Sasaki-Einstein structures and their compactification

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Based on: **G-.**, Neusser, Willse, *arXiv:1803.09531*

and e.g.
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Sasaki Geometry

A (pseudo-)Riemannian manifold (M^n, g) is Sasakian if its standard metric cone is (pseudo-)Kähler (so $n = 2m+1$). It is Sasaki-Einstein if also the metric g is Einstein.

Definition

A *Sasaki structure* on a manifold M ($n \geq 5$ here and throughout) consists of a (pseudo-)Riemannian metric $g_{ab} \in \Gamma(S^2 T^* M)$ and a Killing field $k^a \in \Gamma(TM)$ of g (i.e. $\mathcal{L}_k g = 0$) such that

1. $g_{ab} k^a k^b = 1$
2. $\nabla_a \nabla_b k^c = -g_{ab} k^c + \delta^c{}_a k_b$, $(\star)$

where ∇ denotes the Levi-Civita connection of g.

• Can replace $(\star)$ by: $R_{bc}{}^a{}_d k^d = 2\delta^a{}_{[b} k_{c]}$.

• In fact k_a is a contact form and $J_a^b := \nabla_a k^b$ determines complex structure on the contact distribution. So this data of Sasaki determines a *CR structure*.

Q: If g indef. sig., complete, non-compact: right way to compactify?

Projective DG – a less rigid structure

Definition

On a manifold $M^{n\geq 2}$ a *projective structure* is an equivalence class $\mathbf{p}$ of torsion-free affine connections that share the same geodesics as unparametrised curves.

- $\nabla, \widehat{\nabla} \in \mathbf{p} \Leftrightarrow \widehat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$ where Υ a 1-form

On a general $(M, \boldsymbol{p})$ there is no distinguished ∇ on TM. But there is on the **tractor bundle** $\mathcal{T}$ which extends TM:

$$0 \to \mathcal{E}(-1) \xrightarrow{X^A} \mathcal{T}^A \xrightarrow{Z^a_A} TM(-1) \to 0,$$

given by

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \nu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \rho \delta^b_a \\ \nabla_a \rho - P_{ab}\nu^b \end{pmatrix}. \leftarrow$$ **standard tractor connection**

Here $(\Lambda^n TM)^2 = \mathcal{E}(2n+2)$ and $\mathcal{E}(w)$ are roots.

Sasaki $\rightsquigarrow$ projective

On Sasaki (M, g, k), Levi-Civita ∇^g determines $\boldsymbol{p} = [\nabla^g]$,

$$R_{ab}{}^{c}{}_{d} = \underbrace{W_{ab}{}^{c}{}_{d}}_{tf\ \ and\ \ \textbf{projectively invariant}} + \delta^c{}_a \mathrm{P}_{bd} - \delta^c{}_b \mathrm{P}_{ad},$$

Theorem

A (pseudo-)Riemannian manifold (M, g) is Sasaki if and only if $\exists$ $k^a \in \Gamma(TM)$ s.t

1. $\nabla_{(a}k_{b)} = 0 \leftarrow$ **projectively invariant**
2. $W_{ab}{}^{c}{}_{d} k^d = 0 \leftarrow k$ *in* **projective Weyl nullity**
3. $g_{ab}k^a k^b = 1,$
4. $\mathrm{P}_{ab}k^a k^b = 1,$

where $\mathrm{P}_{ab} = \frac{1}{n} Ricci_{ab}$ is the projective Schouten tensor of ∇^g.

Sasaki Einstein manifolds

There is a particularly simple result for Sasaki-Einstein structures:

Theorem

A Sasaki-Einstein manifold (M, g, k) *(of signature* $(2p-1, 2q)$*), canonically carries a parallel Hermitian structure on the projective tractor bundle* $\mathcal{T}$*. That is, it carries a tractor metric* $h \in \Gamma(S^2\mathcal{T}^*)$ *(of signature* $(2p, 2q)$*) and a tractor complex structure* $\mathbb{J} \in \Gamma(\mathrm{End}\,\mathcal{T})$ *compatible in the sense that* $h(\,\cdot\,,\,\cdot\,) = h(\mathbb{J}\,\cdot\,,\mathbb{J}\,\cdot\,)$ *and both are parallel for the tractor connection of* $\boldsymbol{p} = [\nabla^g]$.

