Singularities of Hessenberg Varieties

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Based on joint works with Martha Precup and Alexander Yong.

Where is FGCU?

Geometry of the Full Flag Variety

The Bruhat decomposition of $GL_n(\mathbb{C})$ is:

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in \mathfrak{S}_n} BwB$$

where $w \in \mathfrak{S}_n$ is identified with the corresponding permutation matrix. This implies:

$$GL_n(\mathbb{C})/B = \bigsqcup_{w \in \mathfrak{S}_n} BwB/B$$

$C_w := BwB/B$ is the Schubert cell.

- A **Hessenberg function** is a function $h\colon [n] \to [n]$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$.
- We often represent h as a tuple $(h(1), h(2), \ldots, n)$.
- To a Hessenberg function h we associate a subspace of $\mathfrak{gl}_n(\mathbb{C})$ (the vector space of $n \times n$ complex matrices) defined as

$$H(h) := \{(a_{i,j})_{i,j \in [n]} \in \mathfrak{gl}_n(\mathbb{C}) \mid a_{i,j} = 0 \text{ if } i > h(j)\}, \quad (1)$$

which we call the **Hessenberg subspace** $H(h)$.

Visualize the **Hessenberg subspace** $H(h)$ as a configuration of boxes on a square grid of size $n \times n$ whose shaded boxes correspond to the $a_{i,j}$ which are allowed to be non-zero (see Figure 1).

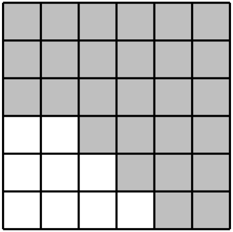

Figure: The picture of $H(h)$ for $h = (3, 3, 4, 5, 6, 6)$.

Definition

Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and $h\colon [n] \to [n]$ a Hessenberg function. The **Hessenberg variety** associated to A and h is defined to be

$$\mathrm{Hess}(A, h) = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid g^{-1}Ag \in H(h)\}. \qquad (2)$$

- nilpotent $\mathrm{Hess}(N, h)$ tend to be more singular, and not very symmetric
- semisimple $\mathrm{Hess}(S, h)$ tend to be smoother, with more group actions

Examples

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- When S_r is a regular semisimple matrix (diagonal with distinct eigenvalues) and $h = (2, 3, 4, ..., n, n)$, then $\text{Hess}(S_r, h)$ is the toric variety associated to the Weyl chambers.

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- For any Hessenberg function and regular semisimple element S_r, there an $\mathfrak{S}_n$-action on $H^*(\mathrm{Hess}(S_r, h))$ called the dot action.

Hessenberg-Schubert cells

Let $C_w \cap \mathrm{Hess}(A, h)$ denote the intersection of a Schubert cell with the Hessenberg variety.

Theorem (Tymoczko 06, Precup 12)

The Hessenberg-Schubert cells form a paving by affines of $\mathrm{Hess}(A, h)$.

The singular locus of the Peterson variety

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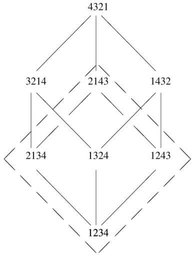
Theorem (I.-Yong 2012)

Let N be regular nilpotent and $h = (2, 3, \ldots, n, n)$ so $\mathrm{Hess}(N, h)$ is the Peterson variety. A point $gB \in (C_w \cap \mathrm{Hess}(N, h))$ is singular if the torus-fixed point wB is singular in $\mathrm{Hess}(N, h)$. Moreover, there are only 3 nonsingular torus-fixed points in $\mathrm{Hess}(N, h)$.



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Theorem (Abe, DeDieu, Galetto, Harada 2018)

If N is regular nilpotent, then $\mathrm{Hess}(N, h)$ is a local complete intersection.

