Sparse Approximation for Nonlinear Dynamics and Stationary Processes

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joint work with

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Problem Set-up

- Given: (possibly noisy or corrupted) samples of a nonlinear continuous function $\mathbf{y} = \mathbf{f}(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^n$

$$\left(\mathbf{x}^1, \mathbf{y}^1\right), \ldots, (\mathbf{x}^m, \mathbf{y}^m)$$

- Goal: recover the underlying equation $\mathbf{f} = (f_1, f_2, \ldots, f_n)$

- Suppose $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ are multivariate polynomials of maximal degree p:

$$f_k(\mathbf{x}) = \sum_{\alpha_1 + \cdots + \alpha_d \leq p} c_\alpha^k x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}.$$

- Goal is then to recover polynomial coefficients $\{c_\alpha^k\}_{k,\alpha}$

Problem Set-up (cont'd)

Problems of interest:

- Nonlinear dynamical systems with bifurcation
- High-dimensional nonlinear dynamical systems
- Chaotic systems in 3D with corrupted data
- Stationary processes (identically distributed + concentration inequality)

Main ideas: sparse optimization + compressed sensing

Problem Set-up (cont'd)

- Learning nonlinear dynamics:
  - Nonlinear dynamical systems with bifurcation
  - High-dimensional nonlinear dynamical systems
  - Chaotic systems in 3D with corrupted data
- Given m samples (possibly noisy or corrupted) of snapshots:

$$(\mathbf{x}(t_1), \dot{\mathbf{x}}(t_1)), \ldots, (\mathbf{x}(t_m), \dot{\mathbf{x}}(t_m))$$

- Goal: learn the multivariate polynomial $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$ s.t.

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=t_i} = \mathbf{f}(x(t_i)), \quad i = 1, \ldots, m.$$

Example 1: Nonlinear Systems with Bifurcation

- Given multiple data sets that follow the same physical law, what can we say about its governing equation? For example,

$$\begin{cases} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= -x_1 x_3 + (24 - 4\lambda)x_1 + \lambda\, x_2 \\ \dot{x}_3 &= x_1 x_3 - \dfrac{8}{3}x_3, \end{cases}$$

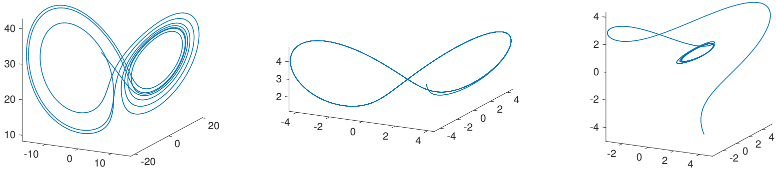

Figure: State space plots for $\lambda = -1$ (left), $\lambda = 7.075$ (middle), and $\lambda = 7.73$ (right), where the dynamics are chaos, a pitchfork bifurcation, and limit cycles, respectively.

Example 2: High-Dimensional Nonlinear Systems

- **Lorenz 96:** a canonical family of ODEs for approximating dynamics of atmosphere:

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = x_{k+1}x_{k-1} - x_{k-2}x_{k-1} - x_k + F, \quad k = 1, \cdots, d,$$

  where $x_{-1} = x_{d-1},\ x_0 = x_d$, and $x_{d+1} = x_1$ and F is a forcing constant.

- Finite difference discretization of many PDEs with applications range from population dynamics to combustion physics.
  - For example, Fisher's equation can be written as

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = x_{k+1} - 2x_k + x_{k-1} + \gamma(x_k - x_k^2), \quad k = 1, \cdots, d.$$

"Kernel Trick" to Linearize Problem

- Form data and velocity matrices from given snapshots:

$$X = \begin{bmatrix} | & \cdots & | \\ X_1 & \cdots & X_d \\ | & \cdots & | \end{bmatrix} = \begin{bmatrix} x_1(t_1) & \cdots & x_d(t_1) \\ x_1(t_2) & \cdots & x_d(t_2) \\ \vdots & \cdots & \vdots \\ x_1(t_m) & \cdots & x_d(t_m) \end{bmatrix}_{m \times d}, \quad \dot{X} = \begin{bmatrix} | & \cdots & | \\ \dot{X}_1 & \cdots & \dot{X}_d \\ | & \cdots & | \end{bmatrix}_{m \times d}$$

- Construct *dictionary matrix* from data:

$$\Phi = \begin{bmatrix} | & | & & | & | & | & & | & \\ 1 & X_1 & \cdots & X_d & X_1^2 & X_1 X_2 & \cdots & X_d^2 & \cdots \\ | & | & & | & | & | & & | & \end{bmatrix}_{m \times N}$$

where $N = \binom{p+d}{d}$ is number of multivariate monomials of degree $\leq p$.

"Kernel Trick" to Linearize Problem

- Recovering poly. coefficients $\mathcal{C} = [c_\alpha^1, c_\alpha^2, \dots c_\alpha^d]_{|\alpha| \le p} \in \mathbb{R}^{N \times d}$ as solution to the linear inverse problem [1]

$$\dot{X} = \Phi\, \mathcal{C}.$$

- In the presence of measurement errors (in data, time-derivative approximations,...), the problem becomes

$$\dot{X} = \Phi\, \mathcal{C} + \mathcal{E}.$$

- We will investigate properties of the matrix Φ in various type of input data.

