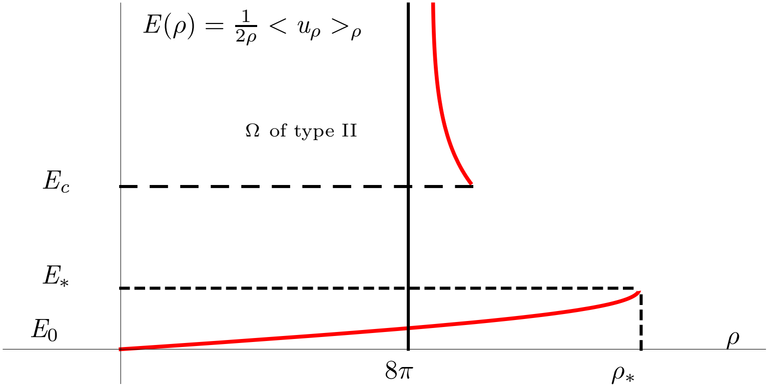
Theorem 4 [BJLY2]

Let Ω be a convex domain of type II. Then there exists $E_c > 0$ and $\rho_c > 8\pi$, such that:
(**i**) solutions of $(\mathbf{P}_{\rho,\Omega})$ whose energy satisfy $E \geq E_c$ form a smooth branch $\Gamma_c = \{(\rho, u_\rho),\ \rho \in (8\pi, \rho_c]\}$;
(**ii**) along Γ_c solutions can be parametrized by the energy $\rho = \rho(E)$, $E \in [E_c, +\infty)$, where ρ is smooth, strictly decreasing for $E \in [E_c, +\infty)$ and $\rho(E) \to 8\pi^+$, as $E \to +\infty$.

We remark that this is also a statement of uniqueness of solutions as parametrized by the energy.

Corollary

Let Ω be a convex domain of type II. Then there exists $E_c > 0$ such that for any $E \geq E_c$ there exists a unique solution of $(\mathbf{P}_{\rho,\Omega})$ whose energy is E.



The monotonicity of the lower branch (local minimizers) has been proved in [B]. Indeed, ρ_* is the "first" value along the branch where σ_1 vanishes.

Please observe that uniqueness fails for fixed $\rho > 8\pi$.

Monotonicity of $\rho = \rho(E)$ for large E on convex domains of type II

In **[BJLY2]** Theorem 4 is stated in a different form, whose concern is with respect to the underlying microcanonical variational principle. The interest in that formulation comes from the surprising thermodynamic behaviour of the vorticity in the supercritical regime, as discussed in **[B]**. Indeed, from the physical point of view, the fact that $\rho(E)$ is decreasing means that we have a negative specific heat state. In other words, **unlike classical thermodynamic systems, the temperature decreases as the energy increases**.

The statement presented here is the analytical version of that result and is mainly concerned with the description of the bifurcation diagram.

The argument is not trivial since, even with the aid of the estimates in **[CL1]** as refined in **[CCL]** for $m = 1$ and Theorems 2,3, it is still hard to establish with a direct computation the **monotonicity** of $\rho(E)$ as a function of E. The workaround we have found goes as follows.

STEP 1

Let u_n be any sequence of solutions of $(\mathbf{P}_{\rho,\Omega})\big|_{\rho=\rho_n}$ with $E = E(\rho_n) \to +\infty$. By using: the Pohozaev identity, the fact that u_n is uniformly bounded near $\partial\Omega$ (moving plane) and that the m-vortex Hamiltonian on a convex domain has no critical points for $m \geq 2$ (M. Grossi, F. Takahashi **J.F.A. (2010)**) then we conclude that, possibly along a subsequence, u_n is a **1-point blow up sequence, whose blow up point is the unique and non degenerate (since Ω is convex) critical point of H_1, say $p \in \Omega$**.

In particular, by using the **characterization of domains of type II** **[CCL]**, **[BLin]** we conclude that $D_\Omega(p) > 0$ **and that** $\rho_n \to 8\pi^+$.