The Bruhat Graph

Let s_i denote the simple transposition in $\mathfrak{S}_n$ exchanging i and $i+1$. The length of $w \in \mathfrak{S}_n$, denoted $\ell(w)$, is the minimum number of simple transpositions in any reduced word

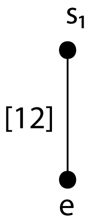
$$w = s_{i_1} s_{i_2} \cdots s_{i_k}.$$

and $\ell(u) \leq \ell(w)$.

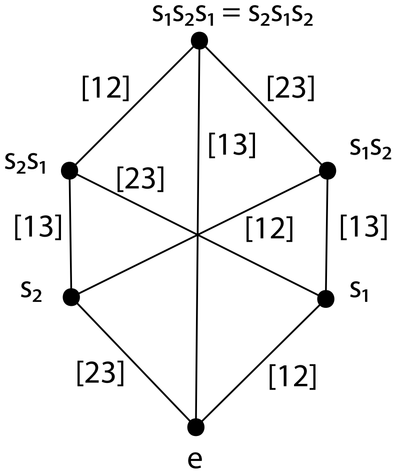
The Bruhat graph of $\mathfrak{S}_n$ is a directed graph with vertex set $\mathfrak{S}_n$ and (labeled) edges:

for all $u, w \in \mathfrak{S}_n$ such that $w = su$ for the transposition s which exchanges i and j and $\ell(u) \leq \ell(w)$.

Example: Bruhat Graphs for $\mathfrak{S}_2$ and $\mathfrak{S}_3$

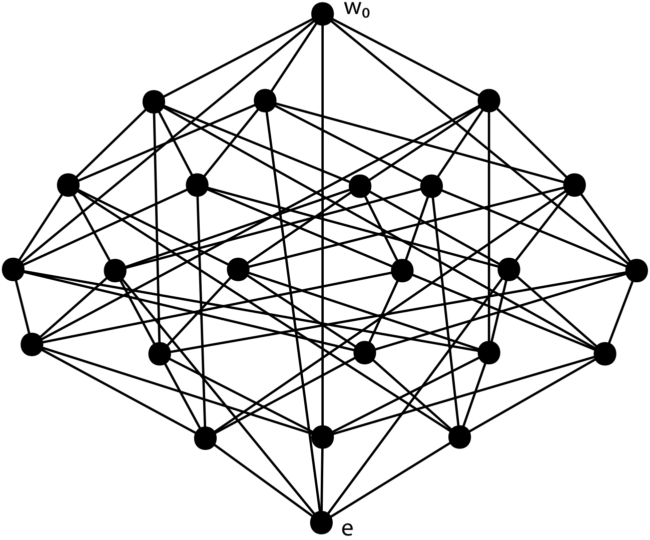


S_2



S_3

Example: Bruhat Graph for $\mathfrak{S}_4$



Motivating questions posed by Tymoczko in 2006.

- Let X be any linear operator. If the Hessenberg space is in banded form (such as the standard Hessenberg space), is $\mathrm{Hess}(X, h)$ pure-dimensional?

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- Let X be any linear operator. If the Hessenberg space is in banded form (such as the standard Hessenberg space), is $\mathrm{Hess}(X, h)$ pure-dimensional?
- Are all semisimple Hessenberg varieties smooth?

Properties of semisimple Hessenberg varieties

Hess. fun.	Jordan Blocks of S	Singular	Irreduc.	Pure-Dim?
(2,3,4,4)	(3,1)	Yes	No	Yes
(2,4,4,4)	(3,1)	Yes	No	No
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(2,3,4,4)	(2,2)	Yes	No	No
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- $\text{Hess}(S, h)$ are not smooth in general.

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Take Aways:

- $\text{Hess}(S, h)$ are not smooth in general.
- $\text{Hess}(S, h)$ is not pure-dimensional for $h = (2, 3, 4, 4)$.
- $\text{Hess}(S, h)$ can have singular irreducible components.

