Reaction-diffusion equations in periodic media.

Influence of the geometry of the domain

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Joint work with Luca Rossi

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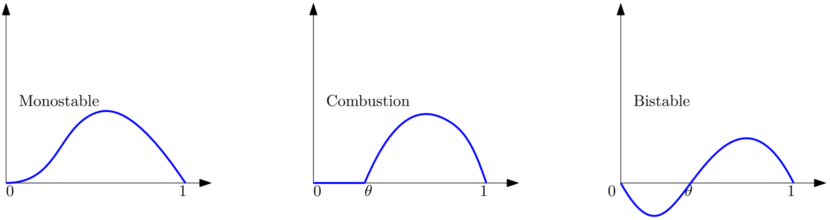
Reaction-diffusion equations

Homogeneous reaction-diffusion equation

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0$$

Reaction terms f :



$\longrightarrow$ Long-time behavior determined by the sign of $\int_0^1 f$

Fronts

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0$$

Theorem (Kolmogorov, Petrovski, Piskunov (1937), Kanel' (1964), Aronson, Weinberger (1978))

There is $c^\star \in \mathbb{R}$ such that there exist traveling fronts with speed $c^\star$ in the direction $e \in \mathbb{S}^{N-1}$.
Moreover, $c^\star$ has the sign of $\int_0^1 f$.

$\longrightarrow$ $c^\star$: minimal speed.

The invasion phenomenon

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0$$

Definition (Invasion)

Invasion occurs for the initial datum u_0 if

$$u(t,x) \underset{t\to+\infty}{\longrightarrow} 1, \quad \text{locally uniformly in } \ x.$$

In the sequel, we always consider initial data that are **compactly supported, non-zero, non-negative**.

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Definition (Invasion)

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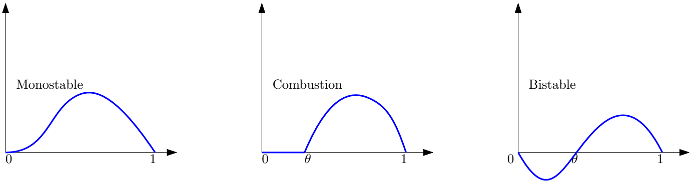
Definition (Speed of invasion)

Invasion occurs with the speed $w \in \mathbb{R}$ in the direction $e \in \mathbb{S}^{N-1}$ if

$$u(t,cte) \underset{t\to+\infty}{\longrightarrow} \begin{cases} 1 & \text{if } \ c \in [0,w) \\ 0 & \text{if } \ c > w \end{cases}$$

The invasion phenomenon

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0$$



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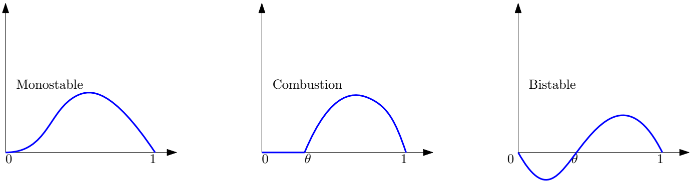


Observation

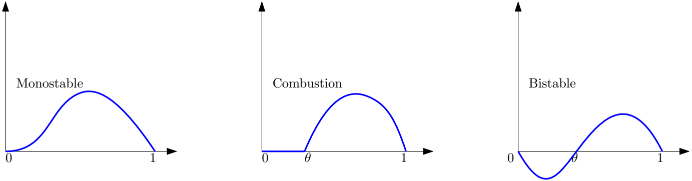
If f is of the combustion or bistable type and if $u_0 \leq \theta$, then

$$u(t,x) \underset{t\to+\infty}{\longrightarrow} 0$$

Invasion can not hold for every initial data !

The invasion phenomenon

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0$$



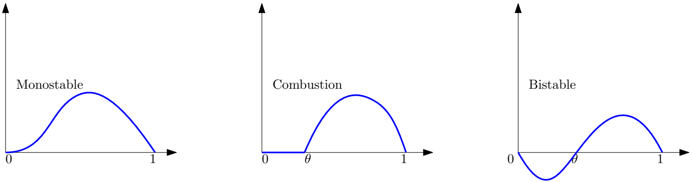
Definition (Invasion for large initial data)

We say that **invasion occurs for large enough initial data** provided that, for every $\eta \in (\theta, 1)$, there is $R > 0$ such that invasion occurs for initial data u_0 satisfying

$$u_0 \geq \eta \mathbb{1}_{B_R}$$

The invasion phenomenon

$$\partial_t u = d\Delta u + f(u), \quad x \in \mathbb{R}^N, \ t > 0$$



Theorem (Aronson, Weinberger (1978))

Invasion occurs for large enough initial data provided

$$\int_0^1 f > 0$$

Moreover, $w = c^\star$.