STEP 2

We think at u_ρ as $u^{(\varepsilon)}$, where $u^{(\varepsilon)}$ is a solution of the Gel'fand problem $-\Delta u^{(\varepsilon)} = \varepsilon^2 e^{u^{(\varepsilon)}}$, in Ω $u^{(\varepsilon)} = 0$ on $\partial\Omega$, where

$$\rho_\varepsilon = \varepsilon^2 \int_\Omega e^{u^{(\varepsilon)}}.$$

Therefore, for each $(\varepsilon, u^{(\varepsilon)})$ we have a solution of $(\mathbf{P}_{\rho,\Omega})$ with $\rho = \rho_\varepsilon$ and $u_\rho = u^{(\varepsilon)}$. Since Ω is convex, then p is the unique and non degenerate maximum point of H_1. In this situation, by known results about the Gel'fand problem (T. Suzuki, **L.N.M. 1540 (1993)**, M. Grossi, H. Ohtsuka, T. Suzuki, **Adv. Diff. Eq. (2011)**) there exists a smooth and non degenerate branch $(\varepsilon, u^{(\varepsilon)})$, $\varepsilon \in (0, \varepsilon_c]$ of solutions of the Gel'fand problem which makes 1-point blow up, that is $\rho_\varepsilon \to 8\pi$, as $\varepsilon \to 0^+$.

STEP 3

Since we know that $D_\Omega(p) > 0$, then $\rho_\varepsilon \to 8\pi^+$, as $\varepsilon \to 0^+$.
As a consequence, by using **Theorem 2**, then we can prove that the (smooth) map $\varepsilon \mapsto \rho_\varepsilon$ **is strictly increasing in** $(0, \varepsilon_c]$. **Please observe that on B_1 this map is strictly decreasing.** Therefore is well defined **the inverse map $\varepsilon = \varepsilon(\rho)$ (which is a priori only continuous and differentiable a.e.)** and $\rho_c = \lim\limits_{\varepsilon \nearrow \varepsilon_c} \rho_\varepsilon > 8\pi$.

At this point, by using **STEP 1 and Theorem 2**, and taking a smaller $\rho_c > 8\pi$ if necessary, then it is not difficult to see that, for E_c large enough, any solution of $(\mathbf{P}_{\rho,\Omega})$ whose energy satisfy $E \geq E_c$ takes the form,

$$u_\rho = \left. u^{(\varepsilon)} \right|_{\varepsilon = \varepsilon(\rho)}, \ \rho \in (8\pi, \rho_c].$$

Then **Theorem 3** implies that (ρ, u_ρ), $\rho \in (8\pi, \rho_c]$ is a smooth branch, which proves $(\mathbf{i})$.

Idea of the proof of Theorem 4

Since ρ_ε is smooth and strictly increasing in $(0, \varepsilon_c]$ and since $\rho_\varepsilon \to 8\pi^+$ as $\varepsilon \to 0^+$, then it is enough to prove that $\varepsilon = \varepsilon(E)$ is a smooth and strictly decreasing function of E in $[E_c, +\infty)$ with $\varepsilon(E) \to 0^+$ as $E \to +\infty$. Indeed, we can prove that $E = E_\varepsilon$ is a smooth function of ε, with $\dfrac{dE_\varepsilon}{d\varepsilon} < 0$ in $(0, \varepsilon_c]$ and $E_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. This is done by observing that,

$$E_\varepsilon = \frac{\varepsilon^2}{2\rho_\varepsilon^2} \int\limits_\Omega e^{u^{(\varepsilon)}} u^{(\varepsilon)},$$

is smooth. In particular, by using basics facts about blow up solutions we see that $E_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. Finally, by taking the derivatives of E_ε and after some integration by parts with a suitable test function in the Gel'fand equation, then we conclude that $\dfrac{dE_\varepsilon}{d\varepsilon} \to -\infty$ as $\varepsilon \to 0^+$, which proves (**ii**).