The Geometry of Semisimple Hessenberg Varieties

Recalling the Bruhat decomposition,

$$\mathrm{GL}/B = \bigsqcup_{w \in \mathfrak{S}_n} C_w \Rightarrow \mathrm{Hess}(S,h) = \bigsqcup_{w \in \mathfrak{S}_n} (C_w \cap \mathrm{Hess}(S,h)).$$

We call $C_w \cap \mathrm{Hess}(S,h)$ a Hessenberg-Schubert cell.

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- $\overline{C_w \cap \mathrm{Hess}(S,h)} = ?$
- What is the singular locus of $\mathrm{Hess}(S,h)$?

Goal: To answer the above questions for $h = (2,3,4,...,n,n)$.

The standard Hessenberg space

From now on we fix $h = (2, 3, \ldots, n, n)$, i.e., the one defining the Peterson varieties and the toric varieties, and we vary the (conjugacy class of the) semisimple operator S.

Group Actions

Let M be the centralizer of S in $GL_n(\mathbb{C})$. M is a block-diagonal subgroup.

Example

If $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ then $M = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$

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The GKM-graph of Hess(S, h):

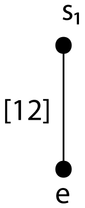
- Vertices are indexed by $\mathfrak{S}_n$.
- Remove any edge from the GKM-graph of $\mathrm{GL}_n(\mathbb{C})/B$ labeled by $[ij]$ such that $E_{ij} \notin M$ and $w^{-1}(i) > h(w^{-1}(j)) = w^{-1}(j) + 1$.

The Regular Semisimple Case

Theorem: (De Mari, Procesi, Shayman 1992) Let S_r be a regular semisimple matrix. Then $\text{Hess}(S_r, h)$ is a smooth, irreducible variety. It is the toric variety associated to the Weyl chambers.

Example: Let $S_r = \text{diag}(1, -1)$. In this case, $M = T$.

Remove all edges $[ij]$ such that $w^{-1}(i) > w^{-1}(j) + 1$



In this case, $\text{Hess}(S_r, h) = \mathbb{P}^1$.

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Example: Let $S_r = \mathrm{diag}(1, 2, 3)$. In this case, $M = T$.

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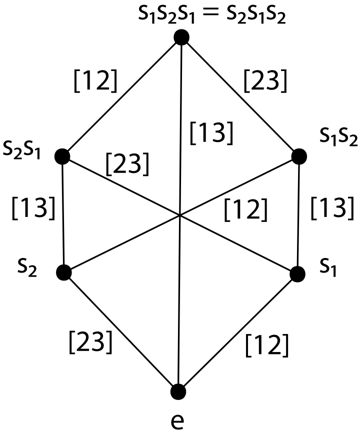

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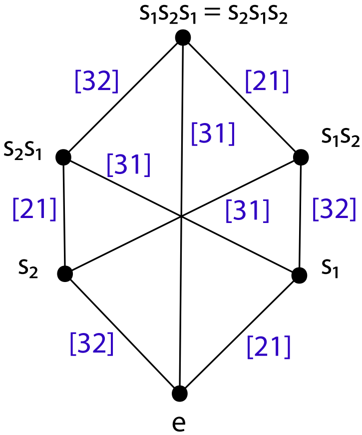

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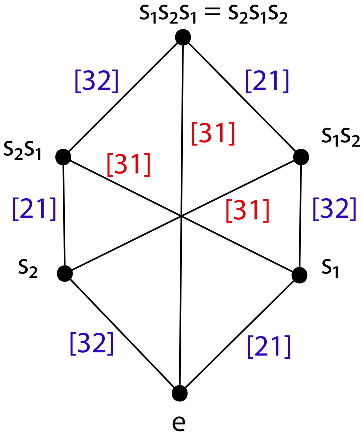

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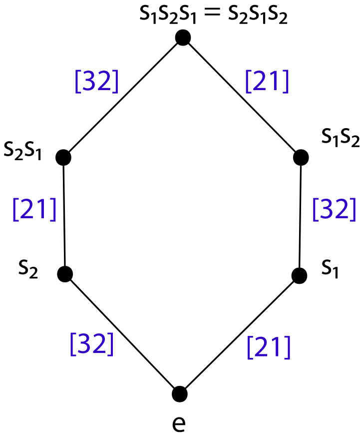