Proof.

In the scale of the metric

$$h := \begin{pmatrix} 1 & 0 \\ 0 & g_{ab} \end{pmatrix} \quad \text{and} \quad \mathbb{J}^A{}_B =: \begin{pmatrix} 0 & -k_b \\ k^a & \nabla_b k^a \end{pmatrix},$$

now use the properties of k and the formula for the tractor connection. □

The converse: Projective with $SU(p,q)$-holonomy

Theorem

Let $(M, \mathbf{p})$ be a projective manifold equipped with compatible parallel tractor metric h and tractor complex structure $\mathbb{J}$. Then M is stratified into a disjoint union of submanifolds

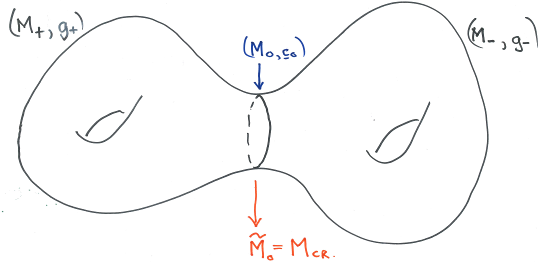
$$M = M_+ \cup M_0 \cup M_-,$$

according to the strict sign of $\tau := h(X, X)$:

1. *The submanifolds $M_\pm$ are open and (if nonempty) resp. equipped with Sasaki-Einstein structures $(g_\pm, k)$ with $Ric^{g_\pm} = 2mg_\pm$ where g_+ has signature $(2p-1, 2q)$, and g_- has signature $(2q-1, 2p)$ and $[\nabla^g] = \boldsymbol{p}$.*
2. *The submanifold M_0 is (if nonempty) a smooth separating hypersurface and is equipped with an oriented Fefferman conformal structure of signature $(2p-1, 2q-1)$.*

$(M, \boldsymbol{p})$ with $SU(2p)$ holonomy $\Leftrightarrow$ Sasaki-Einstein manifold s.t. g +ve def.

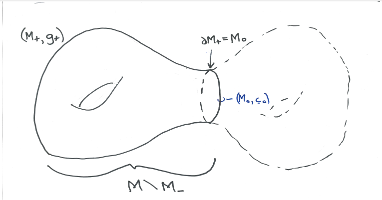
In other signatures something even more interesting can happen:



M_0 has a conformal structure that is locally an S^1 bundle over a CR manifold: $S^1 \to M_0 \to \tilde{M}_0 = M_{\mathrm{CR}}$

Theorem

Assume the setting above. Then the manifold with boundary $(M \setminus M_{\mp}, \mathbf{p}, \mathbb{J}, h)$ is an order 2 projective compactification of the Sasaki-Einstein manifold, respectively, $(M_\pm, g_\pm, k)$.



This means that locally near the boundary the metric looks like

$$g = c \cdot \frac{dt^2}{t^2} + \frac{g_0}{t}, \qquad g_0|_{TM_0 \times TM_0} \quad \text{gives conformal str on } M_0$$

where t a defining function for $M_0 = \partial M$, so ∂M at infinity for geodesics of g, but projective structure extends to ∂M. We understand asymptotics . . .

Another link to CR geometry

It is well known that Sasaki-Einstein manifolds locally fibre over Kähler-Einstein structures. So

$$\text{Sasaki-Ein.} \quad (M_\pm, g_\pm, k) \to (\tilde{M}_\pm, \tilde{g}_\pm, J) \quad \text{Kähler-Ein..}$$

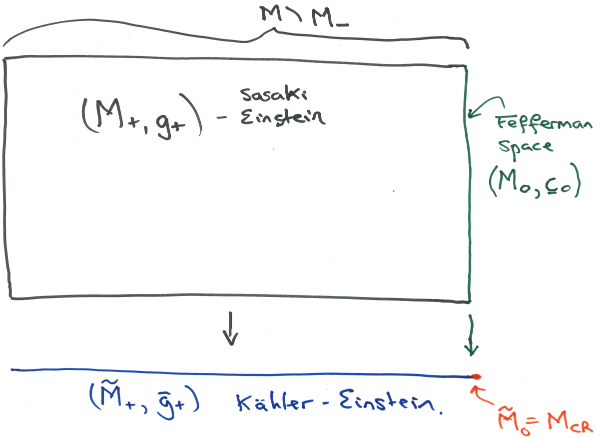
This is compatible with the Fefferman fibration $M_0 \to \tilde{M}_0 = M_{\mathrm{CR}}$:

Theorem

Assume the setting of above with $M_0 \neq \emptyset$. The construction produces a manifold with boundary $(\widetilde{M} \setminus \widetilde{M}_\mp, J)$ that is an (order 2) c-projective compactification of the Kähler-Einstein manifold $(\widetilde{M}_\pm, \tilde{g}_\pm)$ with CR boundary $\tilde{M}_0$.

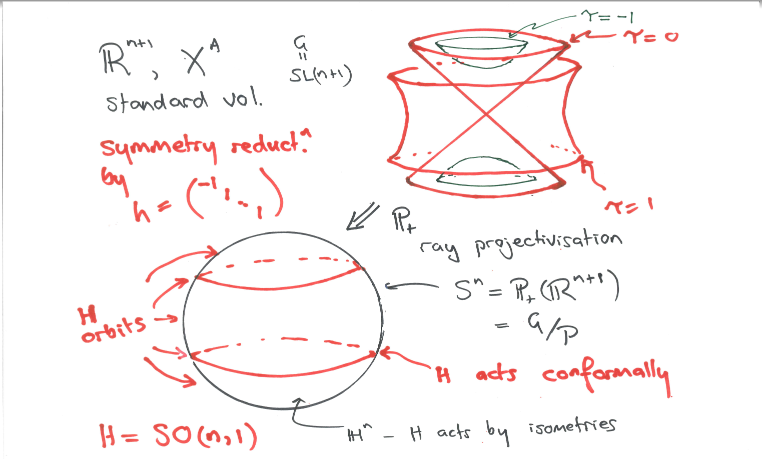
The notion of c-projective compactification is the analogue of projective compactification, based around c-projective geometry, that is suitable for Kähler compactification.

The big picture



This encodes the singularity of the (Cheng-Yau type) Kähler-Einstein metric for CR manifolds compatibly with the singularity of the Sasaki geometry on the "metric cone".

Background: $H = SO(n,1)$ orbits on the sphere

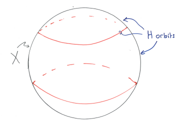


$S^n = \mathbb{P}_+(\mathbb{R}^{n+1} \setminus \{0\})$ is **model of flat projective geometry**.

Symmetry reduction by h (plus time↑): ⇒ North polar cap is projective compactification of $\mathbb{H}^n$; $\tau = 0$ projective ∞ with conformal str.

NB: Embeddings relate the orbits – but these encoded in $H \hookrightarrow G$.

Problem: Suppose a Lie group H acts on a manifold X with a finite number of orbits. Then: (i) understand and relate the different (Klein) geometries on the orbits; and (ii) construct and treat a well defined curved version of this theory.



If $H < G$ and the Lie gp G transitive on X. There is a nice route:

Theorem (Cartan, Tanaka, $\cdots$)

If P is a parabolic subgroup of a semisimple Lie group G then there is a **canonical** *notion of geometry*

$$\begin{array}{ccc} \mathcal{G} & \leftarrow & P \\ \downarrow & & \\ M & & \end{array} \quad \text{modelled on} \quad \begin{array}{ccc} G & \leftarrow & P \\ \downarrow & & \\ G/P & & \end{array}$$

where $\mathcal{G}$ is equipped with a Cartan connection ω – viz. a suitably equivariant Lie(G)-valued 1-form, cf. Maurer-Cartan form on G.

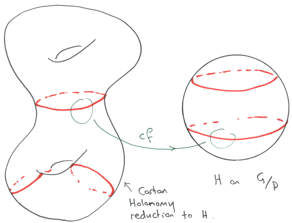
Projective DG: $G = SL(\mathbb{R}^{n+1})$, & $P < G$ stabilises a ray in $\mathbb{R}^{n+1}$.