Idea of the proof of Theorem 2

Arguing by contradiction, let $v_n^{(j)} = u_n^{(j)} - \log\left(\int\limits_\Omega he^{u_n^{(j)}}\right)$, $j = 1, 2$, then the normalized difference,

$$\xi_n = \frac{v_n^{(1)} - v_n^{(2)}}{\|v_n^{(1)} - v_n^{(2)}\|_\infty},$$

is a solution of the problem,

$$-\Delta \xi_n = \rho_n h(x) c_n(x) \xi_n, \text{ in } \Omega, \quad \xi_n = d_n \text{ on } \partial\Omega,$$

where

$$c_n(x) = \frac{e^{v_n^{(1)}} - e^{v_n^{(2)}}}{v_n^{(1)} - v_n^{(2)}}, \quad \int\limits_\Omega h(x) c_n(x) \xi_n = 0,$$

and d_n is a constant satisfying $|d_n| \leq 1$.

By using the estimates at hand, we wish to prove that $\xi_n \to 0$ uniformly in $\overline{\Omega}$, which is of course a contradiction to $\|\xi_n\|_\infty = 1$. However the proof is not straightforward.

STEP 1: $\xi_n(x) = -b_0 + \mathrm{o}(1)$, in $\overline{\Omega} \setminus \cup_i^m B_r(p_i)$, for some $b_0 \in [-1, 1]$. This is a consequence of the estimates in **[L]**.
Major problem: it turns out that this constant b_0 is closely related to the radial part of ξ_n near the blow up points! To prove that $b_0 = 0$ is far from trivial.

STEP 2: Let $\lambda_{n,i}^{(j)} = \max\limits_{B_r(p_i)} v_n^{(j)}$. We need to prove that $\left|\lambda_{n,i}^{(1)} - \lambda_{n,i}^{(2)}\right|$ is very small and this is not at all obvious: it can be done by using **the estimates in [CL1], $\rho_n^{(1)} = \rho_n^{(2)}$, the fact that $\underline{\mathbf{p}}$ is a non degenerate critical point of H_m and that either $\Lambda_\Omega(\underline{\mathbf{p}}) \neq 0$ or $D_\Omega(\underline{\mathbf{p}}) \neq 0$,**

STEP 3: By using **STEP 2** we can prove that for any fixed $R > 0$, along a subsequence, in a shrinking disk of radius $R\delta_{n,i} \to 0$, where $\delta_{n,i} = e^{-\frac{\lambda^{(1)}_{n,i}}{2}}$, the normalized difference ξ_n converges in C^2-norm, to a linear combination of bounded solutions of the linearized Liouville equation $-\Delta\phi - \frac{8}{(1+|x|^2)^2}\phi = 0$ in $\mathbb{R}^2$, that is, as $n \to +\infty$,

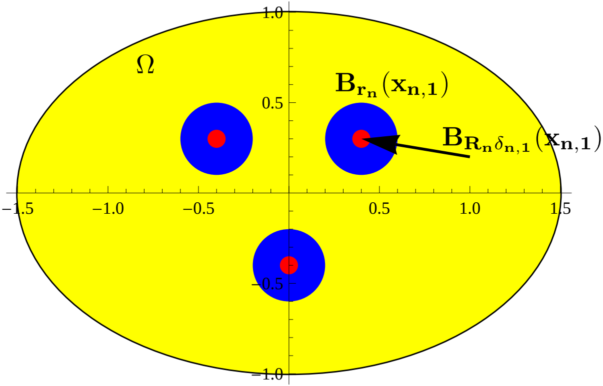
$$\xi_n(x) - b_{i,0}\phi_{i,0}(x) - b_{i,1}\phi_{i,1}(x) - b_{i,2}\phi_{i,2}(x) \to 0, \;\; \textbf{in} \;\; C^2(\overline{B_{R\delta_{n,i}}(p_i)}),$$

where $\phi_{i,0}$ is radial, while $\phi_{i,1}, \phi_{i,2}$ are non radial.

Idea of the proof of Theorem 2

$0 < R\delta_{n,i} << r_n \to 0^+$

blue region = neck region = $\cup_i B_{r_n}(x_{n,i}) \setminus B_{R\delta_{n,i}}(x_{n,i})$



It turns out however that we are still far from our goal, since, even with the estimates obtained so far, it is not easy to prove that **all those coefficients vanish**. Moreover, even if we would succeed in doing this step, we would still miss the **neck region**. A solution to these problems has been suggested in **[LY]** in the context of the analysis of the Chern-Simons-Higgs model, which leads to a different elliptic problem on the flat two torus. The analysis in our case presents some extra difficulties.