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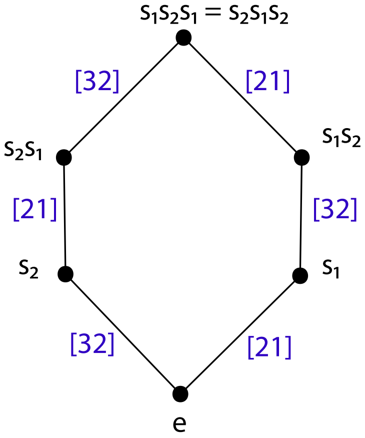
Example: Let $S_r = \mathrm{diag}(1,2,3)$. In this case, $M = T$.

$\dim(C_w \cap \mathrm{Hess}(S_r, h))$ is given by the number of edges incident to w and $y \leq w$



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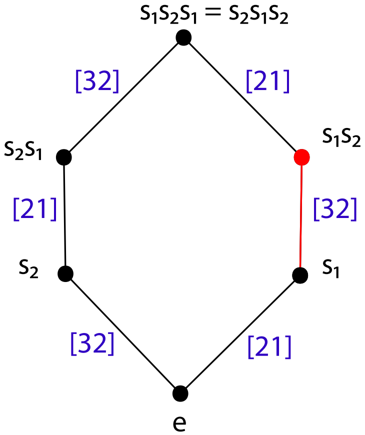

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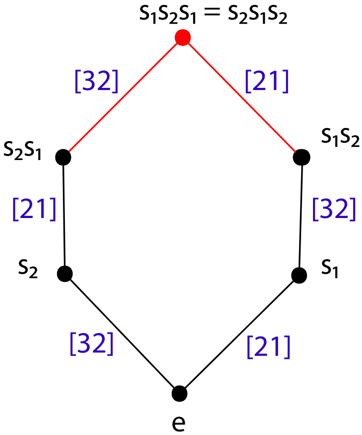
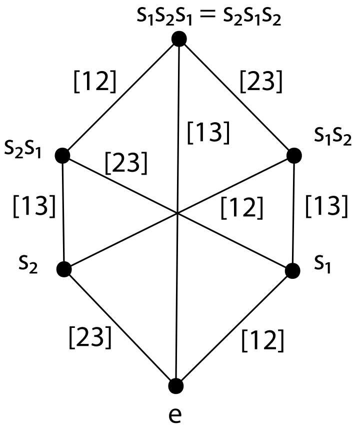
$\mathrm{Hess}(S_r, h) = \overline{C_{w_0} \cap \mathrm{Hess}(S_r, h)}$



The Non-regular Case

Example: Let $S = \text{diag}[1, 1, -2]$ so $M = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$.

Remove all edges $[ij]$ such that $E_{ij} \notin M$ and $w^{-1}(i) > w^{-1}(j) + 1$



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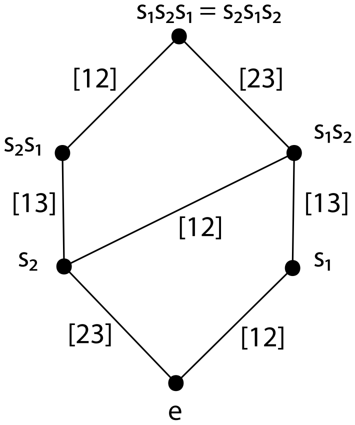

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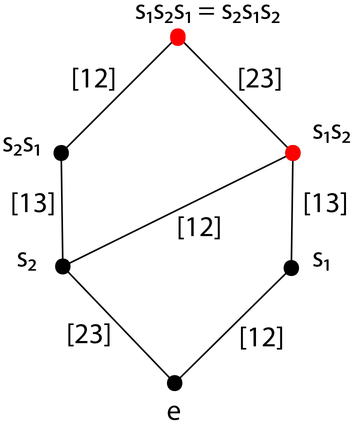
Both $C_{s_1 s_2 s_1} \cap \text{Hess}(S, h)$ and and $C_{s_1 s_2} \cap \text{Hess}(S, h)$ have dimension 2



Preview of Main Results

Theorem (I.-Precup)

The irreducible components of $\mathrm{Hess}(S, h)$ *are of the form*

$$M \cdot (\overline{C_v \cap \mathrm{Hess}(S, h)})$$

for v *in a certain subset of* ${}^{M}W$.