[1] Brunton, Proctor, and Kutz, "Discovering governing equations from data by sparse identification of nonlinear dynamical systems", *PNAS* 2016.

Sparse Optimization and Bifurcation

Nonlinear Dynamics with Bifurcation

- Consider the Lorenz system with a single bifurcation parameter λ:

$$\begin{cases} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= -x_1 x_3 + (24 - 4\lambda)x_1 + \lambda\, x_2 \\ \dot{x}_3 &= x_1 x_3 - \dfrac{8}{3} x_3, \end{cases}$$

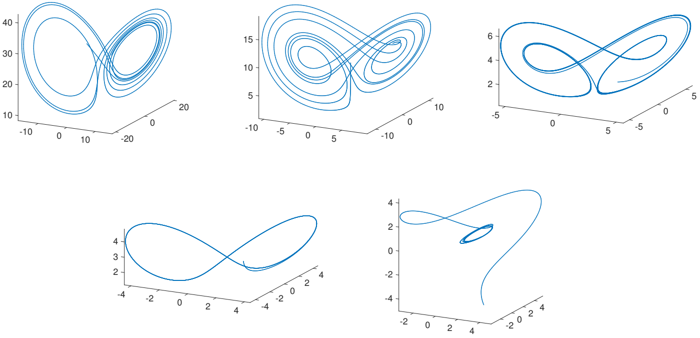

Figure: State space plots for different λ with $dt = 0.005$.

Sparse Group Penalization

- Denote the coefficient matrix

$$C_j = \begin{bmatrix} | & | & | & | \\ c_j^{(1)} & c_j^{(2)} & \dots & c_j^{(m)} \\ | & | & | & | \end{bmatrix}_{\bar{n}\times m}$$

- Observe: The vectors $c_j^{(i)}$ have the same support set for all i!
- Solve the following group-sparse optimization problem:

$$\boxed{\min_{C_j} \sum_{i=1}^{m} \|\Phi^{(i)} c_j^{(i)} - V_j^{(i)}\|_2^2 + \gamma \|C_j\|_{2,0}}$$

where the $\ell^{2,0}$ penalty is defined as:

$$\|A\|_{2,0} := \# \left\{ k : \left(\sum_{\ell} |a_{k,\ell}|^2 \right)^{1/2} \neq 0 \right\}.$$

Numerical Method

- Proximal descent method + Hard-iterative thresholding
- Step 1: Gradient descent

$$\left(\widetilde{c^{(i)}}\right)^{k+1} = \left(c^{(i)}\right)^{k} - (\Phi^{(i)})^{T}\left(\Phi^{(i)}\left(c^{(i)}\right)^{k} - V^{(i)}\right)$$

- Step 2: Hard-iterative thresholding

$$(c^{(i)})^{k+1} = \begin{cases} 0, \quad \text{if} \quad \|row\| < \sqrt{\gamma} \\ \underset{c^{(i)}}{\operatorname{argmin}} \|\Phi^{(i)} c^{(i)} - V^{(i)}\|_2^2 \quad \text{otherwise.} \end{cases}$$

Group Hard-Iterative Thresholding Algorithm

Given: initialization matrix C^0, *tol* and parameters γ.

while $\|C^{k+1} - C^k\|_\infty > tol$ **do**

for $i = 1$ **to** m:

$$\left(\widetilde{c^{(i)}}\right)^{k+1} = \left(c^{(i)}\right)^k - (\Phi^{(i)})^T \left(\Phi^{(i)} \left(c^{(i)}\right)^k - V^{(i)}\right)$$

end for

$$S^{k+1} = \text{supp}\left(H_{\sqrt{\gamma}}\left[\widetilde{c^{(1)}}, \widetilde{c^{(2)}}, \cdots, \widetilde{c^{(m)}}\right]\right)$$

for $i = 1$ **to** m:

$$(c^{(i)})^{k+1} = \underset{c^{(i)}}{\operatorname{argmin}} \|\Phi^{(i)} c^{(i)} - V^{(i)}\|_2^2 \quad \text{s.t.} \ \ \text{supp}(c^{(i)}) \subset S^{k+1}$$

end for

end while

Convergence Guarantees

$$\min_{C} \; F(C) := \sum_{i=1}^{m} \|\Phi^{(i)} c^{(i)} - V^{(i)}\|_2^2 + \gamma \|C\|_{2,0}$$

Theorem

Let C^k be the sequence generated by the proposed numerical scheme, then $F(C^{k+1}) \leq F(C^k)$ and there are subsequences that converge to local minimizers. In addition, if

$$D := diag[\Phi^{(1)}, \ldots, \Phi^{(m)}]$$

is coercive then the sequence C^k converges to a local minimizer.

Convergence Guarantees: General Bound

Proposition (General bound)

Suppose, for each i, $\overline{n} \leq \ell_i$, there exists a subset $S \subset [\ell_i]$ of size $|S| = \overline{n}$ such that $\{X^{(i)}(k, -) \mid k \in S\}$ do not belong to a common algebraic hypersurface of degree $\leq p$.

