A POSTERIORI ERROR ESTIMATES ON ANISOTROPIC MESHES

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- For singly perturbed *semilinear reaction-diffusion* equations

$$-\varepsilon^2 \triangle u + f(x,u) = 0$$

where $x \in \Omega \subset \mathbb{R}^2$, subject to $u = 0$ on $\partial\Omega$

$$f(x,u) - f(x,v) \geqslant C_f[u-v] \text{ whenever } u \geqslant v, \boxed{\varepsilon^2 + C_f \gtrsim 1}$$

we look for *residual-type a posteriori error estimates*

$$\boxed{\|\text{error}\|_* \leqslant \text{function}\big(\text{mesh, comp.sol-n}\big)}$$

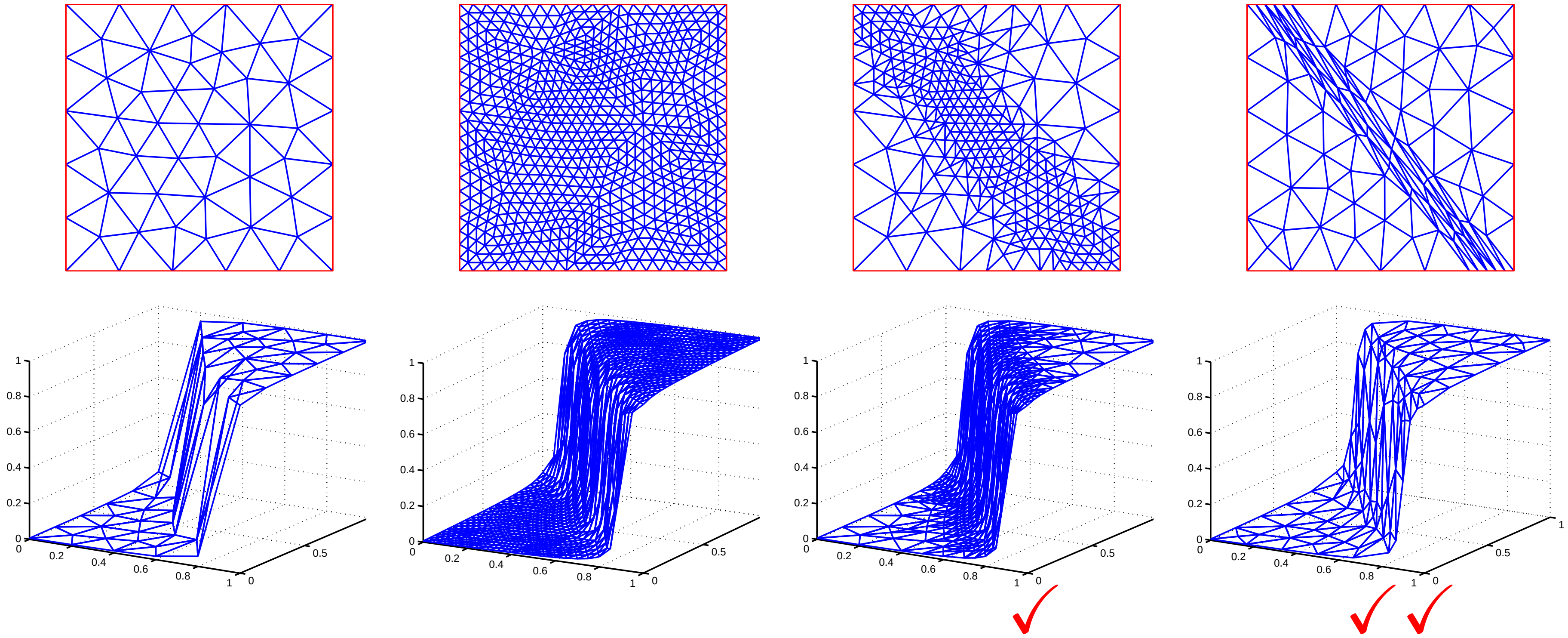
where $\|\cdot\|_*$ is the ***maximum norm*** or the ***energy norm***

on **anisotropic meshes**

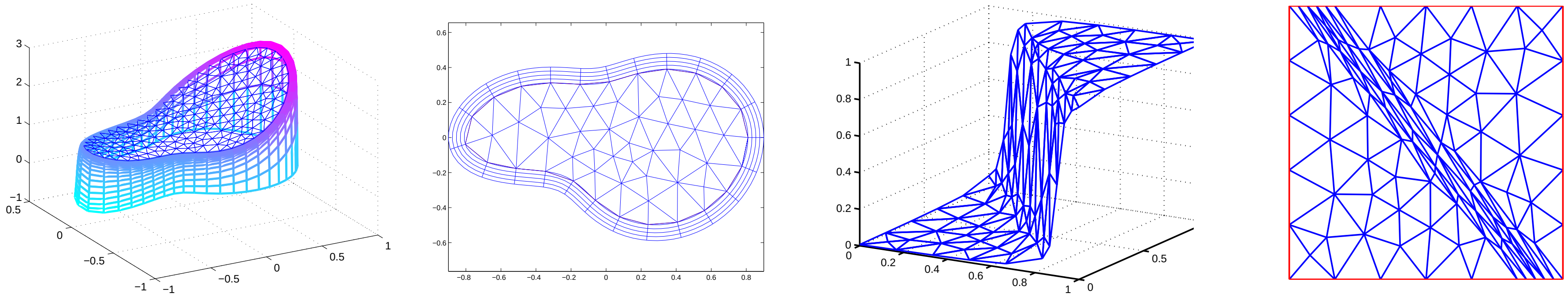
- Interpolation error bounds $\Rightarrow$

anisotropic meshes are superior for layer solutions

(a) Standard mesh. (b) Fine mesh. (c) Shape-regular refinement. (d) *Anisotropic* ref-nt.



- **anisotropic meshes are superior for layer solutions**

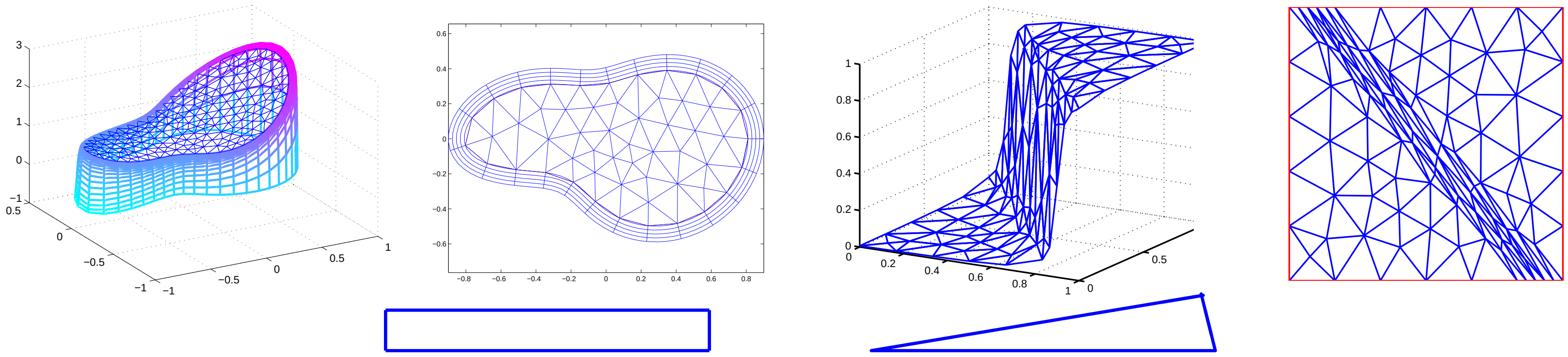


(i) **fine in layer regions ; coarse outside**

(ii) **maximum mesh aspect ratio $\sim$ (layer width)$^{-1} \gg 1$**

BUT **theoretical difficulties within the FEM framework...**

- anisotropic meshes are superior for layer solutions



BUT **theoretical difficulties within the FEM framework...**

- It's not just about working hard and tracing all the constants very carefully
- New tricks are required...

ALSO **Perceptions and expectations t.b. adjusted for anisotropic meshes**

Section A

Perceptions & expectations t.b. adjusted for anisotropic meshes

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Section B

Part 1 **Reaction-Diffusion eq. — a posteriori estimates on anisotropic meshes**

— Problem addressed (more detail)

— Mesh assumptions + preview of results

— Error represenation ⇒ From the L_∞ to the energy norm??

Part 2 **A bit of analysis**: 3 technical issues addressed

Part 3 **Some Numerics**

Section C

Efficiency, i.e. lower estimator: also problematic on anisotropic meshes...

