An introduction to C*-algebras

Workshop Model Theory and Operator Algebras
BIRS, Banff

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We will denote by $\mathcal{H}$ a complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$, and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators $\mathcal{H} \to \mathcal{H}$. It becomes a Banach algebra with the operator norm.

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Recall

For $a \in \mathcal{B}(\mathcal{H})$, the adjoint operator $a^* \in \mathcal{B}(\mathcal{H})$ is the unique operator satisfying the formula

$$\langle a\xi_1 \mid \xi_2 \rangle = \langle \xi_1 \mid a^*\xi_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{H}.$$

Then the adjoint operation $a \mapsto a^*$ is an involution, i.e., it is anti-linear and satisfies $(ab)^* = b^*a^*$.

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Then the adjoint operation $a \mapsto a^*$ is an involution, i.e., it is anti-linear and satisfies $(ab)^* = b^*a^*$.

Observation

One always has $\|a^*a\| = \|a\|^2$.

Proof: Since $\|a^*\| = \|a\|$ is rather immediate from the definition, "$\leq$" is clear. For "$\geq$", observe

$$\|a\xi\|^2 = \langle a\xi \mid a\xi \rangle = \langle \xi \mid a^*a\xi \rangle \leq \|a^*a\xi\|, \quad \|\xi\| = 1.$$

Definition

An (abstract) C^*-algebra is a complex Banach algebra A with an involution $a \mapsto a^*$ satisfying the C^*-identity

$$\|a^*a\| = \|a\|^2, \quad a \in A.$$

We say A is unital, if there exists a unit element $\mathbf{1} \in A$.

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A concrete C^*-algebra is a self-adjoint subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, which is closed in the operator norm.

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As the operator norm satisfies the C^*-identity, every concrete C^*-algebra is an abstract C^*-algebra.

Example

For some compact Hausdorff space X, we may consider

$$\mathcal{C}(X) = \{\text{continuous functions } X \to \mathbb{C}\}.$$

With pointwise addition and multiplication, $\mathcal{C}(X)$ becomes a **commutative** abstract C^*-algebra if we equip it with the adjoint operation

$$f^*(x) = \overline{f(x)}$$

and the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

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Fact (Spectral theory)

*As an abstract C^*-algebra, $\mathcal{C}(X)$ **remembers** X.*

The **goal for this lecture** is to go over the spectral theory of Banach algebras and C^*-algebras, culminating in:

Theorem (Gelfand–Naimark)

Every (unital) commutative C^*-algebra is isomorphic to $\mathcal{C}(X)$ for some compact Hausdorff space X.

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Theorem (Gelfand–Naimark–Segal)

Every abstract C^*-algebra can be expressed as a concrete C^*-algebra.

The **goal for tomorrow** is to cover examples and advanced topics.

From now on, we will assume that A is a Banach algebra with unit. We identify $\mathbb{C} \subseteq A$ as $\lambda \mapsto \lambda \cdot \mathbf{1}$.

Observation (Neumann series)

If $x \in A$ with $\|\mathbf{1} - x\| < 1$, then x is invertible. In fact

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Proof: $x \sum_{n=0}^{\infty} (\mathbf{1} - x)^n = \sum_{n=0}^{\infty} \left((\mathbf{1} - x)^n - (\mathbf{1} - x)^{n+1} \right) = \mathbf{1}.$

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Observation

The set of invertibles in A is open.

Proof: If z is invertible and x is any element with $\|z - x\| < \|z^{-1}\|^{-1}$, then $\|\mathbf{1} - z^{-1}x\| < 1$. By the above $z^{-1}x$ is invertible, but then x is also invertible.

Definition

For an element $x \in A$, its spectrum is defined as

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda - x \text{ is not invertible in } A\} \ \subseteq \ \mathbb{C}.$$

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Observation

The spectrum $\sigma(x)$ is a compact subset of $\{\lambda \mid |\lambda| \leq \|x\|\}$. One defines the spectral radius of x as $r(x) = \max\limits_{\lambda \in \sigma(x)} |\lambda| \leq \|x\|$.

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Theorem

The spectrum $\sigma(x)$ of every element $x \in A$ is non-empty.

(The proof involves a non-trivial application of complex analysis.)

Definition

A character on A is a non-zero multiplicative linear functional $A \to \mathbb{C}$.

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Observation

A character $\varphi : A \to \mathbb{C}$ is automatically continuous, in fact $\|\varphi\| = 1$.

Proof: As φ is non-zero, we have $0 \neq \varphi(\mathbf{1}) = \varphi(\mathbf{1})^2$, hence $\varphi(\mathbf{1}) = 1$. If x were to satisfy $|\varphi(x)| > \|x\|$, then $\varphi(x) - x$ is invertible by the Neumann series trick. However, it lies in the kernel of φ, which yields a contradiction.