Convergence Guarantees: General Bound

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Suppose, for each i, $\overline{n} \leq \ell_i$, *there exists a subset* $S \subset [\ell_i]$ *of size* $|S| = \overline{n}$ *such that* $\{X^{(i)}(k, -) \mid k \in S\}$ *do not belong to a common algebraic hypersurface of degree* $\leq p$.

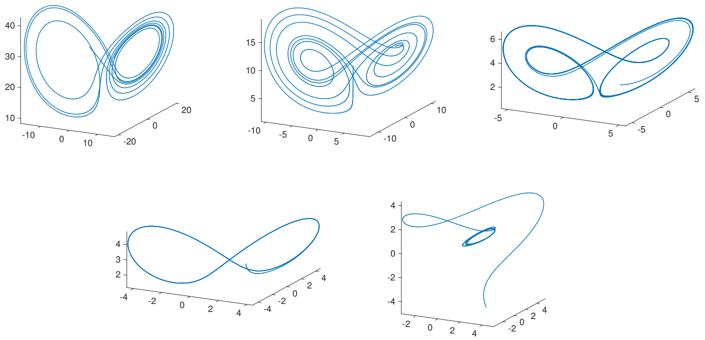
This is a necessary and sufficient condition for the dictionary matrix D *to be full rank: for each* $D^{(i)}$, *there exists a* $\delta_i > 0$ *such that*

$$\inf_u \frac{\|\Phi^{(i)} u\|_2}{\|u\|_2} \geq \delta_i. \tag{1}$$

Numerical Results: Lorenz 3D

- Consider the Lorenz system with a single bifurcation parameter λ:

$$\begin{cases} \dot{x}_1 &= 10(x_2 - x_1) \\ \dot{x}_2 &= -x_1 x_3 + (24 - 4\lambda)x_1 + \lambda\, x_2 \\ \dot{x}_3 &= x_1 x_3 - \dfrac{8}{3} x_3, \end{cases}$$



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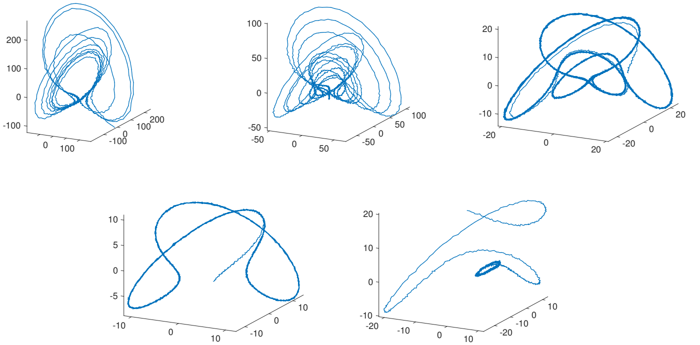

Figure: Noisy velocity space plots corresponding to the data given in Figure 3 with noise level $\sigma_{noise} = 0.5\%$.

Numerical Results: Lorenz 3D

Coeff.	Set 1	Set 2	Set 3	Set 4	Set 5
1	0	0	0	0	0
x_1	28.0232 (28)	5.2104 (5.2)	-3.6068 (-3.6)	-4.2960 (-4.3)	-6.9246 (-6.92)
x_2	-1.0093 (-1.0)	4.6970 (4.7)	6.9020 (6.9)	7.0719 (7.075)	7.7310 (7.73)
x_3	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_1 x_3$	-1.0002 (-1)	-1.0003 (-1)	-0.9989 (-1)	-1.0002 (-1)	-0.9992 (-1)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_3^4	0	0	0	0	0

Recovered coefficients from all five sets for $\frac{dx_2}{dt}$. The true values are highlighted in (red).

Sparse Approximation in High-Dimensional Nonlinear Dynamical Systems

Sparsity-of-Effect Principle

- For high-dimensional systems ($d \gg 1$), system is usually dominated by main effects and first- and second-order interactions.
  - For d large, consider polynomial dictionary only up to degree $p = 2$, then D is $m \times N$ where $N = \binom{2+d}{d} = \frac{(d+1)(d+2)}{2}$.
- In systems of practical interest, low-order interactions also sparse – exploit!
- Reformulate as a basis pursuit problem for an underdetermined system ($m \ll N$):

$$\boxed{\min \|\mathcal{C}\|_1, \quad \text{s.t} \quad \|\dot{X} - D\,\mathcal{C}\| \leq \sigma}$$

where σ represents error in time-derivative approximations.

Sparse Approximation in High-Dimensional Systems

$$\boxed{\min \|\mathcal{C}\|_1, \quad \text{s.t} \quad \|\dot{X} - D\,\mathcal{C}\| \leq \sigma}$$

- Limitation in theory for high-dimensional nonlinear dynamical system compared to rich (but technical) theory for ergodicity/chaos in 3D nonlinear systems.

- In our work, we show that in many scenarios, we can obtain sparse recovery for $\mathcal{C}$.
  - Main idea: random initializations + multiple trajectories.