One Perception: **the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...**

- Quasi-uniform meshes, linear elements

$$\|u - u_h\|_{L_\infty(\Omega)} \leqslant \ln(C + \varepsilon/h) \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega)}$$

  - Schatz, Wahlbin, *On the quasi-optimality in L_∞ of the $\overset{\circ}{H}{}^1$-projection into finite element spaces*, Math. Comp. 1982: $-\triangle u = f$,
  - Schatz, Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems* ..., Math. Comp., 1983: $-\varepsilon^2 \triangle u + au = f$,

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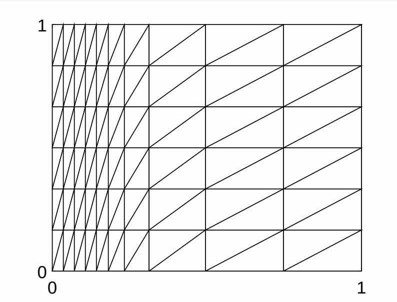
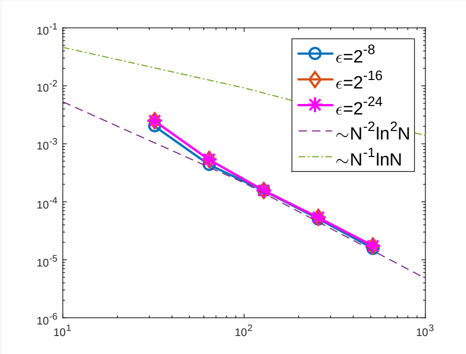
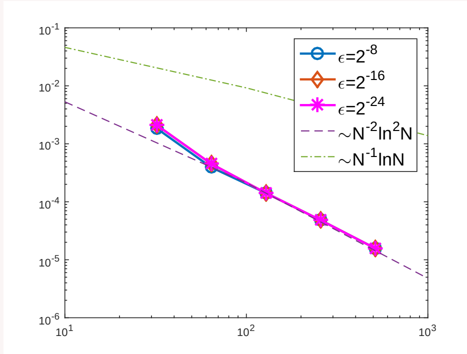
- **Strongly-anisotropic triangulations:** no such result
  - BUT this is frequently considered a reasonable heuristic conjecture t.b. used in the anisotropic mesh adaptation (Hessian-related metrics...)
  - IN FACT, this is **NOT true** (see next)

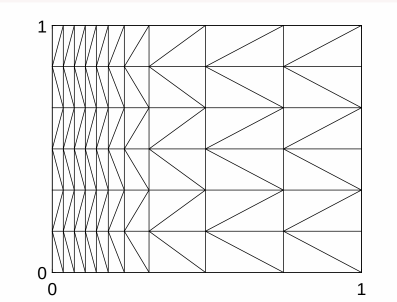
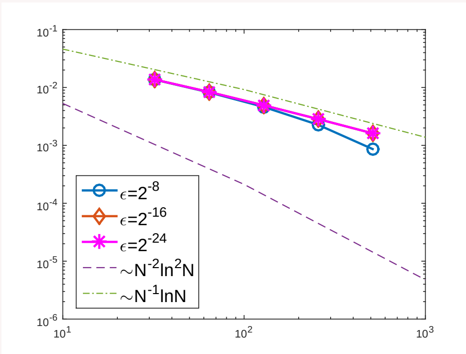
Example: $-\varepsilon^2 \triangle u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a sharp boundary layer

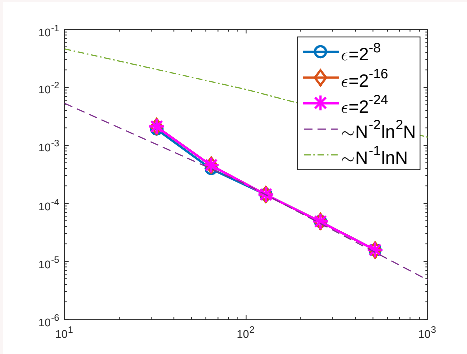
Observation #1: **Mass Lumping may be superior on anisotropic meshes**

Standard linear FEM | Mass Lumping





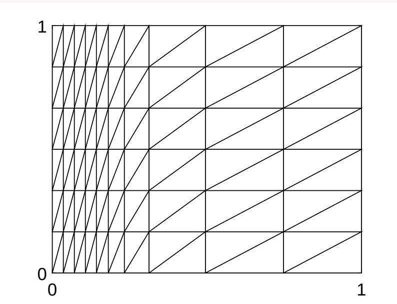
Here we use a Shishkin mesh: piecewise-uniform, $DOF \simeq N^2$, mesh diameter $\simeq N^{-1}$

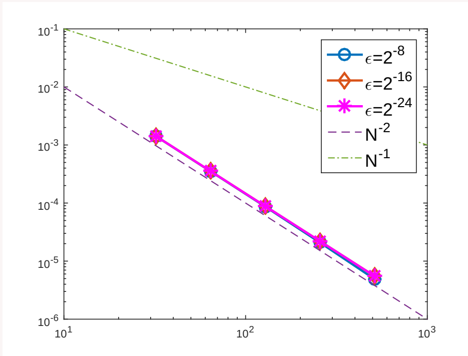
$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2}\ln^2 N \simeq DOF^{-1}\ln(DOF)$$

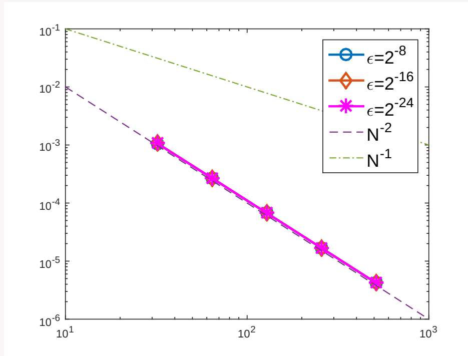
<u>Same Example:</u> $-\varepsilon^2\triangle u + u = 0$ with $u = e^{-x/\varepsilon}$ exhibiting a sharp boundary layer

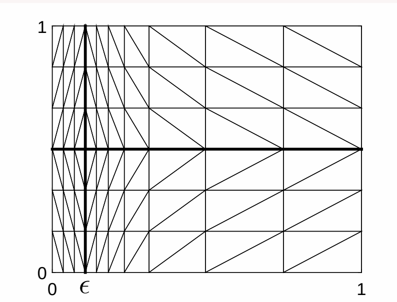
<u>Observation #2:</u> **Convergence Rates may depend on the mesh structure** (even for mass lumping), **NOT ONLY on the interpolation error**

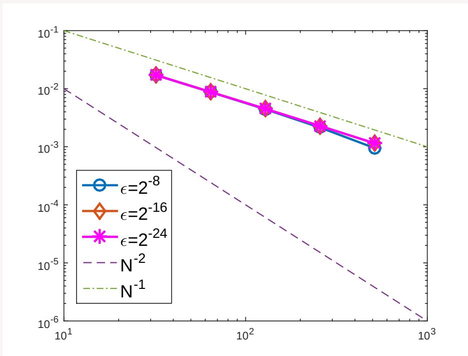
Standard linear FEM   Mass Lumping

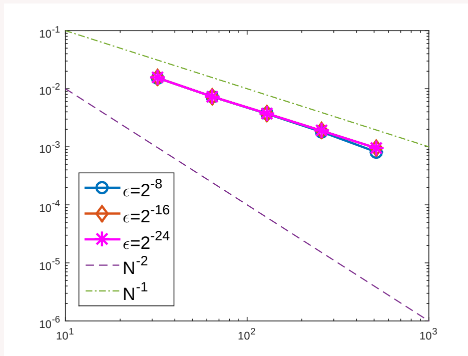












Here we use a graded Bakhvalov mesh:

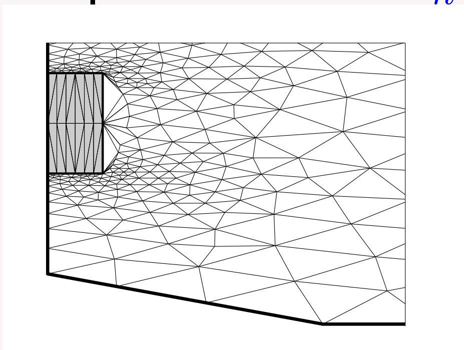
$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

- A theoretical explanation of the above phenomena is given in:

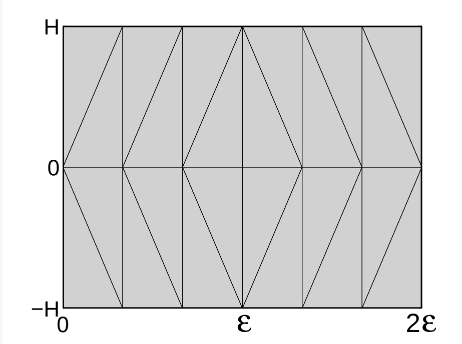
  N.Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp., 2014.

What happens in $\mathring{\Omega} := (0, 2\varepsilon) \times (-H, H)$
with the tensor-product mesh $\mathring{\omega}_h := \{x_i = \varepsilon \frac{i}{N_0}\}_{i=0}^{2N_0} \times \{-H, 0, H\}$??