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Definition

For **commutative** A, we define its spectrum (aka character space) as

$$\hat{A} = \{\text{characters } \varphi : A \to \mathbb{C}\}.$$

Due to the Banach-Alaoglu theorem, we see that the topology of pointwise convergence turns $\hat{A}$ into a compact Hausdorff space.

Observation

If $J \subset A$ is a maximal ideal in a (unital) Banach algebra, then J is closed. If A is commutative, then $A/J \cong \mathbb{C}$ as a Banach algebra.

Proof: Part 1: Since the invertibles are open, there are no non-trivial dense ideals in A. So $\overline{J}$ is a proper ideal, hence $J = \overline{J}$ by maximality.

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Part 2: The quotient is a Banach algebra in which every non-zero element is invertible. If it has a non-scalar element $x \in A/J$, then $\lambda - x \neq 0$ is invertible for all $\lambda \in \mathbb{C}$, which is a contradiction to $\sigma(x) \neq \emptyset$.

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Observation

For commutative A, the assignment $\varphi \mapsto \ker \varphi$ is a 1-1 correspondence between $\hat{A}$ and maximal ideals in A.

Proof: Clearly the kernel of a character is a maximal ideal as it has codimension 1 in A. Since we have $\varphi(\mathbf{1}) = 1$ for every $\varphi \in \hat{A}$ and $A = \mathbb{C}\mathbf{1} + \ker \varphi$, every character is uniquely determined by its kernel. Conversely, if $J \subset A$ is a maximal ideal, then $A/J \cong \mathbb{C}$, so the quotient map gives us a character.

A is still commutative.

Theorem

Let $x \in A$. Then

$$\sigma(x) = \left\{ \varphi(x) \mid \varphi \in \hat{A} \right\}.$$

Proof: Let $\lambda \in \mathbb{C}$. If $\lambda = \varphi(x)$, then $\lambda - x \in \ker(\varphi)$, so $\lambda - x$ is not invertible. Conversely, if $\lambda - x$ is not invertible, then it is inside a (proper) maximal ideal. By the previous observation, this means $(\lambda - x) \in \ker\varphi$ for some $\varphi \in \hat{A}$, or $\lambda = \varphi(x)$.

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Theorem (Spectral radius formula)

For any Banach algebra A and $x \in A$, one has

$$r(x) = \lim_{n \to \infty} \sqrt[n]{\|x^n\|}.$$

Proof: The "$\leq$" part follows easily from the above (for A commutative). The "$\geq$" part is another clever application of complex analysis.

For commutative A, consider the usual embedding

$$\iota : A \hookrightarrow A^{**}, \quad \iota(x)(f) = f(x).$$

Since every element of A^{**} is a continuous function on $\hat{A} \subset A^*$ in a natural way, we have a restriction mapping $A^{**} \to \mathcal{C}(\hat{A})$. The composition of these two maps yields:

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Observation

The Gelfand transform is norm-contractive. In fact, for $x \in A$ we have $\hat{x}(\hat{A}) = \sigma(x)$ and hence $\|\hat{x}\| = r(x) \leq \|x\|$ for all $x \in A$.

Definition

Let A be a unital C^*-algebra. An element $x \in A$ is

1. normal, if $x^*x = xx^*$.
2. self-adjoint, if $x = x^*$.
3. positive, if $x = y^*y$ for some $y \in A$.
   Write $x \geq 0$.
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Any element $x \in A$ can be written as $x = x_1 + ix_2$ for the self-adjoint elements

$$x_1 = \frac{x + x^*}{2}, \quad x_2 = \frac{x - x^*}{2i}.$$

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Observation

If $x \in A$ is self-adjoint, then it follows for all $t \in \mathbb{R}$ that

$$\|x + it\|^2 = \|(x - it)(x + it)\| = \|x^2 + t^2\| \leq \|x\|^2 + t^2.$$

Proposition

If $x \in A$ is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.

Proof: Step 1: The spectrum of x inside A is the same as the spectrum of x inside its bicommutant $A \cap \{x\}''$.[1] As x is self-adjoint, this is a commutative C^*-algebra. So assume A is commutative.

[1] This holds in any Banach algebra.

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Step 2: For $\varphi \in \hat{A}$, we get

$$|\varphi(x) + it|^2 = |\varphi(x + it)^2| \leq \|x\|^2 + t^2, \quad t \in \mathbb{R}.$$

But this is only possible for $\varphi(x) \in \mathbb{R}$, as the left-hand expression will otherwise outgrow the right one as $t \to (\pm)\infty$.[2]

[1] This holds in any Banach algebra.