Theorem (I.-Precup)

Each irreducible component of $\mathrm{Hess}(S, h)$ *is smooth. Therefore the singularities of* $\mathrm{Hess}(S, h)$ *occur exactly where two irreducible components intersect.*

Using the M-orbit

Fact: If g_1B and g_2B are in the same M-orbit of $\mathrm{Hess}(S, h)$, then g_1B is singular if and only if g_2B is.

Let $W_M = \langle s_i : s_i \in M \rangle$. For each $w \in \mathfrak{S}_n$ there exists a unique $v \in \mathfrak{S}_n$ and $y \in W_M$ such that

$$w = yv \quad \text{and} \quad \ell(w) = \ell(y) + \ell(v).$$

We say that v is the shortest coset representative for $W_M \backslash \mathfrak{S}_n$. Denote the subset of shortest coset representatives by ${}^M W$.

Example

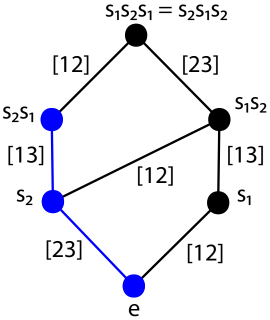
$$M = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \text{ so } W_M = \{e, s_1\} \text{ and } {}^M W = \{e, s_2, s_2 s_1\}.$$

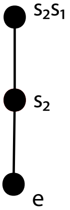
M-orbit on Hess(S,h)

If $v \in {}^M W$, then

$$M \cdot C_v = \bigsqcup_{y \in W_M} C_{yv} \Rightarrow M \cdot (C_v \cap \mathrm{Hess}(S,h)) = \bigsqcup_{y \in W_M} (C_{yv} \cap \mathrm{Hess}(S,h)).$$

Example: Let $S = \mathrm{diag}[1,1,-2]$ so $M = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$.





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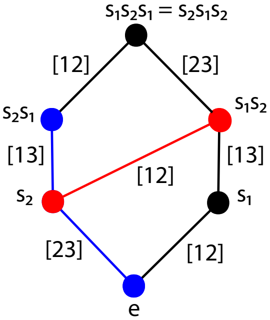
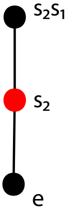
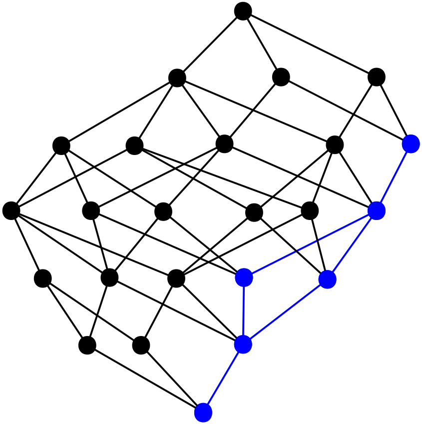




Another Example

Let $S = \text{diag}[1, 1, -1, -1]$ so $M = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$.