Sparse Approximation in High-Dimensional Systems

- Given snapshots from K different trajectories:

$$\{\mathbf{x}(t_1,1),\ldots,\mathbf{x}(t_m,1)\},\quad \{\dot{\mathbf{x}}(t_1,1),\ldots,\dot{\mathbf{x}}(t_m,1)\},\ldots$$
$$\{\mathbf{x}(t_1,K),\ldots,\mathbf{x}(t_m,K)\},\quad \{\dot{\mathbf{x}}(t_1,K),\ldots,\dot{\mathbf{x}}(t_m,K)\}$$

- Form dictionary matrix D of size $mK \times N$ and solve for $\mathcal{C}$.

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Theorem (Schaeffer, **T'**, and Ward, 2017)

Assume each component of $f(x) = (f_1(x),\ldots,f_d(x))$ is quadratic and has at most s non-zero polynomial coefficients; the K initializations $\{\mathbf{x}(t_1,1),\ldots,\mathbf{x}(t_1,K)\}$ are drawn i.i.d. uniformly from $[-1,1]^d$; and the number of bursts $K \geq 9c_*\, s \log N \log(\varepsilon^{-1})$.

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$$\{\mathbf{x}(t_1,K),\ldots,\mathbf{x}(t_m,K)\},\quad \{\dot{\mathbf{x}}(t_1,K),\ldots,\dot{\mathbf{x}}(t_m,K)\}$$

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Assume each component of $f(x) = (f_1(x), \ldots, f_d(x))$ is quadratic and has at most s non-zero polynomial coefficients; the K initializations $\{\mathbf{x}(t_1,1),\ldots,\mathbf{x}(t_1,K)\}$ are drawn i.i.d. uniformly from $[-1,1]^d$; and the number of bursts $K \geq 9c_* \, s \log N \log(\varepsilon^{-1})$.

Then with probability $1-\varepsilon$, $\mathcal{C}$ is the unique solution to the ℓ_1-minimization problem:

$$\min \|\mathcal{C}\|_1 \quad \textit{subject to} \quad \dot{X} = D\,\mathcal{C},$$

and recovery is stable with respect to inexact sparsity and robust with respect to additive noise (as from approximating derivatives).

Sparse Approximation in High-Dimensional Systems

- The initializations could be taken to be i.i.d. from other distributions such as Gaussian or Chebyshev distribution
  - Choose appropriate orthonormal monomials w.r.t different distributions: uniform dist. vs Legendre polynomials, Gaussian dist. vs Hermite polynomials, ...
- The reconstruction guarantees can be extended to
  - higher-order polynomial systems (the constant in the theoretical result will increase)
  - other bounded orthonormal basis such as sines and cosines,...
- The basis pursuit problem can be solved using *spgl1*[2], *SpaRSA*[3], or *cvx*.

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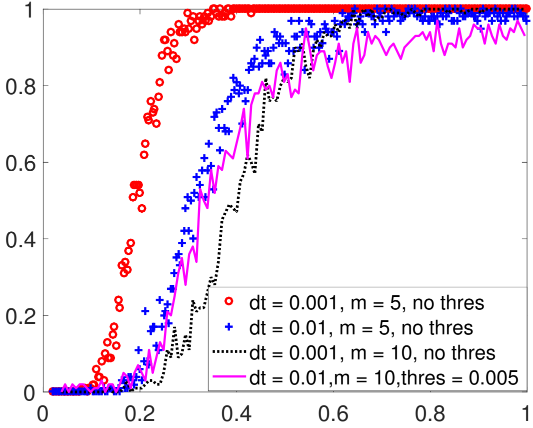
[2] Van Den Berg and Friedlander, "Probing the Pareto frontier for basis pursuit solutions", SIAM Journal on Scientific Computing, 2008.

[3] Wright, Nowark, and Figueiredo, "Sparse reconstruction by separable approximation", IEEE Trans. on Signal Processing, 2009.

Example 1: Lorenz 96 – Phase Transition

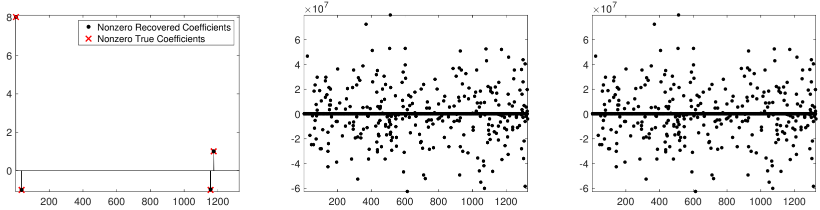
$$\text{Lorenz 96:} \quad \frac{dx_k}{dt} = -x_{k-2}\,x_{k-1} + x_{k-1}\,x_{k+1} - x_k + F, \quad k = 1, \ldots, d$$



Probability of exact recovery vs the undersampling rate K/N with $N = 1326$, $F = 8$.

For dt $= 0.001$, $K = 80$ is needed to achieve 90% prob. of success for both m.