$\mathcal{T}$ in Ω:



$\mathcal{T}_0$ in $\Omega_0 \subset \Omega$:



Mass lumping, $U_i := u_h(x_i, 0)$ and $U_i^{\pm} := u_h(x_i, \pm H)$:

$$\frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_i U_i = 0$$

with $\gamma_i = 1$ for $i \neq N_0$, and $\gamma_{N_0} = \frac{2}{3}$

$\varepsilon \ll H \quad \Rightarrow \quad \frac{\varepsilon^2}{h^2}[-U_{i-1} + 2U_i - U_{i+1}] + \frac{\varepsilon^2}{H^2}[-U_i^- + 2U_i - U_i^+] + \gamma_i U_i = 0$

Implications of the above example:

- Theoretical:

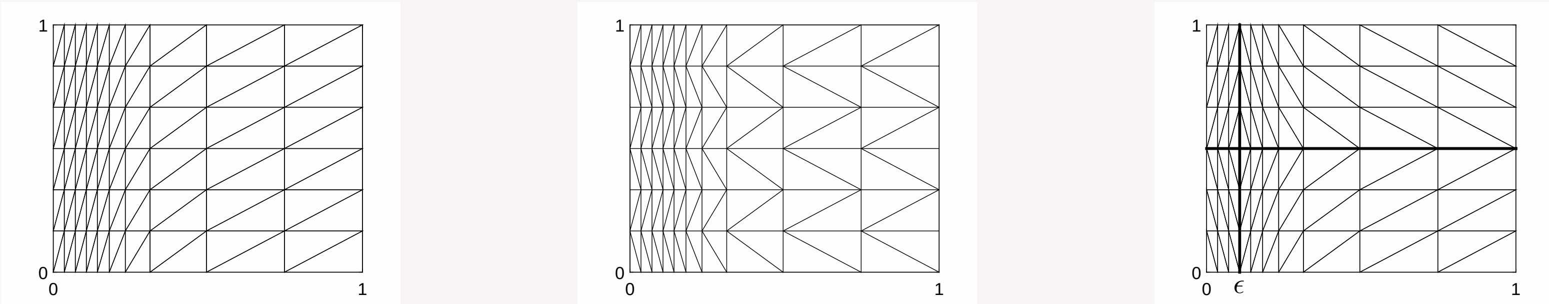
  if one tries to prove ”standard” (almost) second-order a priori/a posteriori error estimate in the maximum norm on a general anisotropic mesh, this may be impossible...

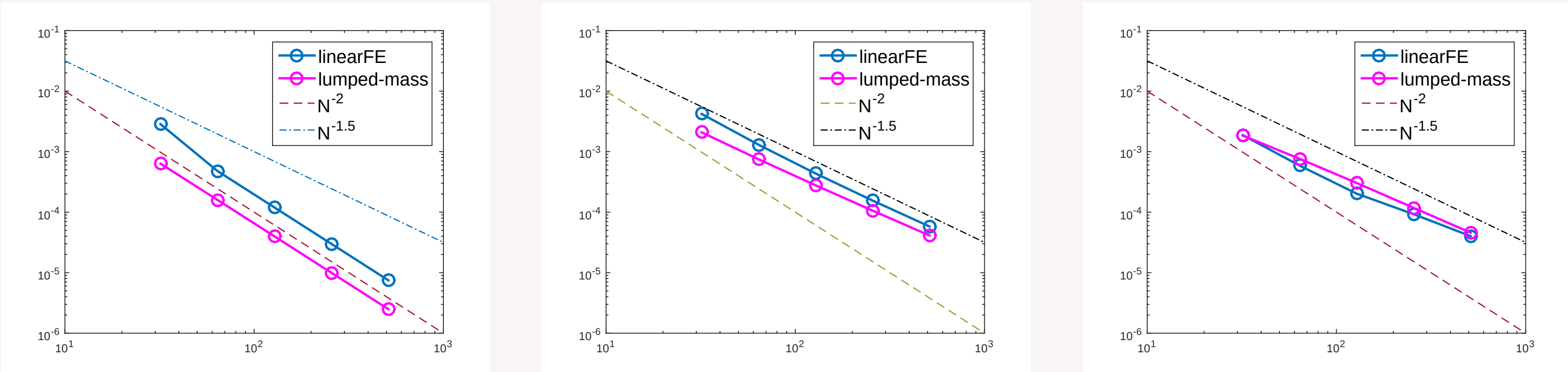
- Anisotropic mesh adaptation (Hessian-related metrics...):

  One needs to be careful with the heuristic conjecture that the computed-solution error in the maximum norm is closely related to the corresponding interpolation error...

Non-singularly-perturbed EXAMPLE [Nochetto et al, Numer. Math., 2006]:

$$-\triangle u + f(u) = 0 \text{ with } f(u) \sim -u^{-3} \text{ and } u = \sqrt{x}$$





Graded mesh: $\{(i/N)^6\}_{i=0}^N$:

$$\|u - u^I\|_{L_\infty(\Omega)} \simeq N^{-2} \simeq DOF^{-1}$$

Mesh transition parameter: $\epsilon = 0.1$

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Efficiency, i.e. lower estimator: also problematic on anisotropic meshes...

Laplace equation $-\triangle u = f(x)$, linear elements, shape-regular mesh

[Ainsworth & Oden, 2000, Chap. 2]

- H^1 norm [Babuška & Miller, 1987]

$$\|u_h - u\|_{H^1(\Omega)} \lesssim \left\{ \sum_{T\in\mathcal{T}} \Bigg(\underbrace{\|h_T\, f\|^2_{L_2(T)}}_{\sim\|h_T \triangle u\|^2_{L_2(T)}} + \underbrace{h_T^2\, \|[\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim\|h_T\, D^2 u\|^2_{L_2(T)}} \Bigg) \right\}^{1/2}$$

$$\sim \|h_T\, D^2 u\|_{L_2(\Omega)} \sim \|\text{linear interpolation error}\|_{H^1(\Omega)}$$

Laplace equation $-\triangle u = f(x)$, linear elements, shape-regular mesh

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$$\sim \|h_T\, D^2 u\|_{L_2(\Omega)} \sim \big\|\text{linear interpolation error}\big\|_{H^1(\Omega)}$$

- L_∞ norm [Eriksson, 1994], [Nochetto, 1995]

$$\|u_h - u\|_{L_\infty(\Omega)} \lesssim \ln(h_{\min}^{-1})\, \max_{T\in\mathcal{T}}\Big\{ \underbrace{h_T^2\, \|f\|_{L_\infty(T)}}_{\sim h_T^2\, \|\triangle u\|_{L_\infty(T)}} + \underbrace{h_T\, \|[\![\nabla u_h]\!]\|_{L_\infty(\partial T)}}_{\sim h_T^2\, \|\, D^2 u\|_{L_\infty(T)}} \Big\}$$

$$\sim \|h_T^2 D^2 u\|_{L_\infty(\Omega)} \sim \big\|\text{linear interpolation error}\big\|_{L_\infty(\Omega)}$$

Laplace equation $-\triangle u = f(x)$, linear elements, shape-regular mesh :

- In the H^1 and L_∞ norms:

$$\|\text{error}\|_* \leqslant \text{function}\big(\text{mesh, comp.solution}\big)$$

$$\sim \|\underbrace{\text{linear interpolation error}}_{\text{discrete analogue}}\|_*$$

- Higher-order elements + other norms + other equations have been considered as well.

- PURPOSE of such bounds: to be used in the automatic mesh adaptation...

$-\varepsilon^2 \triangle u + f(x,u) = 0$, **shape-regular mesh**, any-order FEM,
also analogous lower bounds...

- **Energy norm** $|||\text{error}|||_{\varepsilon\,;\Omega} := \varepsilon \|\nabla \text{error}\|_{L_2(\Omega)} + \|\text{error}\|_{L_2(\Omega)}$

  [Verfürth, Numer. Math., 1998, $-\varepsilon^2 \triangle u + u = f(x)$], for linear FEs :

$$\Big\{ \sum_{T\in\mathcal{T}} \Big(\underbrace{\|\min\{1, \tfrac{h_T}{\varepsilon}\}\, f(\cdot, u_h)\|^2_{L_2(T)}}_{\sim\|\varepsilon h_T \triangle u\|^2_{L_2(T)}} + \min\{1, \tfrac{\varepsilon}{h_T}\}\ \underbrace{h_T^2\, \|\varepsilon\, [\![\nabla u_h]\!]\|^2_{L_\infty(\partial T)}}_{\sim\|\varepsilon h_T\, D^2 u\|^2_{L_2(T)}} \Big) \Big\}^{1/2}$$

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- **L_∞ norm** [Demlow & Kopteva, Numer. Math. 2015], for linear FEs :

$$\max_{T\in\mathcal{T}}\Big\{\min\{1,\ell_h\frac{h_T^2}{\varepsilon^2}\}\,\underbrace{\|f(\cdot,u_h)\|_{L_\infty(T)}}_{\sim\varepsilon^2|\triangle_h u_h|+O(h_T^2)} + \min\{\varepsilon,\ell_h h_T\}\,\underbrace{\|[\![\nabla u_h]\!]\|_{L_\infty(\partial T)}}_{\sim h_T|D^2u|}\Big\}$$

where $\ell_h = \ln(2+\varepsilon h_{\min}^{-1})$

$-\varepsilon^2 \triangle u + f(x,u) = 0$, **ANISOTROPIC mesh** :

- **L_∞ norm** [Kopteva, SIAM J. Numer. Anal., 2015, new for $\varepsilon = 1$ and $\varepsilon \ll 1$]

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— [Kunert, Kunert & Verfürth, Numer. Math., 2000, $-\triangle u = f(x)$ and $-\varepsilon^2\triangle u + u = f(x)$]

ISSUE: the error constant involves the so-called *matching function* $\boxed{m(u-u_h,\mathcal{T})}$, which may be as large as the mesh aspect ratio $\frac{H_T}{h_T}$,

which is UNDESIRABLE...