Reaction-diffusion in periodic media

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \nu \cdot \nabla u &= 0, \qquad\qquad\quad x \in \partial\Omega,\ t > 0 \end{cases}$$

Periodicity assumption :

$$\Omega + k = \Omega, \quad \forall k \in \mathbb{Z}^N,$$

Existence of fronts

$$\begin{cases} \partial_t u = \Delta u + f(u), & x \in \Omega,\ t > 0 \\ \nu \cdot \nabla u = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

$\rightarrow$ *No fronts here.* Instead : *pulsating traveling fronts* (introduced by Shigesada, Kawasaki, Teramoto (1986)).

$$u(t,x) := \phi(x \cdot e - ct, x)$$

Theorem (Berestycki, Hamel (2002))

There is $c^\star : \mathbb{S}^{N-1} \to \mathbb{R}^+$ such that there exist pulsating traveling fronts with speed $c^\star(e)$ in the direction $e \in \mathbb{S}^{N-1}$ if f is monostable or combustion.

$\longrightarrow c^\star$: minimal speed.

Reaction-diffusion in periodic media

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad & x \in \Omega,\ t > 0 \\ \nu \cdot \nabla u &= 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

Question

In this heterogeneous framework, when do we have invasion ?

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Theorem (D., Rossi (2017))

Existence of fronts with positive speed in every direction

$\Longrightarrow$

Invasion occurs for large enough initial data

$\rightarrow$ *Generalizes a theorem by Weinberger (2002)*

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Corollary (D., Rossi (2017))

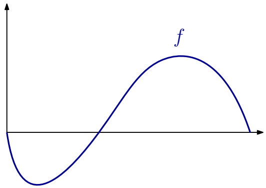
If f is of the combustion or monostable type, then invasion occurs for large enough initial data.

Influence of the geometry of Ω : invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \partial_\nu u &= 0, \qquad x \in \partial\Omega,\ t > 0 \end{cases}$$

Question

What about the bistable case ? (Existence of fronts not known in general)



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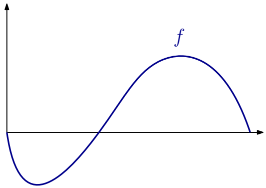
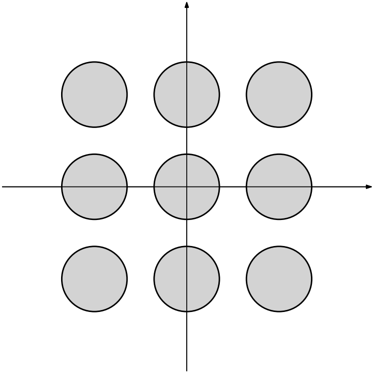


The geometry of Ω plays a role !

Influence of the geometry of Ω : invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \partial_\nu u &= 0, \qquad\qquad\qquad x \in \partial\Omega,\ t > 0 \end{cases}$$

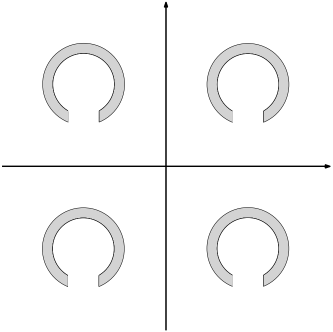
Star-shaped small obstacles $\implies$ Invasion occurs for large enough initial data



Influence of the geometry of Ω : invasion

$$\begin{cases} \partial_t u = \Delta u + f(u), & x \in \Omega,\ t > 0 \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

There are domains where invasion is blocked :



Follows from a theorem by Berestycki, Hamel and Matano.

Reason : Bistable is not "bi"-stable here !

A new phenomenon : oriented invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \partial_\nu u &= 0, \qquad\qquad\quad x \in \partial\Omega,\ t > 0 \end{cases}$$

Theorem (D., Rossi (2017))

There are periodic domains Ω where "invasion" occurs in some directions, and "blocking" in others.

A new phenomenon : oriented invasion

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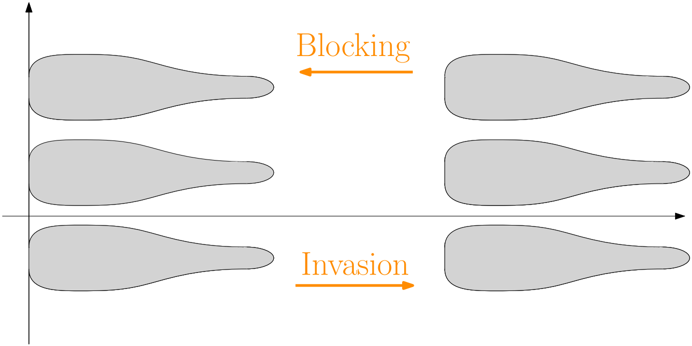

Proof : "sliding" method + Gidas-Ni-Nirenberg theorem +Moving planes+"teleporting"