[2] Notice: this works for any $\varphi \in A^*$ with $\|\varphi\| = \|\varphi(\mathbf{1})\| = 1$!

Proposition

Let A be a commutative C**-algebra. Then every character $\varphi \in \hat{A}$ is $*$-preserving, i.e., it satisfies $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$.*

Proof: Write $x = x_1 + ix_2$ as before and use the above for

$$\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2) = \overline{\varphi(x_1) + i\varphi(x_2)} = \overline{\varphi(x)}.$$

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Corollary

For a commutative C^*-algebra A, the Gelfand transform

$$A \to \mathcal{C}(\hat{A}), \quad \hat{x}(\varphi) = \varphi(x)$$

is a $*$-homomorphism.

Let A be a C^*-algebra and $B \subseteq A$ a C^*-subalgebra.

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Proof: By the above we may assume $x = x^*$. We know $\sigma_B(x) \subset \mathbb{R}$, so $x_n = x + \frac{i}{n} \overset{n\to\infty}{\longrightarrow} x$ is a sequence of invertibles in B. We know $\|x_n - x\| < \|x_n^{-1}\|^{-1}$ implies that x is invertible in B.

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Corollary

We have $\sigma_B(x) = \sigma_A(x)$ *for all* $x \in B$.[3]

[3] This often fails for inclusions of Banach algebras!

Let A be a C^*-algebra.

Observation

$x \in A$ is normal if and only if $\mathrm{C}^*(x, \mathbf{1}) \subseteq A$ is commutative. In this case the spectrum of $\mathrm{C}^*(x, \mathbf{1})$ is homeomorphic to $\sigma(x)$.

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Proposition

For a normal element $x \in A$, we have $r(x) = \|x\|$.

Proof: Observe from the C^*-identity that

$$\|x\|^4 = \|x^*x\|^2 = \|x^*xx^*x\| = \|(x^2)^*x^2\| = \|x^2\|^2.$$

By induction, we get $\|x^{2^n}\| = \|x\|^{2^n}$. By the spectral radius formula, we have

$$r(x) = \lim_{n\to\infty} \sqrt[2^n]{\|x^{2^n}\|} = \|x\|.$$

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Corollary

For all $x \in A$, we have $\|x\| = \sqrt{\|x^*x\|} = \sqrt{r(x^*x)}$.

Theorem (Gelfand–Naimark)

For a commutative C^**-algebra* A*, the Gelfand transform*

$$A \to \mathcal{C}(\hat{A}), \quad \hat{x}(\varphi) = \varphi(x)$$

is an isometric $*$*-isomorphism.*

Proof: We have already seen that it is a $*$-homomorphism.

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For surjectivity, observe that the image of A in $\mathcal{C}(\hat{A})$ is a closed unital self-adjoint subalgebra, and which separates points. By the **Stone–Weierstrass theorem**, it follows that it is all of $\mathcal{C}(\hat{A})$.

Observation

Let $x \in A$ be a normal element in a C^*-algebra. Let $A_x = \mathrm{C}^*(x, \mathbf{1})$ be the commutative C^*-subalgebra generated by x. Then $\hat{A}_x \cong \sigma(x)$ by observing that for every $\lambda \in \sigma(x)$ there is a unique $\varphi \in \hat{A}_x$ with $\varphi(x) = \lambda$. Under this identification $\hat{x} \in \mathcal{C}(\hat{A}_x)$ becomes the identity map on $\sigma(x)$.

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Theorem (functional calculus)

Let $x \in A$ be a normal element in a (unital) C^*-algebra. There exists a unique (isometric) $*$-homomorphism

$$\mathcal{C}(\sigma(x)) \to A, \quad f \mapsto f(x)$$

that sends $\mathrm{id}_{\sigma(x)}$ to x.

Proof: Take the inverse of the Gelfand transform

$$A_x \to \mathcal{C}(\hat{A}_x) \cong \mathcal{C}(\sigma(x)).$$

Theorem

An element $x \in A$ is positive if and only if x is normal and $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$.

Proof: If the latter is true, then $y = \sqrt{x}$ satisfies $y^* y = y^2 = x$. So x is positive. The "only if" part is much trickier.

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Corollary

For $a, b \in A$ positive, the sum $a + b$ is positive.

Proof: Apply the triangle inequality: We have $\|a + b\| \leq \|a\| + \|b\|$ and

$$\big\|(\|a\| + \|b\|) - (a + b)\big\| \leq \big\|\|a\| - a\big\| + \big\|\|b\| - b\big\| \leq \|a\| + \|b\|.$$

Theorem

Every algebraic (unital) $*$-homomorphism $\psi : A \to B$ between (unital) C^*-algebras is contractive, and hence continuous.[4]

Proof: It is clear that $\sigma(\psi(x)) \subseteq \sigma(x)$ for all $x \in A$. By the spectral characterization of the norm, it follows that

$$\|\psi(x)\|^2 = r(\psi(x^*x)) \leq r(x^*x) = \|x\|^2.$$

[4] This generalizes to the non-unital case as well!