Example 1: Lorenz 96 – Comparison



The coefficients learned from basis pursuit method (left), the least-square algorithm (middle), and the sequential thresholding algorithm (right) for the 35th component of the Lorenz 96 with $d = 50, \mathrm{dt} = 0.001$. The threshold parameter for the last two methods is set to be 0.05.

- The least-square and the sequential thresholding solutions have sparsity $s \gg 10$ and coefficients on the order of 10^7.
- Our solution is 5-sparse in the Legendre basis, when transformed back, it is nearly exact (up to a few significant digits)!

Other Sampling Strategies

- Depend on the degree of prior knowledge about the data or the governing equations, the number of initializations can be reduced.
  - Due to the localization of ODE system discretized from a PDE
    $$K \sim c\, s\, \log(\ell) \log(\varepsilon^{-1}).$$
  - Due to a strong decay of correlations of chaotic systems $\Rightarrow$ Sample both at small time-scale (for time-derivative approximation) and at large-scale (for ergodicity).

Sparse Recovery in Low-Dimensional Nonlinear Dynamical Systems

Corrupted Chaotic Systems

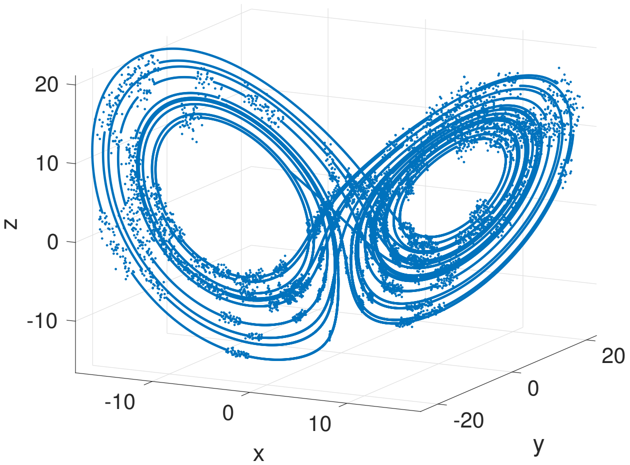


Figure: Lorenz System with 15% Corrupted Data

Problem Formulation

- Under same assumptions as before, now observe data corresponding to the time-Δ map, corrupted by outliers:

$$\mathbf{x}_j^o = \mathbf{x}(t_j) + O_{1,j}, \quad t_j = j\Delta, \quad j = 0, 1, \ldots, m,$$
$$\dot{\mathbf{x}}_j^o = \dot{\mathbf{x}}(t_j) + O_{2,j}, \quad t_j = j\Delta, \quad j = 0, 1, \ldots, m$$

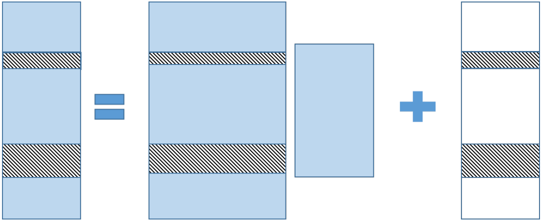
  where $\#\{j : |O_{1,j}| > 0\} + \#\{j : |O_{2,j}| > 0\} \leq s$ for $s < m$ but support is unknown a priori.

- In this setting, given the corrupted data matrices X^o and $\dot{X}^o$, the difference matrix

$$\dot{X}^o - \Phi(X^o)\mathcal{C}$$

  will have at most $2s$ nonzero rows, the locations of which are unknown a priori.

Problem Formulation (cont'd)



- Proposed reconstruction in corrupted data setting: solve the jointly sparse optimization problem

$$(\mathcal{C}, \mathcal{E}) = \operatorname*{argmin}_{\mathcal{C},\mathcal{E}} \|\mathcal{E}\|_{2,1} = \operatorname*{argmin}_{\mathcal{C},\mathcal{E}} \sum_{i=1}^{m} \|\mathcal{E}(i,:)\|_2,$$

$$\text{subject to} \quad \dot{X}^o = \Phi(X^o)\mathcal{C} + \mathcal{E} \quad \text{and} \;\; \mathcal{C} \text{ is sparse.}$$

Numerical Scheme

$$(\mathcal{C}, \mathcal{E}) = \underset{\mathcal{C},\mathcal{E}}{\operatorname{argmin}} \|\mathcal{E}\|_{2,1} = \underset{\mathcal{C},\mathcal{E}}{\operatorname{argmin}} \sum_{i=1}^{m} \|\mathcal{E}(i,:)\|_2,$$

$$\text{subject to} \quad \Phi(X^o)\mathcal{C} + \mathcal{E} = \dot{X}^o \quad \text{and } \mathcal{C} \text{ is sparse.}$$

- The corresponding augmented Lagrangian is of the form

$$(\mathcal{C}, \mathcal{E}, b) = \underset{\mathcal{C},\mathcal{E},b}{\operatorname{argmin}} \sum_{i=1}^{m} \|\mathcal{E}(i,:)\|_2 + \frac{\mu}{2}\|\Phi(X^o)\mathcal{C} + \mathcal{E} - \dot{X}^o + b\|_F^2,$$
$$\text{subject to } \mathcal{C} \text{ is sparse.} \tag{2}$$

- It can be solved via alternating directional method of multipliers (ADMM)/Split Bregman,...