Oriented invasion in a cylinder

Speed of invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \nu \cdot \nabla u &= 0, \qquad x \in \partial\Omega,\ t > 0 \end{cases}$$

Question

Can we compute the speed of invasion ?

$$w : \mathbb{S}^{N-1} \to \mathbb{R}\ ?$$

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Can we compute the speed of invasion ?

$$w : \mathbb{S}^{N-1} \to \mathbb{R} \ ?$$

Theorem (D. (2017))

If there are fronts with positive speed in every direction, there holds

$$w(e) = \min_{\xi \cdot e > 0} \frac{c^\star(\xi)}{\xi \cdot e}$$

Proof : follows a geometric argument by Rossi

The Freidlin-Gartner formula

For KPP nonlinearities :

- Freidlin and Gartner (1979) : probability theory
- Berestycki, Hamel and Nadin (2008) : PDE techniques
- Berestycki and Nadin (2016) : homogenization (see also Evans, Souganidis)

For more general nonlinearities :

- Weinberger (2002) : abstract monotone systems. Works for $f \geq 0$ and $\Omega \neq \mathbb{R}^N$.
- Rossi (2017) : regularity theory and geometric interpretation. On $\mathbb{R}^N$.

Influence of the geometry of Ω on the speed of invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad x \in \Omega,\ t > 0 \\ \partial_\nu u &= 0, \qquad x \in \partial\Omega,\ t > 0 \end{cases}$$

If $\Omega = \mathbb{R}^N$, then $w \equiv c^\star$, that is, **invasion occurs with its maximal possible speed.**

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Question (Berestycki, Hamel, Nadirashvili (2005))

Are there domains Ω where this is not the case, i.e., where

$$w \not\equiv c^\star?$$

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Equivalent problem (D. (2017))

Are there domains Ω where w is not constant ?

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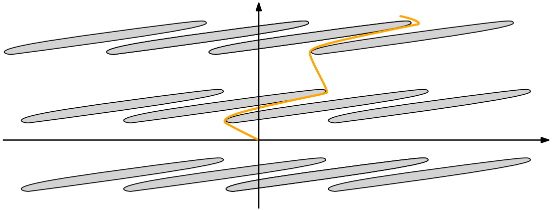

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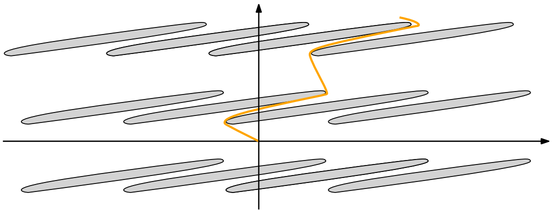

Proposition (D. (2017))

$$w(e) \leq 2\sqrt{\max_{u\in[0,1]} \frac{f(u)}{u} C_\Omega(e)},$$

where

$$C_\Omega(e) := \liminf_{\lambda\to+\infty} \frac{\lambda}{d_\Omega(0,\lambda e)} \leq 1.$$

Influence of the geometry of Ω on the speed of invasion

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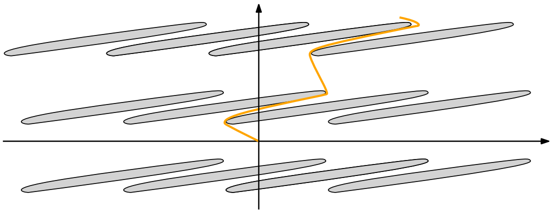

Theorem (D. (2017))

If f is monostable or combustion, there are periodic domains Ω where

$$w \not\equiv c^\star$$

Symmetries of Ω and invasion

$$\begin{cases} \partial_t u &= \Delta u + f(u), \qquad & x \in \Omega,\ t > 0 \\ \partial_\nu u &= 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

Question

On the contrary, when does the invasion occur at the speed of fronts ?

A simple observation

If e_{min} minimizes $c^\star$, then

$$w(e_{min}) = c^\star(e_{min})$$

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Theorem (Berestycki, Hamel, Nadirashvili (2005))

If f is of the KPP type and if Ω is invariant by translations in the direction e, then

$$w(e) = c^\star(e)$$

Symmetries of Ω and invasion

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Question

On the contrary, when does the invasion occur at the speed of fronts ?

Theorem (D. (2017))

If f is of the KPP type and if there is $T \in \mathcal{O}_N$ such that

- *T leaves Ω invariant, i.e., $T\Omega = \Omega$,*
- *There is $e \in \mathbb{S}^{N-1}$ such that $Te = e$ and $Ker(T - I_N) = \mathbb{R}e$,*

then

$$w(e) = c^\star(e)$$

$\Longrightarrow$ In the direction of axes of symmetry, invasion occurs at the maximal speed.

Thank you for your attention !