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Observation

For $x \in A$ normal and $f \in \mathcal{C}(\sigma(x))$, we have $\psi(f(x)) = f(\psi(x))$.

Proof: Clear for $f \in \{$*-polynomials$\}$. The general case follows by continuity of the assignments $[f \mapsto f(x)]$ and $[f \mapsto f(\psi(x))]$ and the Weierstrass approximation theorem.

[4] This generalizes to the non-unital case as well!

Theorem

Every injective $*$-homomorphism $\psi : A \to B$ is isometric.

Proof: By the C^*-identity, it suffices to show $\|\psi(x)\| = \|x\|$ for positive $x \in A$. Suppose we have $\|\psi(x)\| < \|x\|$. Choose a non-zero continuous function $f : \sigma(x) \to \mathbb{R}^{\geq 0}$ with $f(\lambda) = 0$ for $\lambda \leq \|\psi(x)\|$.

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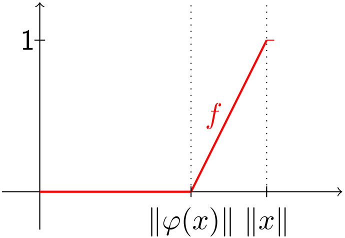

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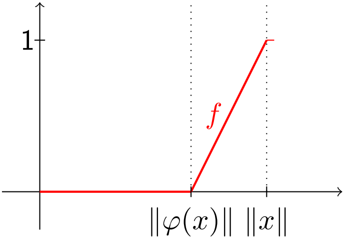

Then $f(x) \neq 0$, but

$$\psi(f(x)) = f(\psi(x)) = 0,$$

which means ψ is not injective.

Definition

Let A be a C^*-algebra. A representation (on a Hilbert space $\mathcal{H}$) is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$.

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Observation

Every positive functional $\varphi : A \to \mathbb{C}$ is continuous.

Proof: Suppose not. By functional calculus, every element $x \in A$ can be written as a linear combination of at most four positive elements

$$x = (x_1^+ - x_1^-) + i(x_2^+ - x_2^-)$$

with norms $\|x_1^+\|, \|x_1^-\|, \|x_2^+\|, \|x_2^-\| \leq \|x\|$. So φ is unbounded on the positive elements.

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Given $n \geq 1$, one may choose $a_n \geq 0$ with $\|a_n\| = 1$ and $\varphi(a_n) \geq n2^n$. Then $a = \sum_{n=1}^\infty 2^{-n} a_n$ is a positive element in A. By positivity of φ, we have $\varphi(a) \geq \varphi(2^{-n} a_n) \geq n$ for all n, a contradiction.

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Corollary

For a positive functional φ, the assignment $(x, y) \mapsto \varphi(y^* x)$ defines a positive semi-definite, anti-symmetric, sesqui-linear form. In particular, it is subject to the ***Cauchy–Schwarz*** *inequality*

$$|\varphi(y^* x)|^2 \leq \varphi(x^* x)\varphi(y^* y).$$

Theorem

Let A be a unital C^*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is positive if and only if $\|\varphi\| = \varphi(\mathbf{1})$.

Proof: For the "only if" part, observe for $\|y\| \leq 1$ that

$$|\varphi(y)|^2 = |\varphi(\mathbf{1}y)|^2 \leq \varphi(\mathbf{1})\varphi(y^*y) \leq \varphi(\mathbf{1})\|\varphi\|.$$

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$$\varphi(a) \notin \{\lambda \in \mathbb{C} \mid |\lambda_0 - \lambda| \leq \rho\} \supseteq \sigma(a),$$

where $\lambda_0 = \frac{1}{2}(\max \sigma(a) + \min \sigma(a))$ and $\rho = \frac{1}{2}(\max \sigma(a) - \min \sigma(a))$.

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where $\lambda_0 = \frac{1}{2}(\max \sigma(a) + \min \sigma(a))$ and $\rho = \frac{1}{2}(\max \sigma(a) - \min \sigma(a))$. Then $y = \lambda_0 - a$ is self-adjoint, hence $\|y\| = r(y) = \rho$, but $\varphi(y) = \lambda_0 - \varphi(a) > \rho$, a contradiction to $\|\varphi\| = 1$.

Corollary

For an inclusion of (unital) C^*-algebras $B \subseteq A$, every positive functional on B extends to a positive functional on A.