A short "proof" of nearly everything

Theorem (Curved orbit decomposition - Čap,G., Hammerl)

Suppose $(\mathcal{G}, \omega) \to M$ is a Cartan geometry (modelled on $G \to G/P$) endowed with a **parallel tractor field** *h giving a Cartan holonomy reduction with* **holonomy group** *H. Then:*
*(1) M is canonically stratified $M = \bigcup_{i \in H \backslash G/P} M_i$ in a way locally diffeomorphic to the the H-***orbit decomposition** *of G/P; and*
(2) there $\exists$ a **Cartan geometry on** *M_i of the same type as the model.*

Thus there is a **general way to define a curved analogue of an orbit decomposition of a homogeneous space.**



The resolution

Well known useful move. Can consider a Kähler structure from different perspectives. namely as:
• symplectic manifold equipped also with a compatible complex structure; or
• a complex manifold equipped with a suitable Hermitian metric; or
• a Riemannian manifold with a complex structure that is compatible with the metric and parallel for the Levi-Civita connection.

The **analogue here** is to note $\mathrm{SU}(p,q) = \mathrm{U}(p,q) \cap \mathrm{SL}(m+1,\mathbb{C})$, where $p+q=m+1$, and

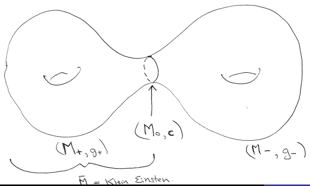
$$\mathrm{U}(p,q) = \mathrm{SO}(2p,2q) \cap \mathrm{Sp}(2m+2,\mathbb{R}) \cap \mathrm{GL}(m+1,\mathbb{C})$$

(in fact intersection of any two will do) and thus consider separately **projective** (Cartan) **holonomy reductions** to $\mathrm{SO}(2p,2q)$, $\mathrm{Sp}(2m+2,\mathbb{R})$, and $\mathrm{GL}(m+1,\mathbb{C})$.

Theorem (Cap,G.,Hammerl)

h **tractor metric** *sig.* $(2p, 2q)$ *and parallel on* $(M, \mathbf{p})$ *implies*

- *If* $q = 0$ *then* $(M, \mathbf{p}, h) \Leftrightarrow (M, g)$ *Einstein with positive scalar curvature.*
- *If* $p, q \neq 0$ *then* M *is stratified* $M = M_+ \cup M_0 \cup M_-$ *according to strict sign of* $\tau = h(X, X)$.
- *If* $M_0 \neq \emptyset$ *then it is a smooth embedded separating hypersurface with a* **conformal structure c** *of signature* $(2p-1, 2q-1)$.
- *On the open submanifolds* $M_\pm$, *h induces metrics* $g_\pm$ *which are positive/negative Einstein of signature* $(2p-1, 2q)$/ *resp.* $(2q-1, 2p)$. *(Complete if M closed.)*



Symplectic holonomy reduction of projective geometries

For a symplectic holonomy reduction we consider a projective manifold of dimension $2m+1$ equipped with a **parallel** and **nondegenerate** skew tractor field

$$\Omega_{AB} \in \Gamma(\wedge^2 \mathcal{T}^*) \quad \text{s.t.} \quad \nabla^{\mathcal{T}}\Omega(x) = 0 \quad \& \quad \wedge^m \Omega(x) \neq 0 \quad \forall x \in M.$$

Theorem

Suppose $(M, \mathbf{p})$ *is a projective (necessarily odd-dimensional) manifold equipped with a parallel symplectic form* $\Omega_{AB} \in \Gamma(\wedge^2 \mathcal{T}^*)$*. Then* $k := \Pi^{\wedge^2 \mathcal{T}^*}(\Omega)$ *satisfies* $\nabla_{(a}k_{b)} = 0$ *and*

$$H := \ker k = \{u^a \in TM : k_a u^a = 0\} \subset TM \tag{1}$$

is a **contact distribution** *and* $\mathbf{p}$ *is* **compatible with** H *in that* H *is totally geodesic. [So* $(M, H, \langle \mathbf{p} \rangle)$ *is a* **contact projective manifold** *and the torsion of* $(M, H, \langle \mathbf{p} \rangle)$ *vanishes identically.]*

Proof.