. . . .

— [Kopteva, Numer. Math., 2017]

extends the framework of [Kopteva, SIAM J. Numer. Anal., 2015]

from the L_∞ to the energy norm... (*NO matching functions!*)

$-\varepsilon^2 \triangle u + f(x,u) = 0$, **ANISOTROPIC mesh** :

- **L_∞ norm** [Kopteva, SINUM, 2015, new for $\varepsilon = 1$ and $\varepsilon \ll 1$] — this talk ✓

- **Energy norm** $|\!|\!|\text{error}|\!|\!|_{\varepsilon\,;\Omega} = \varepsilon \|\nabla \text{error}\|_{L_2(\Omega)} + \|\text{error}\|_{L_2(\Omega)}$

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Part 2 **A bit of analysis**: 3 technical issues addressed

1. Application of a Scaled Trace theorem when estimating the Jump Residual (”long” edges cause problems...)
2. Shaper bounds for the Interior Residual (by identifying connected paths of anisotropic nodes...)
3. Quasi-interpolants (of Clément/Scott-Zhang type) are not readily available for general anisotropic meshes [Apel, Chapt. III]... —(may be of independent interest)

Part 3 **Some Numerics**

For $-\varepsilon^2\triangle u + f(x,u) = 0$, we consider a *standard finite element approximation*

$$\varepsilon^2(\nabla u_h, \nabla v_h) + (f_h^I, v_h) = 0, \quad v_h \in S_h, \quad f_h := f(\cdot, u_h)\,,$$

where $S_h \subset H_0^1(\Omega)$ is a linear finite element space

- Ω is a **polygonal, possibly non-Lipschitz,** domain in $\mathbb{R}^n$, $n = 2$:

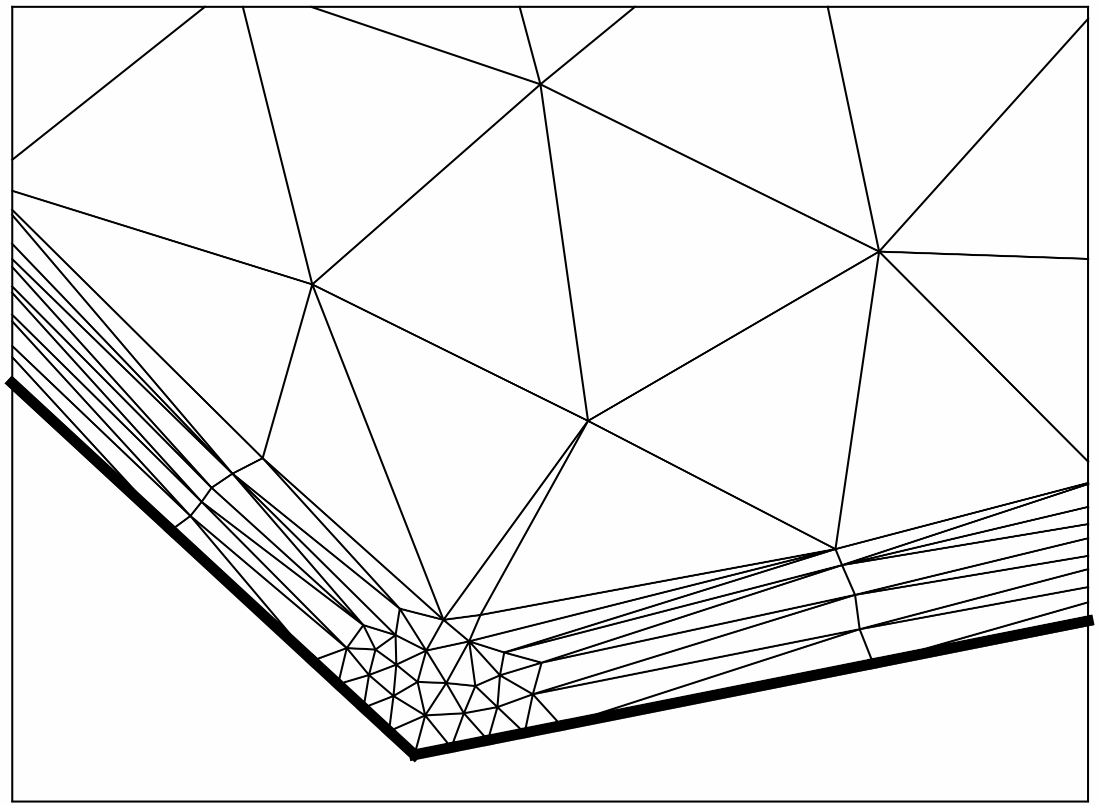
  $\Rightarrow \quad u \in H_0^1(\Omega) \cap C(\bar{\Omega})$;

  to be more precise, $u \in W_l^2(\Omega) \subseteq W_q^1 \subset C(\bar{\Omega})$ for some $l > \frac{1}{2}n$ and $q > n$.

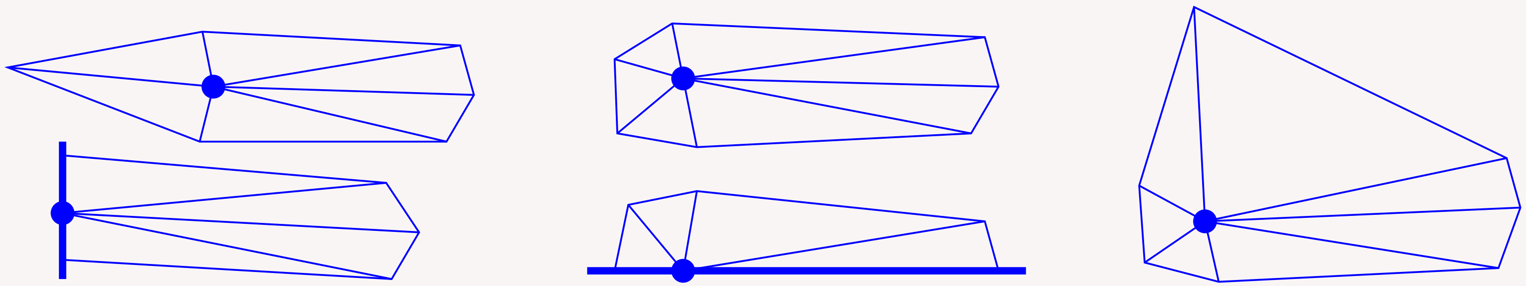
- one-sided-Lipschitz-condition version of $\quad f_u(x,u) \geqslant C_f \geqslant 0$,

  but $f_u \leqslant \bar{C}_f$ NOT assumed

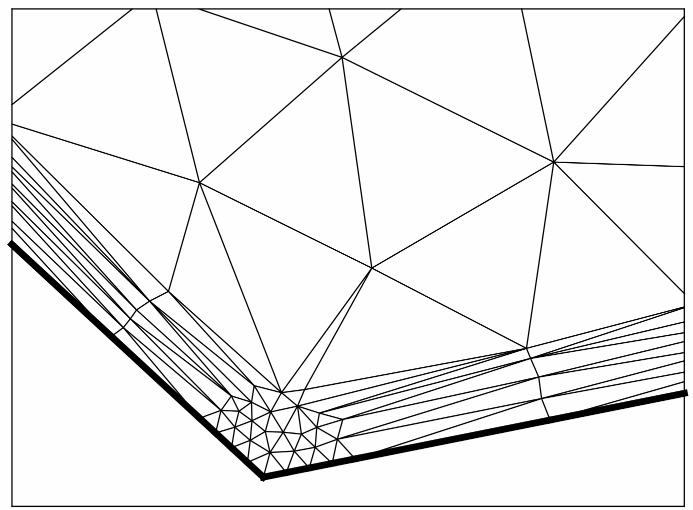
Roughly speaking, want to include meshes of the type:



- Permitted mesh node types:



- Example of a mesh for which the analysis works:

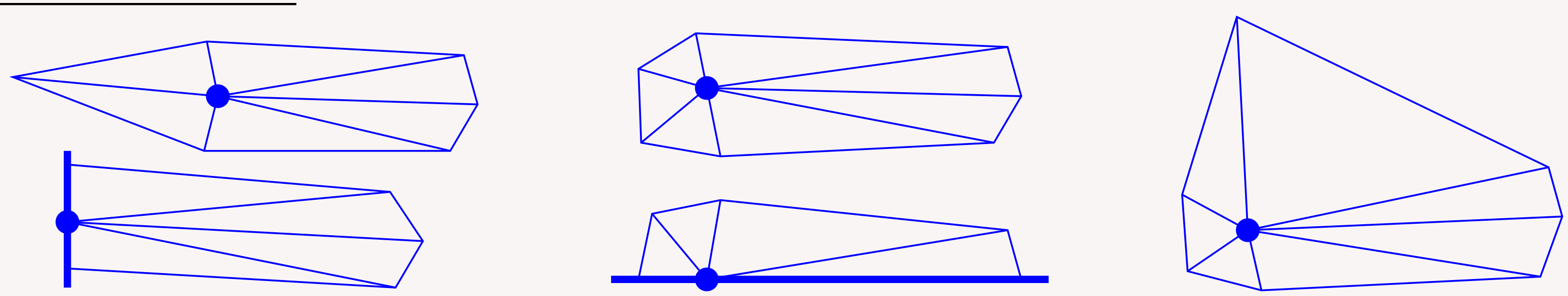


Notation: $H_T := \mathrm{diam}(T)$, $h_T := 2H_T^{-1}|T|$, $H_z := \mathrm{diam}(\omega_z)$, $h_z := \max_{T\subset\omega_z} h_T$

Main Triangulation Assumptions:

- *Maximum Angle condition.*
- *Local Element Orientation condition.* For any $z \in \mathcal{N}$, with the patch ω_z of elements surrounding z, there is a rectangle $R_z \supset \omega_z$ such that $|R_z| \sim |\omega_z|$.
- Also let the number of triangles containing any node be uniformly bounded.