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Observation

For $x \in A$ normal, there is a state φ with $\|x\| = |\varphi(x)|$.

Proof: Pick $\lambda_0 \in \sigma(x)$ with $|\lambda_0| = \|x\|$. We know

$$A_x = \mathrm{C}^*(x, \mathbf{1}) \cong \mathcal{C}(\sigma(x))$$

so that $x \mapsto \mathrm{id}$. The evaluation map $f \mapsto f(\lambda_0)$ corresponds to a state on A_x with the desired property. Extend it to a state φ on A.

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- The order "$\leq$" is compatible with sums.
- For all self-adjoint $a \in A$, we have $a \leq \|a\|$.
- If $a \leq b$ and $x \in A$ is any element, then $x^*ax \leq x^*bx$.

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For proving the last part, write $b - a = c^*c$. Then

$$x^*bx - x^*ax = x^*(b-a)x = x^*c^*cx = (cx)^*cx \geq 0.$$

Given a state φ on A, we have observed that $(x, y) \mapsto \varphi(y^*x)$ forms a positive semi-definite, anti-symmetric, sesqui-linear form.

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For all $a, x \in A$, we have $\varphi(x^*a^*ax) \leq \|a\|^2\varphi(x^*x)$. The null space $N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$ is a closed left ideal in A.

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Observation

The quotient $H_\varphi = A/N_\varphi$ carries the inner product

$$\langle [x] \mid [y] \rangle_\varphi = \varphi(y^*x),$$

and the left A-module structure satisfies $\|[ax]\|_\varphi \leq \|a\| \cdot \|[x]\|_\varphi$ for all $a, x \in A$.

Definition (Gelfand–Naimark–Segal construction)

For a state φ on a C^*-algebra A, let $\mathcal{H}_\varphi$ be the Hilbert space completion $\mathcal{H}_\varphi = \overline{H_\varphi}^{\|\cdot\|_\varphi}$. Then $\mathcal{H}_\varphi$ carries a unique left A-module structure which extends the one on H_φ and is continuous in $\mathcal{H}_\varphi$. This gives us a representation

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The only non-tautological part is that π_φ is compatible with adjoints. For this we observe

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Definition

In the (unital) situation above, set $\xi_\varphi = [\mathbf{1}] \in \mathcal{H}_\varphi$. Then $\|\xi_\varphi\| = 1$ as we have assumed φ to be a state.

Theorem (GNS)

The assignment $\varphi \mapsto (\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is a 1-1 correspondence between states on A and cyclic representations modulo unitary equivalence.

Proof: Let us only check that $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is cyclic. Indeed, $\pi_\varphi(A)\xi_\varphi = \pi_\varphi(A)([\mathbf{1}]) = [A] = H_\varphi \subseteq \mathcal{H}_\varphi$, which is dense by definition.

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Theorem (Gelfand–Naimark)

Every abstract C^*-algebra A is a concrete C^*-algebra. *In particular, there exists a faithful representation $\pi : A \to \mathcal{H}$ on some Hilbert space.*[5]

Proof: For $x \in A$, find φ_x with $\|\varphi_x(x^*x)\| = \|x\|^2$. Then form the cyclic representation $(\pi_{\varphi_x}, \mathcal{H}_{\varphi_x}, \xi_{\varphi_x})$.

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Proof: For $x \in A$, find φ_x with $\|\varphi_x(x^*x)\| = \|x\|^2$. Then form the cyclic representation $(\pi_{\varphi_x}, \mathcal{H}_{\varphi_x}, \xi_{\varphi_x})$. We claim that the direct sum

$$\pi := \bigoplus_{x\in A} \pi_{\varphi_x} \ : \ A \to \mathcal{B}\Big(\bigoplus_{x\in A} \mathcal{H}_{\varphi_x}\Big)$$

does it. Indeed, given any $x \neq 0$ we have

$$\|\pi(x)\|^2 \geq \|\pi(x)\xi_{\varphi_x}\|^2 = \langle [x] \mid [x]\rangle_{\varphi_x} = \varphi_x(x^*x) = \|x\|^2.$$

[5] If A is separable, we may choose $\mathcal{H}$ to be separable!

Let us now discuss noncommutative examples of C^*-algebras:

Example

The set of $\mathbb{C}$-valued $n \times n$ matrices, denoted M_n, becomes a C^*-algebra. By linear algebra, $M_n \cong \mathcal{B}(\mathbb{C}^n)$.

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For numbers $n_1, \dots, n_k \geq 1$, the C^*-algebra

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Every finite-dimensional C^*-algebras has this form.