The tractor Ω **parallel** and **skew** implies

$$\Omega_{AB} \stackrel{\nabla}{=} \begin{pmatrix} 0 & -k_b \\ k_a & \nabla_b k_a \end{pmatrix} \quad \text{and} \quad \nabla_{(a} k_{b)} = 0.$$

Then Ω **non-deg** implies $k \wedge (dk) \wedge \cdots \wedge (dk) \neq 0 \ \forall x$. So k_a is a contact form. The restriction of $\nabla_{(a} k_{b)}$ to the contact distribution H is the second fundamental form, and this vanishes. □

Complex holonomy reduction of projective geometries

A projective manifold $(M, \mathbf{p})$ with a parallel tractor complex structure

$$\mathbb{J}^A{}_B \in \Gamma(\mathrm{End}\, \mathcal{T}),$$

that is, a parallel tractor endomorphism $\mathbb{J}^A{}_B$ satisfying $\mathbb{J}^2 = -\,\mathrm{id}_{\mathcal{T}}$ – is oriented and odd dimensional. The tractor holonomy group must be in $\mathrm{SL}(2m+2, \mathbb{R}) \cap \mathrm{GL}(m+1, \mathbb{C}) \cong \mathrm{SL}(m+1, \mathbb{C}) \times \mathrm{U}(1)$, but one can show that, if M is simply connected, there is a parallel tractor complex volume form, so the holonomy group is (in) $\mathrm{SL}(m+1, \mathbb{C})$.
In special scales (∇ s.t. $\nabla_a k^a = 0$) we have

$$\mathbb{J}^A{}_B = \begin{pmatrix} 0 & -\mathrm{P}_{bc} k^c \\ k^a & \nabla_b k^a \end{pmatrix}.$$

k^a is nowhere zero: $SL(m+1, \mathbb{C})$ acts transitively on $\mathbb{C}^{m+1} \setminus \{0\}$ and hence on the projective model $\mathbb{P}_+^{\mathbb{R}}(\mathbb{C}^{m+1}) = \mathbb{P}_+(\mathbb{R}^{2m+2})$. So in the curved case there can be only one curved orbit.

Assembling

- (locally) when $k = \Pi(\mathbb{J})$ is a projective symmetry the leaf space $\tilde{M}$ gets a *complex structure* J on $T\tilde{M}$ and a compatible **c-projective structure** $\tilde{\boldsymbol{p}} = [\tilde{\nabla}]$ – this is a certain equivalence class of affine connections preserving J.
- Along $(M_0, \boldsymbol{c}_0)$, from the othogonal holonomy reduction, the parallel $\mathbb{J}$ is also parallel for the conformal tractor connection. Thus by a characterisation (Čap+G., Leitner) this is a Fefferman Space that fibres over a CR manifold $\tilde{M}_0$.
- The leaf space $\tilde{M}_0$ is a hypersurface in $\tilde{M}$ so also gets a CR structure from J. It is straightforward to show these agree.
- The parallel tractor fields on M descend to parallel tractors for the c-projective geometry $(\tilde{M}, [\tilde{\boldsymbol{p}}])$ – thus the latter has a parallel tractor hermitian form and hence an Einstein (pseudo-)Kähler structure (cf.CENM) in the parts $\tilde{M}_\pm$ off $\tilde{M}_0$. By results of Čap+G. and Čap+G.+Hammerl, we then know that the CR structure $\tilde{M}_0$ is the c-projective infinity of these. It is a curved orbit decomposition, now downstairs.

Thank you for Listening

The projective geometry of Sasaki-Einstein structures and their compactification

Sasaki geometry is often viewed as the odd dimensional analogue of Kaehler geometry. In particular a Riemannian or pseudo-Riemannian manifold is Sasakian if its standard metric cone is Kaehler or, respectively, pseudo-Kaehler. We show that there is a natural link between Sasaki geometry and projective differential geometry. The situation is particularly elegant for Sasaki-Einstein geometries and in this setting we use projective geometry to provide the resolution of such structures into less rigid components. This is analogous to usual picture of a Kaehler structure: a symplectic manifold equipped also with a compatible complex structure; or as a complex manifold equipped with a suitable Hermitian metric; or finally as a Riemannian manifold with a complex structure that is compatible with the metric and parallel for the Levi-Civita connection. However the treatment of Sasaki geometry this way is locally more interesting and involves the projective Cartan or tractor connection. This enables us to describe a natural type of compactification of complete non-compact pseudo-Riemannian