Mesh Node Types:



L_∞ norm

For $\varepsilon = 1$, our ESTIMATOR reduces to

$$\|u_h - u\|_\infty \leqslant C\,\ell_h \max_{z\in\mathcal{N}}\Big(H_z \,\|[\![\nabla u_h]\!]\|_{\infty\,;\gamma_z}\Big) + \text{interior-residual terms}$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$.

Here $f_h = f(\cdot, u_h)$,
$\mathcal{N}$ is the set of nodes in $\mathcal{T}$,
$[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge,
ω_z is the patch of elements surrounding any $z \in \mathcal{N}$,
γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$,
$\ell_h = |\ln \underline{h}|$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.

$\mathbf{L}_\infty$ norm

Our FIRST ESTIMATOR reduces to

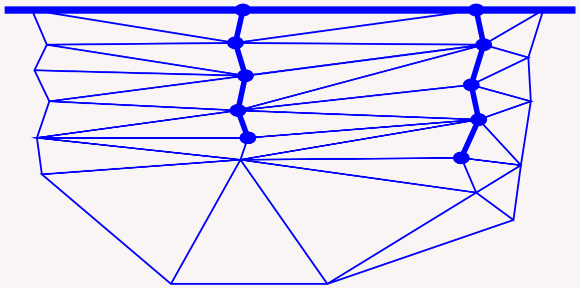
$$\|u_h - u\|_\infty \leqslant C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|[\![\nabla u_h]\!]\big\|_{\infty\,;\gamma_z} + \min\Big\{1, \frac{H_z^2}{\varepsilon^2}\Big\}\,\|f_h^I\|_{\infty\,;\omega_z}\Big) + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon\underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case (on the previous page), but now we assume no shape regularity of the mesh.

L_∞ norm In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes:



Under some additional assumptions on each such sequence (which we call a Path), our SECOND ESTIMATOR

$$\|u_h - u\|_\infty \leqslant C\,\ell_h \Big[\max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\|J_z\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\rm paths}}\Big(\min\{1,\varepsilon^{-2}H_z^2\}\|f_h^I\|_{\infty\,;\omega_z}\Big)$$
$$+ \max_{z\in\mathcal{N}_{\rm paths}}\Big(\min\{\varepsilon, H_z\}\min\{\varepsilon, h_z\}\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z} + \min\{1,\varepsilon^{-2}H_z^2\}\,{\rm osc}(f_h^I;\omega_z)\Big)\Big]$$
$$+ C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $\mathcal{N}_{\rm paths}$ is the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$, $J_z = [\![\nabla u_h]\!]$.

Energy norm

For $\varepsilon = 1$, our ESTIMATOR reduces to

$$\|u_h - u\|_{H^1(\Omega)} \leqslant C \left\{ \sum_{z\in\mathcal{N}} h_z H_z \, \|[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} \right\}^{1/2} + \text{interior-residual terms}$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, and $h_z H_z \sim |\omega_z|$ is the local volume.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Babuška et al], only now we prove it for anisotropic meshes.
- Relation to interpolation error bounds: $|[\![\nabla u_h]\!]|$ may be interpreted as approximating the diameter of ω_z under the metric induced by the squared Hessian matrix of the exact solution.

Energy norm

our FIRST ESTIMATOR reduces to

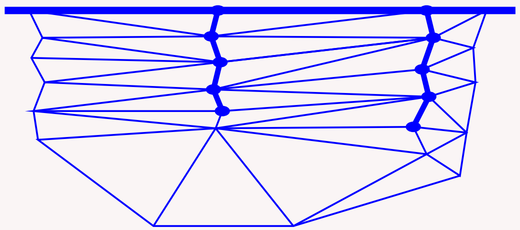
$$|||u_h - u|||_{\varepsilon\,;\Omega} \leqslant C\Big\{\sum_{z\in\mathcal{N}}\Big(\min\{1,\ \tfrac{\varepsilon}{h_z}\}\, h_z H_z \|\varepsilon[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} + \big\|\min\{1,\ \tfrac{H_z}{\varepsilon}\}\, f_h^I\big\|^2_{2\,;\omega_z}\Big)\Big\}^{1/2} + C\,\|f_h - f_h^I\|_{2\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, and $h_z \sim H_z^{-1}|\omega_z|$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Babuška et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case [Verfürth], but now we assume no shape regularity of the mesh.

Energy norm For a sharper (bound for the interior-residual component of the error, we again identify sequences of short edges that connect anisotropic nodes:



Under some additional assumptions on each such sequence (which we call a Path), our SECOND ESTIMATOR

$$|||u_h - u|||_{\varepsilon\,;\Omega} \leqslant C\Big\{\sum_{z\in\mathcal{N}} \min\{1,\, \tfrac{\varepsilon H_z}{h_z^2}\}\, h_z H_z \|\varepsilon[\![\nabla u_h]\!]\|^2_{\infty\,;\gamma_z} + \sum_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}} \|\min\{1,\, \tfrac{H_z}{\varepsilon}\}\, f_h^I\|^2_{2\,;\omega_z}$$

$$+ \sum_{z\in\mathcal{N}_{\text{paths}}} \Big(\|\min\{1,\, \tfrac{h_z}{\varepsilon}\}\, f_h^I\|^2_{2\,;\omega_z} + \|\min\{1,\, \tfrac{H_z}{\varepsilon}\}\, \mathrm{osc}(f_h^I;\omega_z)\|^2_{2\,;\omega_z}\Big)\Big]$$

$$+ C\,\|f_h - f_h^I\|_{2\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $\mathcal{N}_{\text{paths}}$ is the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$

- For a solution u and any $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$,

$$[u_h - u](x) = \varepsilon^2(\nabla u_h, \nabla G(x,\cdot)) + (f(\cdot, u_h), G(x,\cdot))$$

HINT: using the standard linearization $f(x, u_h) - f(x, u) = p(x)[u_h - u]$

with $p = \int_0^1 f_u(\cdot, u + [u_h - u]s)\, ds \geqslant C_f \geqslant 0$

- For each fixed $x \in \Omega$, the Green's function $G = G(x, \cdot)$ solves the problem

$$\begin{aligned} L^*G = -\varepsilon^2 \Delta_\xi G + p(\xi)\, G &= \delta(x - \xi), \qquad \xi \in \Omega, \\ G(x;\xi) &= 0, \qquad \xi \in \partial\Omega. \end{aligned}$$

(NOTE: similar to the dual problem...)

- For a solution u and any $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$,

$$u_h - u = \varepsilon^2(\nabla u_h, \nabla G) + (f(\cdot, u_h), G)$$

- THEOREM [Demlow, Kopteva, 2015] For any $x \in \Omega$,

$$\|G(x,\cdot)\|_{1;\Omega} + \varepsilon\|\nabla G(x,\cdot)\|_{1;\Omega} \lesssim 1.$$

For the ball $B(x,\varrho)$ of radius ϱ centered at $x \in \Omega$, and $\ell_\varrho := \ln(2 + \varepsilon\varrho^{-1})$,

$$\begin{aligned}
\|G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho^2\,\ell_\varrho,\\
\|\nabla G(x,\cdot)\|_{1,B(x,\varrho)\cap\Omega} &\lesssim \varepsilon^{-2}\varrho,\\
\|D^2 G(x,\cdot)\|_{1,\Omega\setminus B(x,\varrho)} &\lesssim \varepsilon^{-2}\ell_\varrho
\end{aligned}$$

- For a solution u and <u>any</u> $u_h \in H_0^1(\Omega) \cap W_1^q(\Omega)$ with $q > n = 2$, using the monotonicity of f and $C_f + \varepsilon^2 \geqslant 1$, one gets

$$\begin{aligned} |||u_h - u|||^2_{\varepsilon\,;\Omega} &\lesssim \varepsilon^2 \langle \nabla(u_h - u), \nabla(u_h - u)\rangle + \langle f(\cdot; u_h) - f(\cdot; u), u_h - u\rangle \\ &= \varepsilon^2 \langle \nabla u_h, \nabla(u_h - u)\rangle + \langle f(\cdot; u_h), u_h - u\rangle\,, \end{aligned}$$

where we also used $-\varepsilon^2 \triangle u + f(x,u) = 0$.