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Example

For a Hilbert space $\mathcal{H}$, the set of compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ forms a norm-closed, $*$-closed, two-sided ideal. If $\dim(\mathcal{H}) = \infty$, then it is a proper ideal and a **non-unital** C^*-algebra.

Notation (ad-hoc!)

Let $\mathcal{G}$ be a countable set, and let $\mathcal{P}$ be a family of (noncommutative) $*$-polynomials in finitely many variables in $\mathcal{G}$ and coefficients in $\mathbb{C}$. We shall understand a relation $\mathcal{R}$ as a collection of formulas of the form

$$\|p(\mathcal{G})\| \leq \lambda_p, \quad p \in \mathcal{P}, \quad \lambda_p \geq 0.$$

A representation of $(\mathcal{G} \mid \mathcal{R})$ is a map $\pi : \mathcal{G} \to A$ into a C^*-algebra under which the relation becomes true.

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Example

The expression $xyx^* - z^2$ for $x, y, z \in \mathcal{G}$ is a noncommutative $*$-polynomial. The relation could mean

$$\|xyx^* - z^2\| \leq 1.$$

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Observation

Up to isomorphism, a C^*-algebra B as above is unique. One writes $B = \mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$ and calls it the universal C^*-algebra for $(\mathcal{G} \mid \mathcal{R})$.

Example

Given $n \geq 1$, one can express M_n as the universal C^*-algebra generated by $\{e_{i,j}\}_{i,j=1}^n$ subject to the relations

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Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. Then one can express $\mathcal{K}(\mathcal{H})$ as the universal C^*-algebra generated by $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ subject to the relations

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(Here e_{ij} represents a rank-one operator sending the i-th vector in an ONB to the j-th vector.)

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"if" part: The isomorphism classes of separable C^*-algebras form a set. There exist set-many representations $\pi : \mathcal{G} \to A_\pi$ of $(\mathcal{G} \mid \mathcal{R})$ on separable C^*-algebras up to conjugacy. Denote this set by I, and consider

$$\mathfrak{A} = \prod_{\pi \in I} A_\pi \quad \text{and} \quad \pi_u : \mathcal{G} \to \mathfrak{A},\ \pi_u(x) = (\pi(x))_{\pi \in I}.$$

By compactness, π_u is a well-defined representation of $(\mathcal{G} \mid \mathcal{R})$. Then check that $B = \mathrm{C}^*(\pi_u(\mathcal{G})) \subseteq \mathfrak{A}$ is universal.

Example

The universal C^*-algebra for the relation $\|xyx^* - z^2\| \leq 1$ does not exist.

Proof: Suppose we have such $x, y, z \neq 0$ in a C^*-algebra, e.g., all equal to the unit. For $\lambda > 0$, replace $y \to \lambda y$ and $x \to \lambda^{-1/2}x$, and let $\lambda \to \infty$.

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Remark

All of this generalizes to more general relations (including functional calculus etc.) and a more flexible notion of generating sets.

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Every separable C^*-algebra A is the universal C^*-algebra for a countable set of equations involving $*$-polynomials of degree at most 2.

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Proof: Start with some countable dense $\mathbb{Q}[i]$-$*$-subalgebra $C \subset A$. By inductively enlarging C, we may enlarge it to another countable dense $\mathbb{Q}[i]$-$*$-subalgebra $D \subset A$ with the additional property that if $x \in D$ is a contraction, then $y = \mathbf{1} - \sqrt{\mathbf{1} - x^*x} \in D$.

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Now let $\mathcal{P}$ be the family of $*$-polynomials that encode all the $*$-algebra relations in D, so

$$X_a X_b - X_{ab},\ \lambda X_a + X_b - X_{\lambda a+b},\ X_a^* - X_{a^*},$$

for $\lambda \in \mathbb{Q}[i]$ and $a, b \in D$. Set $\mathcal{G} = D$, and let $\mathcal{R}$ be the relation where these polynomials evaluate to zero. By construction, representations $(\mathcal{G} \mid \mathcal{R}) \to B$ are the same as $*$-homomorphisms $D \to B$.

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We claim that the inclusion $D \subset A$ turns A into the universal C^*-algebra for these relations. This means that every $*$-homomorphism from D extends to a $*$-homomorphism on A. This is certainly the case if every $*$-homomorphism $\varphi : D \to B$ is contractive.

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Indeed, if $x \in D$ is a contraction, then $y = \mathbf{1} - \sqrt{\mathbf{1} - x^*x} \in D_{sa}$ satisfies

$$x^*x + y^2 - 2y = 0.$$

Thus also $\varphi(x)^*\varphi(x) + \varphi(y)^2 - 2\varphi(y) = 0$ in B, which is equivalent to

$$\varphi(x)^*\varphi(x) + (\mathbf{1} - \varphi(y))^2 = \mathbf{1}.$$

Hence $\|\varphi(x)\| \leq 1$ for every contraction $x \in D$, which finishes the proof.