Numerical Scheme (cont'd)

Algorithm

Given: $\mathcal{E}^0, b^0, tol$ and parameters λ, μ.

while $||\mathcal{E}^k - \mathcal{E}^{k-1}||_\infty > tol$ **do**

$$\mathcal{C}^{k+1} = S_h\left((\Phi(X^\circ))^{-1}(\dot{X}^\circ - \mathcal{E}^k - b^k), \lambda\right)$$

$$\mathcal{E}^{k+1} = S_2\left(\dot{X}^\circ - b^k - \Phi(X^\circ)\mathcal{C}^{k+1}, \mu\right)$$

$$b^{k+1} = b^k + \Phi(X^\circ)\mathcal{C}^{k+1} + \mathcal{E}^{k+1} - \dot{X}^\circ$$

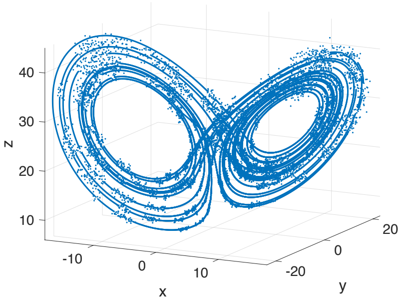
end while

where

$$S_h(u,\gamma) := u \cdot I_{|u|\geq\gamma} = \begin{cases} u & \text{if} \quad |u| \geq \gamma \\ 0 & \text{otherwise,} \end{cases}$$

$$S_2(u_j,\gamma) = \max\left(1 - \frac{1}{\gamma\|u\|_2}, 0\right) u_j, \quad \text{for each row } u_j \text{ of } u.$$

Numerical Results: Lorenz System

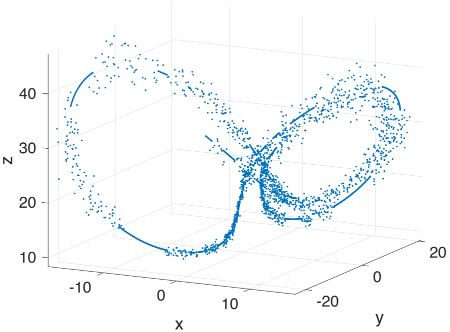


	$\dot{x}$	$\dot{y}$	$\dot{z}$
1	0	0	0
x	-9.9999	27.9995	0
y	9.9999	-0.9999	0
z	0	0	-2.6666
x^2	0	0	0
xy	0	0	0.9999
xz	0	-0.9999	0
y^2	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
z^4	0	0	0

Lorenz system $\dot{x} = -10x + 10y$, $\dot{y} = 28x - y - xz$, $\dot{z} = -2.66z + xy$, with 19.19% corrupted data, 40000 measurements, $\Delta t = 0.0005$. The model recovers the coefficients with 0.0096% error and detect exactly the locations of the outliers after 24 iterations.

Lorenz System - Small Sample Size

- 5000 measurements, $\Delta t = 0.0005$, with 71.89% corruption. The model detects exactly the locations of the outliers and recovers the coefficients with 0.0477% error.



- If $T \leq 2$ (4000 measurements), the scheme doesn't work well
  - It's important to have sufficient amount of measurements.

Lorenz Data with Noise

- Add Gaussian noise to the entire data
- Build the dictionary and approximate the time derivative from noisy + corrupted data

Standard Deviation of Noise	# Times Detect Exactly Outliers (over 100)	Coefficient Error (%)
$0.4\Delta t$	89	$\text{min} = 0.0009, \text{max} = 0.0525$
$0.6\Delta t$	87	$\text{min} = 0.0006, \text{max} = 0.9395$
$0.8\Delta t$	65	$\text{min} = 0.0012, \text{max} = 1.57$

Table: Different noise levels and the recovery results associated with the Lorenz system, $\Delta t = 0.0005$, 40000 measurements, and around 20% corrupted

Reconstruction Guarantee Analysis

Theorem (**T'** and Ward, 2016)

Suppose we observe corrupted measurements of the time-1 map

$$X^{\circ,t} = x^t + \Theta_{1,t}, \quad \dot{X}^{\circ,t} = \dot{\mathbf{x}}^t + \Theta_{2,t}, \quad t = 1, 2, \ldots, m,$$

where $\mathbf{x}^t = x(t)$ is the flow generated by a strongly ergodic vector field whose time-1 map satisfies the Central Limit Theorem. Assume the governing equations are multivariate polynomials of degree at most p.

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where $\mathbf{x}^t = x(t)$ is the flow generated by a strongly ergodic vector field whose time-1 map satisfies the Central Limit Theorem. Assume the governing equations are multivariate polynomials of degree at most p.