Next, assuming $|||u_h - u|||_{\varepsilon\,;\Omega} > 0$, let

$$G := \frac{u_h - u}{|||u_h - u|||_{\varepsilon\,;\Omega}} \qquad \Rightarrow \qquad |||G|||_{\varepsilon\,;\Omega} = 1$$

$$\Rightarrow \quad \boxed{|||u_h - u|||_{\varepsilon\,;\Omega} \lesssim \varepsilon^2 \langle \nabla u_h, \nabla G\rangle + \langle f(\cdot, u_h), G\rangle}$$

— similar to the case of L_∞ norm, only G is no longer the Green's function...

$$-\varepsilon\triangle u + \bar{a}\cdot\nabla u + bu = f(x)$$

- Ω is a polygonal Lipschitz domain in $\mathbb{R}^n$, $n = 2$, $-\frac{1}{2}\nabla\cdot\bar{a} + b \geqslant \beta \gtrsim \|b\|_{L_\infty(\Omega)}$
- [Verfürth, SINUM, 2005]

energy norm $|||v||| := \left\{\varepsilon\|\nabla v\|^2_{L_2(\Omega)} + \beta\|v\|^2_{L_2(\Omega)}\right\}^{1/2}$,

dual norm $|||\varphi|||_* := \sup\limits_{|||v|||=1} \langle\varphi, v\rangle$,

bilinear form $B(u,v) := \varepsilon\langle\nabla u, \nabla v\rangle + \langle\bar{a}\cdot\nabla u + bu, v\rangle$

$$\inf_u \sup_v \frac{B(u,v)}{\left(|||u||| + |||\bar{a}\cdot\nabla u|||_*\right)|||v|||} \gtrsim 1$$

$$\Rightarrow \quad \boxed{|||(u_h-u)||| + |||\bar{a}\cdot\nabla(u_h-u)|||_* \lesssim \sup_{|||G|||=1} B(u_h-u, G)}$$

$$-\varepsilon\triangle u + \bar{a}\cdot\nabla u + bu = f(x)$$

- So $$|||(u_h - u)||| + \|\bar{a}\cdot\nabla(u_h - u)\|_* \lesssim \sup_{|||G|||=1} B(u_h - u, G)$$

$$\Rightarrow \quad \lesssim \sup_{|||G|||=1}\left\{\varepsilon\langle\nabla u_h, \nabla G\rangle + \langle F(\cdot, u_h), G\rangle\right\}$$

where $F := \bar{a}\cdot\nabla u_h + bu_h - f$

—similar error representation to the Reaction-Diffusion case, so one can use almost the same analysis on anisotropic meshes!

- NOTE a change in the analysis:

$$\lesssim \sup_{|||G|||=1}\left\{\varepsilon\langle\nabla u_h, \nabla(G - G_h)\rangle + \langle F(\cdot, u_h), G - G_h\rangle + \textbf{stblz.-terms}\right\}$$

where stblz.-terms $=$ stabilization-terms$(\cdot, \mathbf{u_h}, \mathbf{G_h})$

NEXT: $$\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h) \quad \forall G_h \in S_h$$

NEXT: $$\boxed{\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h)} \quad \forall G_h \in S_h$$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h)) \, \nabla u_h \cdot \nu - \int_T \triangle u_h \, (G - G_h))$$

SO

$$\|u_h - u\|_{\dots} = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h) [\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \underbrace{\triangle u_h}_{=0}) \, (G - G_h)$$

NEXT: $\|u_h - u\|_{\dots} = \varepsilon^2(\nabla u_h, \nabla(G - G_h)) + (f_h, G - G_h)$ $\forall G_h \in S_h$

NOTE: by the **Divergence Theorem** for each $T \subset \mathcal{T}$,

$$\int_T \nabla u_h \cdot \nabla(G - G_h)) = \int_{\partial T} (G - G_h)) \nabla u_h \cdot \nu - \int_T \triangle u_h (G - G_h))$$

SO

$$\|u_h - u\|_{\dots} = \sum_{S \in \mathcal{S}} \varepsilon^2 \int_S (G - G_h)[\![\nabla u_h]\!] \cdot \nu + \sum_{T \in \mathcal{T}} \int_T (f_h - \varepsilon^2 \underbrace{\triangle u_h}_{=0}) (G - G_h)$$

As $\forall G_h \in S_h$, so replace $(G - G_h)$ by

$$G - G_h - \sum_{z \in \mathcal{N}} \bar{g}_z \phi_z = \sum_{z \in \mathcal{N}} [G - G_h - \bar{g}_z]\phi_z$$

where ϕ_z = the standard hat function associated with a node z

$$\|u_h - u\|_{\dots} = \sum_{z \in \mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu + \sum_{z \in \mathcal{N}} \int_{\omega_z} f_h [G - G_h - \bar{g}_z]\phi_z$$

JUMP RESIDUAL:

$$I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu$$

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **"long" edges** cause this problem).

Scaled Trace Theorem (for anisotropic elements; sharp):

$$\max_{S\in\{\textbf{short edges}\}} \|v\|_{1\,;S} + \frac{\mathbf{h_z}}{\mathbf{H_z}} \max_{S\in\{\textbf{long edges}\}} \|v\|_{1\,;S} \lesssim H_z^{-1}\|v\|_{1\,;\omega_z} + \|\nabla v\|_{1\,;\omega_z}$$

JUMP RESIDUAL:

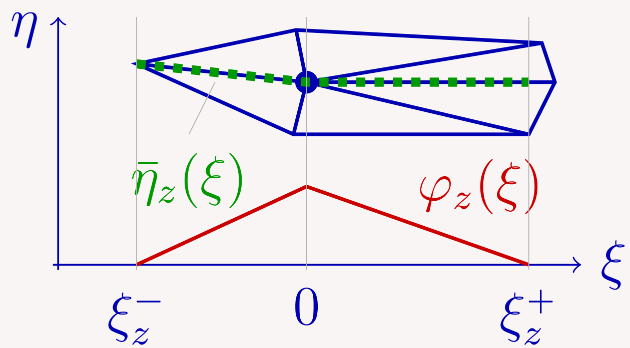
$$I := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} [G - G_h - \bar{g}_z]\phi_z [\![\nabla u_h]\!] \cdot \nu$$

NOTE: An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a **Scaled Trace Theorem** when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator; **”long” edges** cause this problem).

NOTE standard choices: $\bar{g}_z = 0$, or $\int_{\omega_z} (G - G_h - \bar{g}_z)\,\phi_z = 0$ [Nochetto].

Our CHOICE is crucial in addressing this difficulty:

$$\int_{\xi_z^-}^{\xi_z^+} \big[(G - G_h)(\xi, \bar{\eta}_z(\xi)) - \bar{g}_z\big]\,\varphi_z(\xi)\, d\xi = 0$$



Assuming that anisotropic mesh elements are almost non-obtuse ...,
our FIRST ESTIMATOR reduces to

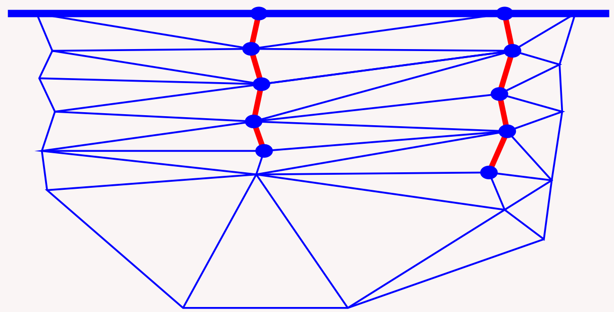
$$\|u_h - u\|_\infty \leqslant C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|[\![\nabla u_h]\!]\big\|_{\infty\,;\gamma_z} + \min\{\varepsilon^2, H_z^2\}\,\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z}\Big) + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

Here $f_h = f(\cdot, u_h)$, $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, $[\![\nabla u_h]\!]$ is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon \underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

- For $\varepsilon = 1$, this gives a standard a posteriori error bound, similar to [Eriksson, Nochetto, Nochetto et al], only now we prove it for anisotropic meshes.
- For $\varepsilon \in (0, 1]$, this is almost identical with our estimator for shape-regular case [Demlow, Kopteva], but now we assume no shape regularity of the mesh.

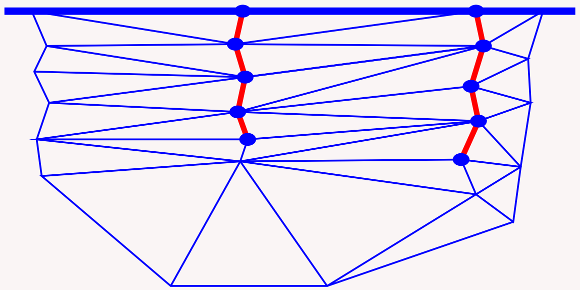
In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes (and call each of them a Path):



Main Additional Assumption: (Curvilinear version also ok...)

- *Path Coordinate-System condition.* For each (semi-)anisotropic path $\mathcal{N}_i$, $i = 1, \ldots, n_{\rm ani} + n_{\rm s.ani}$, let there exist a cartesian coordinate system $(\xi, \eta) = (\xi_i, \eta_i)$ such that $|\sin(\angle(S, {\rm i}_\xi))| \lesssim \frac{h_z}{|S|}$ for any $S \subset \mathcal{S}_z$ of any node $z \in \mathcal{N}_i$ (while, if $\mathcal{N}_i$ is semi-anisotropic a stronger condition $|\angle(S, {\rm i}_\xi)| \lesssim \frac{h_z}{|S|}$ is satisfied).