Definition

Let Γ be a countable discrete group. The universal group C^*-algebra is defined as

$$\mathrm{C}^*(\Gamma) = \mathrm{C}^*\big(\,\{u_g\}_{g\in\Gamma} \mid u_1 = \mathbf{1},\ u_{gh} = u_g u_h,\ u_g^* = u_{g^{-1}}\big).$$

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Fact

If $v \in B$ is any non-unitary isometry in a C^*-algebra, then $\mathrm{C}^*(v) \cong \mathcal{T}$ in the obvious way. In other words, every proper isometry is universal.

Example

For $n \in \mathbb{N}$, one defines the Cuntz algebra in n generators as

$$\mathcal{O}_n = \mathrm{C}^*\Big(s_1, \dots, s_n \mid s_j^* s_j = \mathbf{1},\ \sum_{j=1}^{n} s_j s_j^* = \mathbf{1} \Big).$$

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$$\mathcal{O}_3 = \mathrm{C}^*(s_1, s_2, s_3)$$

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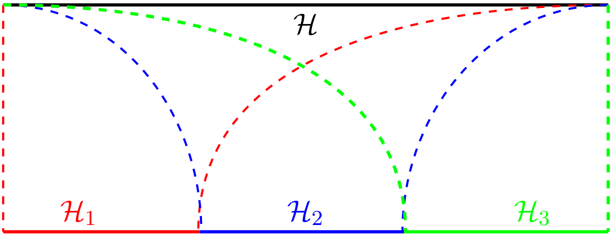

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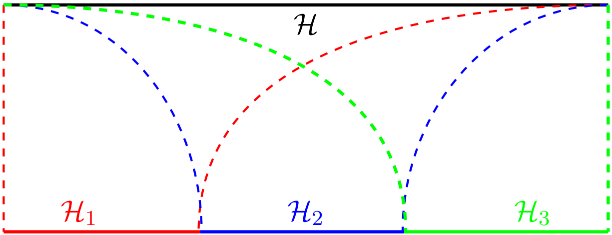

Theorem (Cuntz)

*$\mathcal{O}_n$ is **simple**! That is, every collection of isometries $s_1, \dots, s_n$ in any C^*-algebra as above is universal with this property.*

Fact (Inductive limits)

If

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

is a sequence of C^**-algebra inclusions, then*

$$A = \overline{\bigcup_{n\in\mathbb{N}} A_n}^{\|\cdot\|}$$

exists and is a C^**-algebra.*

Definition

In the above situation, if every A_n is finite-dimensional, we call A an AF algebra. (AF = approximately finite-dimensional)

Example

Consider

$$A_1 = \mathbb{C}, \quad A_2 = M_2, \quad A_3 = M_4 \cong M_2 \otimes M_2, \quad A_4 = M_8 \cong M_2^{\otimes 3}, \quad \dots,$$

with inclusions of the form $x \mapsto x \otimes \mathbf{1}_2 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$.

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This construction can of course be repeated with powers of any other number p instead of 2. $\rightsquigarrow M_{p^\infty}$

$$M_2\left(\begin{array}{c|c} & \\ \hline & \end{array}\right)$$

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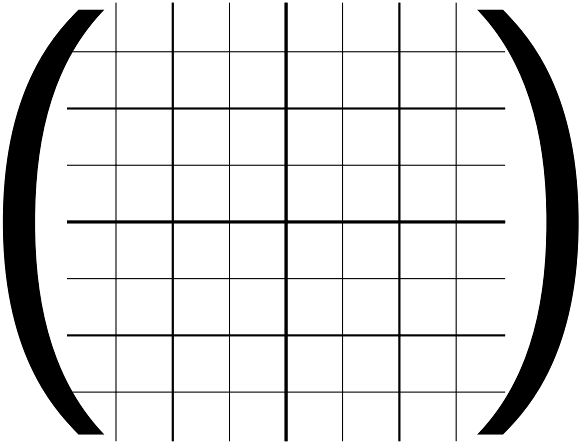
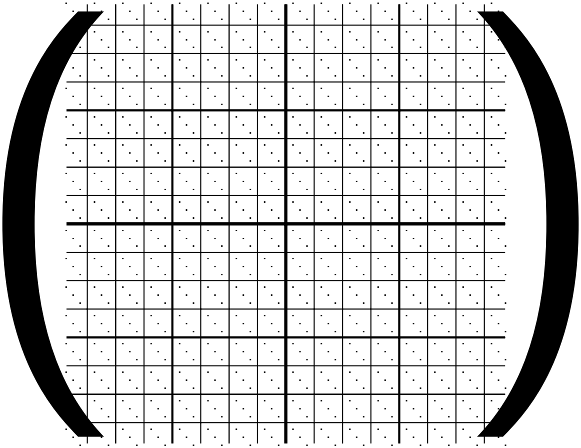


M_{2^∞} 

Let A be a (unital) C^*-algebra and Γ a discrete group.