There are constants C, C' (depending on $\Lambda, \|\Theta\|_\infty$, and ε) such that if $m \geq CN$ and $s \leq C'm^{.9}$, then with probability exceeding $1 - \varepsilon$ with respect to $\mathbf{x}_0 \sim d\mu$, the polynomial coefficients and locations of the outliers can be exactly recovered as the solution to the ℓ_1-minimization problem

$$\min_{\mathcal{C},\mathcal{E}} \|\mathcal{E}\|_1 \quad \textit{subject to} \quad \Phi(X^\circ)\mathcal{C} + \mathcal{E} = \dot{X}^\circ$$

Sketch of the Proof

Result from statistical properties of Lorenz-like systems:

- Lorenz equations support a compact, connected attractor Λ and the flow $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ admits a *physical* measure μ.
- Central Limit Theorem for geometric Lorenz attractors: Fix $\eta > 0$. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a $C^{1+\eta}$ function, and let $Z \sim \mathcal{N}(0, \sigma^2)$. Then there are constants $C_1 > 0$ and $C_{x,\varphi} \geq 0$ such that

$$\left| \frac{1}{m} \sum_{j=0}^{m-1} \varphi(\mathbf{x}^j) - \int_\Lambda \varphi \, d\mu - \frac{\sigma}{\sqrt{m}} Z \right| \leq C_{x,\varphi} m^{-3/4} (\log(m))^{1/2} (\log\log(m))^{1/4}$$

for μ-almost all $x \in \Lambda$, and $\sigma^2 \leq C_1 \|\varphi\|^2_{C^{1+\eta}}$.

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[4] Tucker, ''The Lorenz attractor exists'', *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 1999.

[5] Arajo, Melbourne, and Varandas, ''Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps'', *Comm. in Mathematical Physics*, 2015.

Sketch of the Proof

$$\min_{\mathcal{C},\mathcal{E}} \|\mathcal{E}\|_1 \quad \text{subject to} \quad \Phi(X^\circ)\mathcal{C} + \mathcal{E} = \dot{X}^\circ \quad (*)$$

Result from compressed sensing[6]:

- Every $(\mathcal{C}, \mathcal{E})$, satisfying $A\mathcal{C} + \mathcal{E} = y$ and $\mathcal{E} \in \mathbb{R}^m$ is s-sparse, is the unique solution to (*) **if and only if** A is *full column rank* and for every $v \in \mathcal{R}(A) \setminus \{0\}$, the following holds

$$\sum_{j=1}^{2s} |v_{(j)}| < \frac{1}{2} \|v\|_1$$

- Generally, only random A shown to reach optimal sparsity level $s \asymp \frac{m}{\log(m)}$
- We will show that this is also in the setting where $A = \Phi$ is constructed via data from certain chaotic systems!

[6] Bandeira, Scheinberg, and Vincent, ''On partial sparse recovery'', IEEE Signal Processing Letters 2013.

- Indeed, we can prove that the matrix $A = [\Phi_{m\times r}; I_{m\times m}]$ satisfies the null space property:
    - For every $w \in \ker A \setminus \{\vec{0}\}$ and every set $S \subset \{1,\dots,m+r\}$ of cardinality s, the following holds

$$\|w_S\|_1 < \frac{1}{2}\|w\|_1.$$

- The central limit theorem for chaotic systems can be replaced by other concentration inequalities.
- Extend the proof's technique of [T. and Ward, 2017] to a wider class of data that are not required to be independent.

Learning Functions from Stationary Processes

- Suppose we observe corrupted measurements

$$\left(\mathbf{U}^{(i)} = \mathbf{X}^{(i)} + \mathbf{\Theta}^{(i)}, \mathbf{Y}^{(i)} = f(\mathbf{X}^{(i)})\right)_{i=1}^{m} \subset \mathbb{R}^d \times \mathbb{R},$$

- Assume

$$f(x_1, \cdots, x_d) = \sum_{|\alpha| = \alpha_1 + \ldots + \alpha_d \le p} c^{\alpha} x_1^{\alpha_1} \ldots x_d^{\alpha_d}, \quad j = 1, \ldots, n,$$

- Let $y = [Y^{(1)} \ldots Y^{(m)}]^T$ then $y = \Phi\, c + e$.
- Adding sparsity constraints to the solution:

$$\min_{c,e} \|c\|_1 + \|e\|_1 \quad \text{subject to} \quad y = \Phi\, c + e.$$

Assumptions:

- $\max_i \|\Theta^{(i)}\|_\infty \leq B_\Theta, \quad \text{and} \quad \max_i \|X^{(i)}\|_\infty \leq B_{\mathbf{X}}$
- Assume the common distribution μ of $\{X^{(i)}\}_{i=1}^m$ is non-degenerated, i.e., $\mu(X^{(1)} \in A) = 1$ implies A contains infinitely many elements.
- $\{X^{(i)}\}_{i=1}^m$ satisfies the following concentration inequality

$$\Pr\left(\left|\sum_{i=1}^m \varphi(\mathbf{X}_i) - m\mathbb{E}[\varphi(\mathbf{X})]\right| \geq \zeta\right) \leq C_1 \exp\left(-\frac{\zeta^2}{C_2\omega_m + C_3\zeta\kappa_m}\right),$$

  for any bounded Borel function φ.
- $\sqrt{\omega_m \log m} + \kappa_m \log m = o(m)$

Theorem (Ho, T', and Ward, 2018)