Let $\mathcal{N}_{\text{paths}}$ be the set of mesh nodes that appear in any path, $h_z \sim H_z^{-1}|\omega_z|$, $J_z = [\![\nabla u_h]\!]$.



SECOND ESTIMATOR

$$\|u_h - u\|_\infty \leqslant C\,\ell_h \Big[\max_{z\in\mathcal{N}} \Big(\min\{\varepsilon, H_z\}\,\|J_z\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}} \Big(\min\{1, \varepsilon^{-2}H_z^2\}\,\|f_h^I\|_{\infty\,;\omega_z}\Big)$$
$$+ \max_{z\in\mathcal{N}_{\text{paths}}} \Big(\min\{\varepsilon, H_z\}\ \min\{\varepsilon, h_z\}\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z} + \min\{1, \varepsilon^{-2}H_z^2\}\ \text{osc}(f_h^I;\omega_z)\Big)\Big]$$
$$+ C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,,$$

C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε.

TASK: estimate

$$\bar\Theta := \varepsilon^2 \sum_{T\in\mathcal{T}} \Big(\lambda_T^{p-2}\|\nabla(G-G_h)\|_{p\,;T}^p + \lambda_T^{-2}\|G-G_h\|_{p\,;T}^p\Big),\ \lambda_T := \min\{\varepsilon, H_T\},$$

Aim: $\bar\Theta \lesssim \ell_h$ for $p=1$ for L_∞ norm, or $\bar\Theta \lesssim 1$ for $p=2$...

- It would be convenient to employ a **quasi-interpolant (of Clément/Scott-Zhang type)** with the property

$$|G-G_h|_{k,p\,;T} \lesssim H_T^{j-k}|G|_{j,p\,;\omega_T} \text{ for any } 0 \leqslant k \leqslant j \leqslant 2,\ p=1.$$

T.b. more precise, the estimator involves $\min\{\underbrace{1}_{\text{from } k=j}, \underbrace{\frac{H_T^2}{\varepsilon^2}}_{\text{from } k<j}\}$

- However, such interpolants are not readily available for general anisotropic meshes (see [Apel, Chapt. III] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes).

- It would be convenient to employ a quasi-interpolant (of Clément/Scott-Zhang type) with the property

$$|G - G_h|_{k,p\,;T} \lesssim H_T^{j-k} |G|_{j,p\,;\omega_T} \text{ for any } 0 \leqslant \boxed{k \leqslant j} \leqslant 2,\ \ p = 1.$$

- However, such interpolants are not readily available for anisotropic meshes

- To deal with the maximum norm [Kopteva, 2015]:

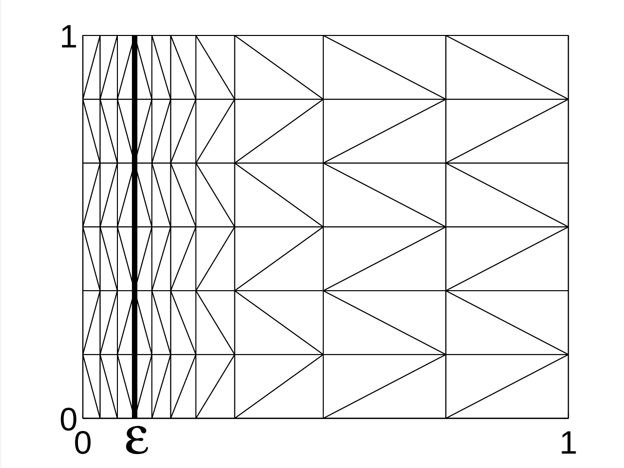
  Because of this difficulty, we employ a less standard interpolant G_h, which gives a version of the **Lagrange interpolant** whenever $H_T \lesssim \varepsilon$, and vanishes whenever $H_T \gtrsim \varepsilon$; however, this construction requires additional mild assumptions on the triangulation...

- To deal with the energy norm[Kopteva, 2017]:

  Quasi-interpolant of Clément/Scott-Zhang type are introduced on anisotropic meshes...

Simple 2d TEST problem: $-\varepsilon^2 \Delta u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

We consider one a-priori-chosen layer-adapted mesh of Bakhvalov type:



- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty\,;\Omega} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from 2 to 1**.

  This phenomenon is noted and explained in

  [N. Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, Math. Comp. 2014.].

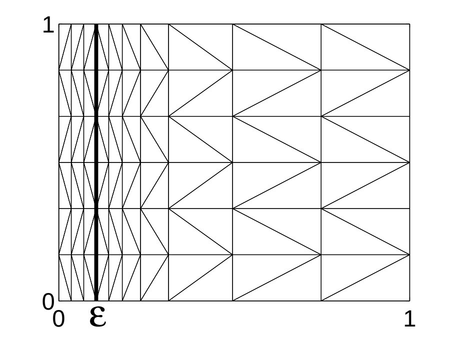
Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **maximum nodal errors** and estimators.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.373e-4	3.723e-3	8.952e-3	8.973e-3	8.973e-3	8.973e-3	8.973e-3
	2.00	1.91	1.01	1.00	1.00	1.00	1.00
128	8.445e-5	9.935e-4	4.446e-3	4.484e-3	4.484e-3	4.484e-3	4.484e-3
	2.00	1.98	1.04	1.00	1.00	1.00	1.00
256	2.112e-5	2.523e-4	2.165e-3	2.236e-3	2.236e-3	2.236e-3	2.236e-3
	FIRST Estimator (odd rows) & **Effectivity Indices** (even rows)						
64	6.810e-3	2.516e-1	9.403e-1	9.981e-1	9.999e-1	1.000e+0	1.000e+0
	20.19	67.59	105.04	111.23	111.44	111.45	111.45
128	1.761e-3	1.120e-1	8.858e-1	9.961e-1	9.999e-1	1.000e+0	1.000e+0
	20.86	112.72	199.26	222.15	222.98	223.01	223.01
256	4.480e-4	4.036e-2	7.901e-1	9.922e-1	9.998e-1	1.000e+0	1.000e+0
	21.21	159.97	365.01	443.82	447.17	447.27	447.28

Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **maximum nodal errors** and estimators.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.373e-4	3.723e-3	8.952e-3	8.973e-3	8.973e-3	8.973e-3	8.973e-3
	2.00	1.91	1.01	1.00	1.00	1.00	1.00
128	8.445e-5	9.935e-4	4.446e-3	4.484e-3	4.484e-3	4.484e-3	4.484e-3
	2.00	1.98	1.04	1.00	1.00	1.00	1.00
256	2.112e-5	2.523e-4	2.165e-3	2.236e-3	2.236e-3	2.236e-3	2.236e-3
	SECOND Estimator (odd rows) & **Effectivity Indices** (even rows)						
64	7.353e-3	1.204e-1	1.224e-1	1.230e-1	1.302e-1	1.302e-1	1.302e-1
	21.80	**32.33**	**13.68**	**14.48**	**14.51**	**14.51**	**14.51**
128	1.885e-3	3.212e-2	6.005e-2	6.621e-2	6.646e-2	6.647e-2	6.647e-2
	22.32	**32.33**	**13.51**	**14.77**	**14.82**	**14.82**	**14.82**
256	4.771e-4	8.268e-3	3.073e-2	3.328e-2	3.354e-2	3.354e-2	3.354e-2
	22.59	**32.77**	**14.20**	**14.89**	**15.00**	**15.00**	**15.00**

We considered one a-priori-chosen layer-adapted mesh of Bakhvalov type:



maximum nodal errors

- The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty\,;\infty} \lesssim N^{-2}$.
- However, **as $\varepsilon \to 0$, the convergence rates deteriorate from 2 to 1**.
- E.g., for the final choice of ε and N, the **aspect ratios** of the mesh elements take values **between 1 and 3.6e+8**.
- Considering these variations, the SECOND estimator performs reasonably well and its efficitivity indices stabilize as $\varepsilon \to 0$.
- By contrast, the FIRST estimator is adequate for $\varepsilon \sim 1$, but its effectivity deteriorates in the singularly perturbed regime.

Table: Bakhvalov mesh, $M = \frac{1}{2}N$: **energy-norm errors** and estimators.