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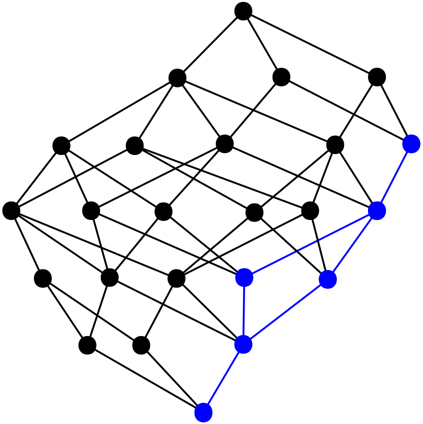

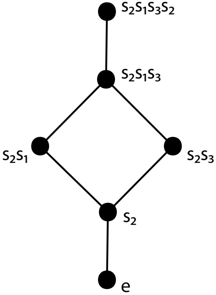

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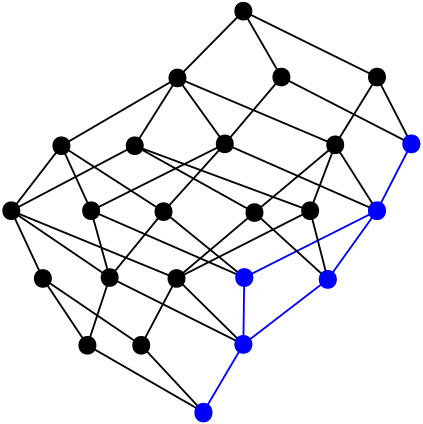

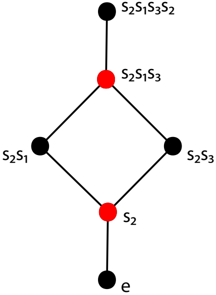

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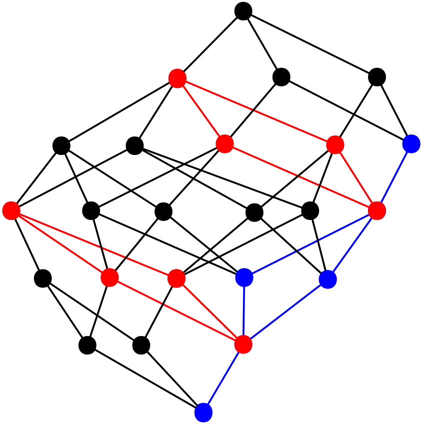
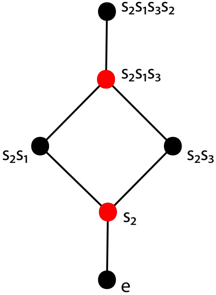




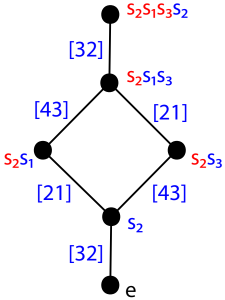
Cell Closures

Given $v \in {}^M W$, let $\Delta_v = \{i : \ell(vs_i) = \ell(v) - 1\}$. Each v can be written uniquely as $v = x_v w_v$ for $w_v \in W_v := \langle s_i : i \in \Delta_v \rangle$.

Example

Let $M = \{e, s_1, s_3\}$ as in the previous slides. Then

$v \in {}^M W$	$v = x_v w_v$	Δ_v
$s_2 s_3 s_1 s_2$	$s_2 s_3 s_1 s_2$	$\{2\}$
$s_2 s_3 s_1$	$s_2 s_3 s_1$	$\{1,3\}$
$s_2 s_1$	$s_2 s_1$	$\{1\}$
$s_2 s_3$	$s_2 s_3$	$\{3\}$
s_2	s_2	$\{2\}$



One More Example

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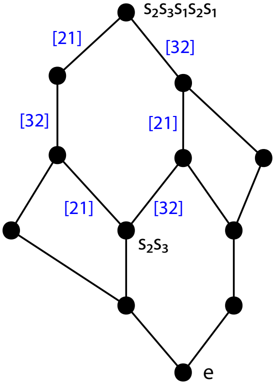
- $\overline{C_v \cap \text{Hess}(S, h)}$ is isomorphic to a smaller rank, regular semisimple Hessenberg variety.
- The singularities of $\text{Hess}(S, h)$ are the M-orbits of the intersections between $\overline{C_v \cap \text{Hess}(S, h)}$ and $\overline{C_u \cap \text{Hess}(S, h)}$ for some $u, v \in {}^{M}W$.