Idea of the proof of Theorem 2

STEP 4: $b_{i,0} = b_0,\ \forall\, i = 1, \cdots, m$, which is obtained as a consequence of a rather subtle and long evaluation which uses **STEP 2** and the Gauss-Green formula (in case $\left|\lambda_{n,i}^{(1)} - \lambda_{n,i}^{(2)}\right| = \mathrm{o}\left(\frac{1}{\lambda_{n,1}^{(1)}}\right)$ as in [**LY**]) or the Green formula if $\frac{1}{C\lambda_{n,1}^{(1)}} \leq \left|\lambda_{n,i}^{(1)} - \lambda_{n,i}^{(2)}\right| \leq \frac{C}{\lambda_{n,1}^{(1)}}$.

In particular we obtain a refinement about the estimate in the neck region, which shows that in fact the radial mean of ξ_n, say ξ_n^*, satisfies $\xi_n^* = -b_0 + \mathrm{o}(1)$ in $\Omega \setminus \cup_i B_{R\delta_{n,i}}(x_{n,i})$.

STEP 5: **This is the more subtle part of the proof, where one shows that,**

$$(\mathbf{5}) \quad b_0 = 0;$$

$$(\mathbf{6}) \quad b_{i,1} = b_{i,2} = 0, \ \forall\, i = 1, \cdots, m.$$

In **[LY]** this is proved by the analysis of two carefully defined Pohožaev-type identities. The underlying idea is to apply, for fixed $i \in \{1, \cdots, m\}$, the Pohožaev "argument" in $B_r(x^{(1)}_{n,i})$, to a (very) carefully defined combination of $w^{(j)}_{n,i}(x) := v^{(j)}_n(x) - \gamma_{n,i}(x)$, where $\gamma_{n,i}$ is an harmonic m-vortex like term, so that for n large, and **after summing up all the expressions obtained in this way**, then some subtle cancellation occurs. In particular, besides the vanishing of $\int\limits_{\Omega} h(x)c_n(x)\xi_n(x)dx$, one has to use $\Lambda_\Omega(\underline{\mathbf{p}}) \neq 0$ or $D_\Omega(\underline{\mathbf{p}}) \neq 0$, which yields $(\mathbf{5})$, and $\det(D^2 H_m(\underline{\mathbf{p}})) \neq 0$ which yields $(\mathbf{6})$.

As a consequence, **the maximum point is eventually trapped in the neck region where $\xi_n^* = -b_0 + \mathrm{o}(1)$ and $b_0 = 0$, which is shown to be impossible by a blow up argument.**

Uniqueness of m-bubbling sequences for the Gel'fand equation.

The argument used in the proof of Theorems 2 and 3 solves another long standing open problem about the Gel'fand equation [**BJLY3**].

Theorem 4 [BJLY3]

Let $u_n^{(1)}$, $u_n^{(2)}$ be a pair of m-bubbling sequences of the Gel'fand problem with blow up points "$\underline{\mathbf{p}}$" and $\varepsilon_n^{(1)} = \varepsilon_n^{(2)}$. Suppose that:
- $\det(D^2 H_m(\underline{\mathbf{p}})) \neq 0$;

Then, for any n large enough, $u_n^{(1)} \equiv u_n^{(2)}$.

- Hopefully we will be able to prove uniqueness and non degeneracy of bubbling solutions blowing up at singular points, $h(x) \simeq |x|^{2\alpha}$, with $\alpha > -1, \alpha \notin \mathbb{N}$.