Definition

Given an action $\alpha : \Gamma \curvearrowright A$, define the crossed product $A \rtimes_\alpha \Gamma$ as the universal C^*-algebra containing a unital copy of A, and the image of a unitary representation $[g \mapsto u_g]$ of Γ, subject to the relation

$$u_g a u_g^* = \alpha_g(a), \quad a \in A,\ g \in \Gamma.$$

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Example

Start from a homeomorphic action $\Gamma \curvearrowright X$ on a compact Hausdorff space. $\rightsquigarrow \mathcal{C}(X) \rtimes \Gamma$.

Observation

For two C^*-algebras A, B, the algebraic tensor product $A \odot B$ becomes a $*$-algebra in the obvious way.

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Can this be turned into a C^*-algebra?

Yes! However, not uniquely in general.

Definition

We say that a C^*-algebra A is nuclear if the tensor product $A \odot B$ carries a unique C^*-norm for every C^*-algebra B. In this case we denote by $A \otimes B$ the C^*-algebra arising as the completion.

Example

Finite-dimensional or commutative C^*-algebras are nuclear. One has $M_n \otimes A \cong M_n(A)$ and $\mathcal{C}(X) \otimes A \cong \mathcal{C}(X, A)$.

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Theorem

If Γ is amenable and A is nuclear, then $A \rtimes \Gamma$ is nuclear for every possible action $\Gamma \curvearrowright A$. So in particular for $A = \mathcal{C}(X)$.

Fact (K-theory)

There is a functor

$$\{\mathrm{C}^*\text{-}\textit{algebras}\} \longrightarrow \{\textit{abelian groups}\}\,, \quad A \mapsto K_*(A) = K_0(A) \oplus K_1(A),$$

which extends the topological K-theory functor $X \mapsto K^*(X)$ for (locally) compact Hausdorff spaces.

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Theorem (Glimm, Bratteli, Elliott)

Let A and B be two (unital) AF algebras. Then

$$A \cong B \quad \Longleftrightarrow \quad (K_0(A), K_0(A)_+, [\mathbf{1}_A]) \cong (K_0(B), K_0(B)_+, [\mathbf{1}_B]).$$

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The sextuple

$$\mathrm{Ell}(A) = \Big(K_0(A), K_0(A)_+, [\mathbf{1}_A], K_1(A), T(A), \rho_A \Big)$$

is called the Elliott invariant and becomes functorial with respect to a suitable target category.

Fact

There is a separable unital simple nuclear infinite-dimensional C^**-algebra* $\mathcal{Z}$ *with* $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, *the Jiang–Su algebra, with* $\mathrm{Ell}(\mathcal{Z}) \cong \mathrm{Ell}(\mathbb{C})$.

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Rough idea: One considers the C^*-algebra

$$\mathcal{Z}_{2^\infty,3^\infty} = \{ f \in \mathcal{C}\big([0,1], M_{2^\infty} \otimes M_{3^\infty}\big) \mid f(0) \in M_{2^\infty} \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes M_{3^\infty} \}$$

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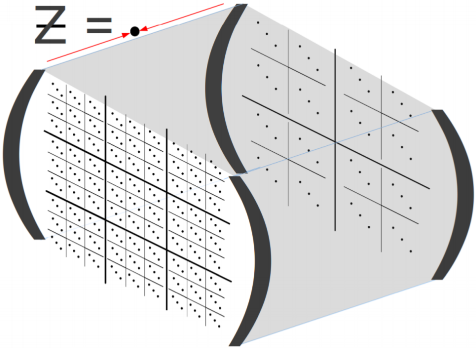
which has the right K-theory but far too many ideals and traces.

One constructs a **trace-collapsing** endomorphism on $\mathcal{Z}_{2^\infty,3^\infty}$ and can define $\mathcal{Z}$ as the stationary inductive limit.

(Graphic created by Aaron Tikuisis.)



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Conjecture (Elliott conjecture; modern version)

Let A and B be two separable unital simple nuclear $\mathcal{Z}$-stable C^*-algebras. Then

$$A \cong B \quad \Longleftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$[6]

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Problem (difficult!)

Determine when $\Gamma \curvearrowright X$ gives rise to a $\mathcal{Z}$-stable crossed product.

[6] To the experts in the audience: No UCT discussion now!

Thank you for your attention!