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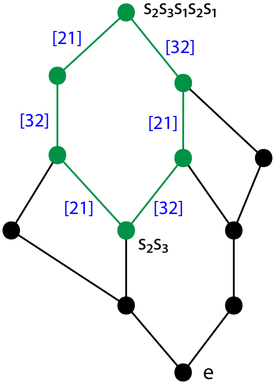

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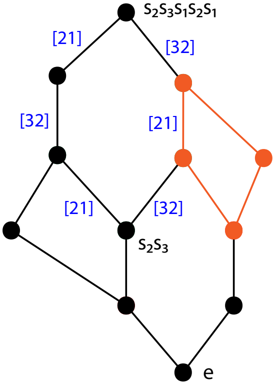

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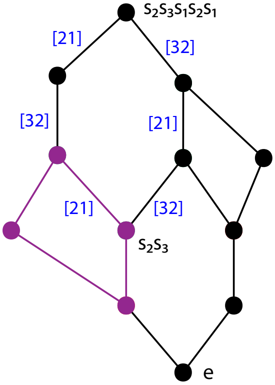

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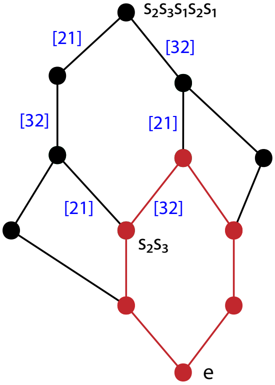

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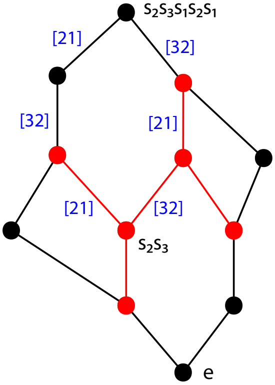


Irreducible Components

Lemma

If $v \in {}^{M}W$ then $\overline{C_v \cap \mathrm{Hess}(S,h)} \cong \mathrm{Hess}_v(S_v, h_v)$ where

- $\mathrm{Hess}_v(S_v, h_v)$ is a regular semisimple Hessenberg variety in a smaller rank flag variety, and
- for all $u, v \in {}^{M}W$, we get $C_u \cap \mathrm{Hess}(S,H) \subseteq \overline{C_v \cap \mathrm{Hess}(S,H)}$ if and only if $x_u = x_v$ and $\Delta_u \subseteq \Delta_v$.

Theorem (I.-Precup)

The irreducible components of $\mathrm{Hess}(S,h)$ *are of the form*

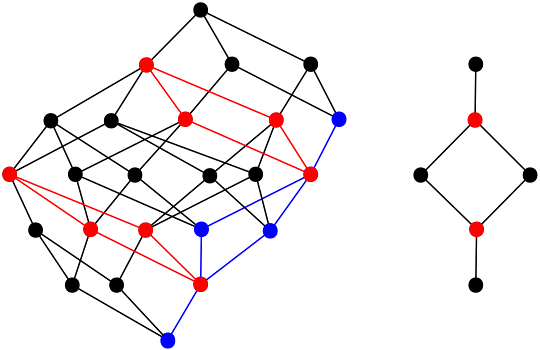
$$M \cdot (\overline{C_v \cap \mathrm{Hess}(S,h)})$$

for v *in a certain subset of* ${}^{M}W$.

Singular Locus

Theorem (I.-Precup)

Each irreducible component of $\mathrm{Hess}(S, h)$ *is smooth. Therefore the singularities of* $\mathrm{Hess}(S, h)$ *occur exactly where two irreducible components intersect.*



Questions

- Classify singular loci of regular nilpotent Hessenberg varieties.
- Are Peterson varieties orbifolds?
- Which Hessenberg varieties are rationally smooth?

Thank you!