To prove (**5**) one uses $\nabla(w_{n,i}^{(1)} + w_{n,i}^{(2)}) \cdot (x - x_{n,i}^{(1)})$ as test function in the equation satisfied by $w_{n,i}^{(1)} - w_{n,i}^{(2)}$, then $\nabla(w_{n,i}^{(1)} - w_{n,i}^{(2)}) \cdot (x - x_{n,i}^{(1)})$ as test function in the equation satisfied by $w_{n,i}^{(1)} + w_{n,i}^{(2)}$, integrate by parts and sum the identities obtained. This particular combination finally "kills" the terms arising from the non radial modes.
After a very lengthy and subtle evaluation one gets, $\Lambda_\Omega(\underline{\mathbf{p}})b_0 = \mathrm{o}_n(1)$ or $D_\Omega(\underline{\mathbf{p}})b_0 = \mathrm{o}_n(1)$, which yields (**5**).

To prove (**6**) one uses $\partial_\ell w_{n,i}^{(1)}$ as test function in the equation satisfied by ξ_n, then $\partial_\ell \xi_n$ as test function in the equation satisfied by $w_{n,i}^{(1)}$, integrate by parts and sum the identities obtained. This particular combination finally "kills" the terms arising from the radial modes.
After a very lengthy and subtle evaluation one gets,
$D^2 H_m(\underline{\mathbf{p}}) \cdot \vec{b} = \mathrm{o}_n(1)$, where $\vec{b} = (b_{1,1}, b_{2,1}, \cdots, b_{1,m}, b_{2,m})$, which yields (**6**).

Appendix

The Pohozaev-type identity which yields to (**5**),

$$\frac{1}{2}\int_{\partial B_r(x^{(1)}_{n,i})} r < Dw^{(1)}_{n,i} + Dw^{(2)}_{n,i}, D\xi_n > \mathrm{d}\sigma$$

$$-\int_{\partial B_r(x^{(1)}_{n,i})} r < \nu, D(w^{(1)}_{n,i} + w^{(2)}_{n,i}) >< \nu, D\xi_n > \mathrm{d}\sigma$$

$$= \int_{\partial B_r(x^{(1)}_{n,i})} r\rho_n h(x) \frac{(e^{v^{(1)}_{n,i}} - e^{v^{(2)}_{n,i}})}{\|w^{(1)}_{n,i} - w^{(2)}_{n,i}\|_\infty} \mathrm{d}\sigma$$

$$-\int_{B_r(x^{(1)}_{n,i})} \frac{\rho_n h(x)(e^{v^{(1)}_{n,i}} - e^{v^{(2)}_{n,i}})}{\|w^{(1)}_{n,i} - w^{(2)}_{n,i}\|_\infty}(2+ < D(\log h + \gamma_{n,i}), x - x^{(1)}_{n,i} >)\mathrm{d}x.$$

The Pohozaev-type identity which yields to (**6**),

$$\int_{\partial B_r(x_{n,i}^{(1)})} < \nu, D\xi_n > D_\ell w_{n,i}^{(2)} \mathrm{d}\sigma + \int_{\partial B_r(x_{n,i}^{(1)})} < \nu, D w_{n,i}^{(2)} > D_\ell \xi_n \mathrm{d}\sigma$$

$$- \frac{1}{2} \int_{\partial B_r(x_{n,i}^{(1)})} < D(w_{n,i}^{(1)} + w_{n,i}^{(2)}), D\xi_n > \frac{\left(x - x_{n,i}^{(1)}\right)_\ell}{|x - x_{n,i}^{(1)}|} \mathrm{d}\sigma$$

$$= - \int_{\partial B_r(x_{n,i}^{(1)})} \rho_n h(x) \frac{(e^{v_{n,i}^{(1)}} - e^{v_{n,i}^{(2)}})}{\|v_{n,i}^{(1)} - v_{n,i}^{(2)}\|_\infty} \frac{\left(x - x_{n,i}^{(1)}\right)_\ell}{|x - x_{n,i}^{(1)}|} \mathrm{d}\sigma$$

$$+ \int_{B_r(x_{n,i}^{(1)})} \rho_n h(x) \frac{(e^{v_{n,i}^{(1)}} - e^{v_{n,i}^{(2)}})}{\|v_{n,i}^{(1)} - v_{n,i}^{(2)}\|_\infty} D_\ell(\log h + \gamma_{n,i}) \mathrm{d}x, \ \ \ell = 1, 2.$$

Thank you very much for your attention!