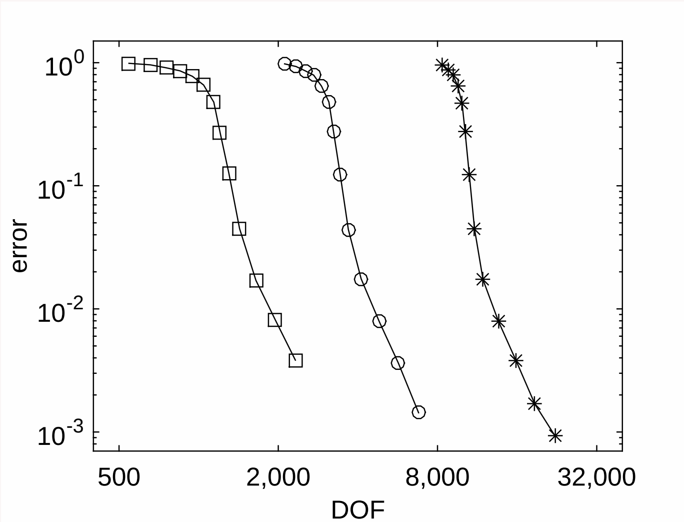
N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.202e-2	5.081e-3	7.993e-4	1.408e-4	2.489e-5	4.399e-6	7.777e-7
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
128	1.602e-2	2.564e-3	3.991e-4	7.028e-5	1.242e-5	2.196e-6	3.882e-7
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
256	8.011e-3	1.289e-3	1.997e-4	3.511e-5	6.207e-6	1.097e-6	1.940e-7
	SECOND Estimator (odd rows) & **Effectivity Indices** (even rows)						
64	1.041e-1	2.102e-2	4.129e-3	7.393e-4	1.308e-4	2.311e-5	4.086e-6
	3.25	**4.14**	**5.17**	**5.25**	**5.25**	**5.25**	**5.25**
128	5.147e-2	1.051e-2	2.050e-3	3.711e-4	6.566e-5	1.161e-5	2.052e-6
	3.21	**4.10**	**5.14**	**5.28**	**5.29**	**5.29**	**5.29**
256	2.559e-2	5.269e-3	1.006e-3	1.858e-4	3.290e-5	5.817e-6	1.028e-6
	3.19	**4.09**	**5.04**	**5.29**	**5.30**	**5.30**	**5.30**

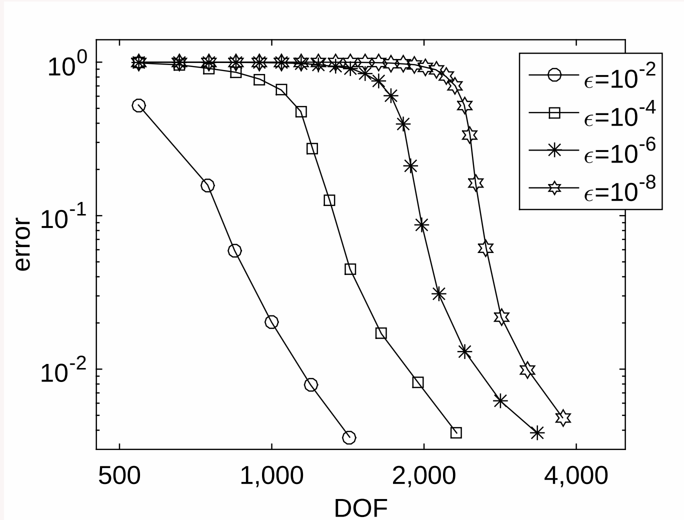
NOTE for $\varepsilon \ll 1$: $\|u_h - u^I\|_{2;\Omega} \simeq \varepsilon \|\nabla u_h - (\nabla u)^I\|_{2;\Omega} \simeq \varepsilon^{1/2} N^{-1}$

$$\Rightarrow \boxed{\|u_h - u\|_{2;\Omega} \simeq |||u_h - u|||_{\varepsilon;\Omega} \simeq \varepsilon^{1/2} N^{-1} + N^{-2}}$$

Simple 2d TEST problem: $-\varepsilon^2 \Delta u + u = F(x)$ in $\Omega = (0,1)^2$ with $\varepsilon^2 = 10^{-6}$,
$u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$

Maximum errors for $\varepsilon = 10^{-4}$ and initial DOF varied (left), and ε varied (right):





In each experiment, we started with a uniform mesh of right-angled triangles of diameter $H_T = 2^{-8}, 2^{-16}, 2^{-32}$, and aspect ratio $\frac{H_T}{h_T} = 2$. At each iteration, we marked for refinement the mesh elements responsible for at least 5% of the overall estimator $\mathcal{E}$, but no more than 15% of the elements. The marked elements were refined only in the x direction using a single or triple green refinement (depending on the orientation of the mesh element). Edge swapping was also employed to improve geometric properties of the mesh and/or possibly reduce $\max_{T\in\mathcal{T}}\{\mathrm{osc}(f_h^I;T)\}$.

Section A

Perceptions & expectations t.b. adjusted for anisotropic meshes

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Section B

Part 1 **Reaction-Diffusion eq. — a posteriori estimates on anisotropic meshes**

- — Problem addressed (more detail)
- — Mesh assumptions + preview of results
- — Error represenation ⇒ From the L_∞ to the energy norm??

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Section C

Efficiency, i.e. lower estimator: also problematic on anisotropic meshes...

Lower Error Estimators on anisotropic meshes in the energy norm???

(consistent with upper estimators?)

- **Standard Bubble Function Approach**

  This approach was employed by [Kunert & Verfürth 2000, Kunert 2001]: let $\varepsilon = 1$,

$$\underline{\mathcal{E}} := \Big\{ \sum_{S \in \mathcal{S} \setminus \partial\Omega} \varrho_S \, J_S^2 + \|h_T f_h^I\|_\Omega^2 \Big\}^{1/2} \lesssim \|u_h - u\|_{H^1(\Omega)} + \|h_T(f_h - f_h^I)\|_\Omega,$$

  For $S = \partial T_1 \cap \partial T_2$:

$$\boxed{\varrho_S = |S| \min\{h_{T_1}, h_{T_2}\}}$$

  We give **a numerical example** (for $\varepsilon = 1$) that clearly demonstrates that

  short-edge jump residual terms in such bounds are not sharp

- So, under additional restrictions on the anisotropic mesh, we shall give a **new bound for the short-edge jump residual terms**, and thus show that at least for some anisotropic meshes the error estimator constructed in the paper is efficient.

  For $\varepsilon = 1$ and $S = \partial T_1 \cap \partial T_2$:

$$\boxed{\varrho_S = |T_1 \cup T_2| = \text{local volume}}$$

Table 1: **Lower error estimators** for test problem with $u = \sin(\pi a x)$ and $\varepsilon = 1$.

	$a = 1$			$a = 3$		
	$N = 20$	$N = 40$	$N = 80$	$N = 20$	$N = 40$	$N = 80$
	Errors $\|u_h - u\|_{H^1(\Omega)}$					
$M = 2N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 8N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 32N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 128N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
	$\underline{\mathcal{E}}$ with $\varrho_S = \|S\| \min\{h_{T_1}, h_{T_2}\}$ (odd rows) & **Effectivity Indices** (even rows)					
$M = 2N$	2.89e-1	1.45e-1	7.24e-2	2.51e+0	1.26e+0	6.33e-1
	2.87	**2.88**	**2.88**	**2.72**	**2.78**	**2.79**
$M = 8N$	1.32e-1	6.59e-2	3.30e-2	1.17e+0	5.86e-1	2.93e-1
	1.31	**1.31**	**1.31**	**1.26**	**1.29**	**1.29**
$M = 32N$	6.27e-2	3.14e-2	1.57e-2	5.62e-1	2.82e-1	1.41e-1
	0.62	**0.62**	**0.62**	**0.61**	**0.62**	**0.62**
$M = 128N$	3.10e-2	1.55e-2	7.75e-3	2.79e-1	1.39e-1	6.97e-2
	0.31	**0.31**	**0.31**	**0.30**	**0.31**	**0.31**

Standard Bubble Function Approach $\Rightarrow$ **Lower Estimator NOT SHARP**

Table 2: **Lower error estimators** for test problem with $u = \sin(\pi a x)$ and $\varepsilon = 1$.

	$a = 1$			$a = 3$		
	$N = 20$	$N = 40$	$N = 80$	$N = 20$	$N = 40$	$N = 80$
	Errors $\|u_h - u\|_{H^1(\Omega)}$					
$M = 2N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 8N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 32N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
$M = 128N$	1.01e-1	5.04e-2	2.52e-2	9.26e-1	4.56e-1	2.27e-1
	$\underline{\mathcal{E}}$ with $\varrho_S = \|T_1 \cup T_2\|$ (odd rows) & **Effectivity Indices** (even rows)					
$M = 2N$	3.00e-1	1.50e-1	7.52e-2	2.61e+0	1.32e+0	6.59e-1
	2.98	**2.98**	**2.98**	**2.82**	**2.89**	**2.90**
$M = 8N$	2.51e-1	1.26e-1	6.28e-2	2.25e+0	1.13e+0	5.64e-1
	2.49	**2.49**	**2.49**	**2.43**	**2.47**	**2.48**
$M = 32N$	2.47e-1	1.23e-1	6.18e-2	2.21e+0	1.11e+0	5.56e-1
	2.45	**2.45**	**2.45**	**2.39**	**2.44**	**2.45**
$M = 128N$	2.46e-1	1.23e-1	6.17e-2	2.21e+0	1.11e+0	5.55e-1
	2.44	**2.45**	**2.45**	**2.39**	**2.43**	**2.45**

New Lower Error Estimator $\Rightarrow$ **EFFICIENT**

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Efficiency, i.e. lower estimator: also problematic on anisotropic meshes...

We use a new approach to prove efficiency under certain mesh assumptions...

- N. Kopteva, *Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations*, *Math. Comp.*, 2014.

- A. Demlow and N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math., 2015.
- N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, *SIAM J. Numer. Anal.*, 2015.
- N. Kopteva, *Energy-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, *Numer. Math.*, 2017

- N. Kopteva, *Fully computable a posteriori error estimator using anisotropic flux equilibration on anisotropic meshes*, 2017, submitted for publication, http://www.staff.ul.ie/natalia/pubs.html.

Thank you!