Under previous assumptions, there are constants C', C'' depending only on B_Θ, B_X, p such that if

$$m \geq C', \quad s \leq C'' m$$

then the polynomial coefficients of f as well as the outlier vector e can be exactly recovered from the unique solution to the ℓ_1-minimization problem

$$\min_{c,e} \|e\|_1 + \|c\|_1 \quad \textit{subject to } e + \Phi c = y.$$

Sparse Recovery for i.i.d. Random Variables

- **Bernstein inequality for i.i.d. random variables** $(\mathbf{X}_i)$ (Modha and Masry, 1996): Suppose $|\psi(X_1) - \mathbb{E}(\psi(X_1))| \leq d_1$ *a.s.*, then we have

$$\Pr\left(\left|\sum_{i=1}^{m} \psi(\mathbf{X}_i) - m\mathbb{E}[\psi(\mathbf{X})]\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{C_2 m + C_3 t}\right)$$

  where

$$C_2 = 2\mathbb{E}(\psi^2(X_1)) - 2(\mathbb{E}(\psi(X_1)))^2,\ C_3 = \frac{2}{3}d_1$$

  and ψ is any bounded Borel function.

- The samples $\{X^{(t)}\}$ satisfy the concentration inequality with $\omega_m = m$ and $\kappa_m = 1$,

$$\sqrt{\omega_m \log m} + \kappa_m \log m = o(m).$$

- So with probability $\left(1 - \frac{1}{m^\delta}\right)$ for some constant $\delta > 0$, m large enough, the associated ℓ_1-minimization problem has a unique solution.

Sparse Recovery for Exponentially Strongly α-Mixing Processes

- Recall: for a stationary stochastic process $\{X_t\}$, define

$$\alpha(s) = \sup_{\substack{-\infty<t<\infty \\ A\in\sigma(X_t^-), B\in\sigma(X_{t+s}^+)}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|$$

- The stochastic process is said to be exponentially strongly α-mixing if

$$\alpha(s) \leq \overline{\alpha}\exp(-c_\alpha s^\beta), \quad s \geq 1,$$

  for some $\overline{\alpha} > 0, \beta > 0$, and $c > 0$, where the constants β and c are assumed to be known.

Sparse Recovery for Exponentially Strongly α-Mixing Processes

- Exponentially strongly α-mixing satisfies the following concentration inequality (Modha and Masry, 1996)

$$\Pr\left(\left|\sum_{i=1}^{m}\psi(\mathbf{X}_i) - m\mathbb{E}[\psi(\mathbf{X})]\right| \geq t\right) \leq C_1 \exp\left(-\frac{t^2}{(C_2 m^2 + C_3 tm)/m_\alpha}\right)$$

  where

$$m_\alpha := \left\lfloor \frac{m}{\lceil (8m/c_\alpha)^{1/(\beta+1)} \rceil} \right\rfloor = \mathcal{O}(m^{\beta/(\beta+1)}),$$

$$C_1 = 2(1+4e^{-2}\overline{\alpha}),\ C_2 = 2\mathbb{E}(\psi^2(X_1)) - 2(\mathbb{E}(\psi(X_1)))^2,\ C_3 = \frac{2}{3}d_1$$

  and ψ is any bounded Borel function.
- The samples $\{X^{(t)}\}$ satisfy the concentration inequality with $\omega_m = m^2/m_\alpha,\ \kappa_m = m/m_\alpha,$

$$\sqrt{\omega_m \log m} + \kappa_m \log m = o(m).$$

- So with probability $\left(1 - \frac{1}{m^\delta}\right)$ for some constant $\delta > 0$, m large enough, the associated ℓ_1-minimization problem has a unique solution.

Conclusions and Future Directions

Conclusions

- Shown that the dictionary matrix generated from the polynomial space satisfies range space property, coercivity,...
- Proved the sparse recovery for high dimensional nonlinear systems, 3D chaotic systems, and stationary process with concentration inequalities
- Presented several numerical examples to validate the proposed methods.

Future directions

- Simulate numerical experiments to validate sparse recovery for stationary processes
- Analyze reconstruct guarantees for other dictionary matrices
- Look for real applications

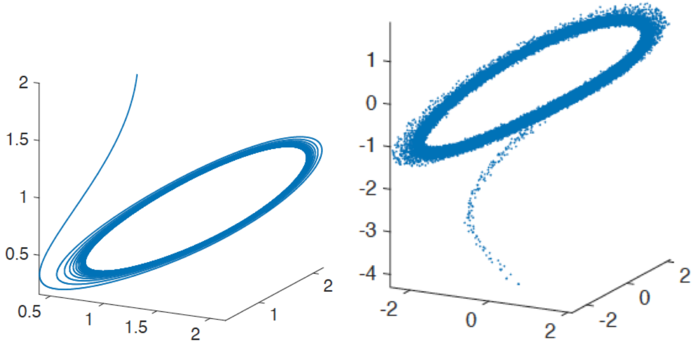

Figure: An example where the state space quickly approaches a limit cycle, which almost stays on a hypersurface of degree 2. The state space is generated from the Lorenz system with $\mu = 7.73$, initialization $U_0 = [1, 1, 2]$, and time step $dt = 0.005$.