Scenario Optimization: the Performance-Risk Tradeoff

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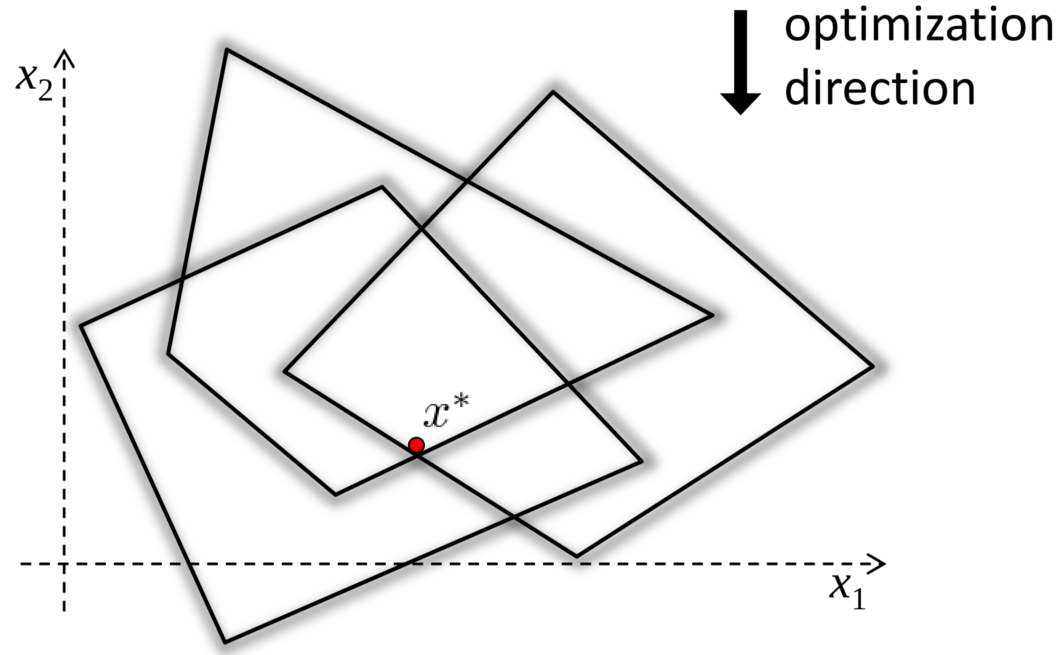
Scenario optimization

$$\min_{x} \quad c(x)$$
$$\text{s.t.} \quad x \in \mathcal{X}_{\delta_i}$$
$$i = 1, \ldots, N$$

solution: x^*



scenarios (data)

$$\begin{cases} \delta_1 & \rightarrow & \mathcal{X}_{\delta_1} \\ \delta_2 & \rightarrow & \mathcal{X}_{\delta_2} \\ & \vdots & \\ \delta_N & \rightarrow & \mathcal{X}_{\delta_N} \end{cases}$$

Problem ingredients

Cost function: $c(x)$

Family of constraint sets: $\mathcal{X}_\delta$

δ stochastic parameter

$(\Delta, \mathcal{F}, \mathbb{P})$ unknown

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For any given x in the optimization domain

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vs.

risk $V(x) = \mathbb{P}\{\delta \in \Delta : \; x \notin \mathcal{X}_\delta\}$

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Family of constraint sets: $\mathcal{X}_\delta$

δ stochastic parameter

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For any given x in the optimization domain

performance $c(x)$ → accessible

vs.

risk $V(x) = \mathbb{P}\{\delta \in \Delta : \; x \notin \mathcal{X}_\delta\}$ → not accessible

Scenario optimization theory

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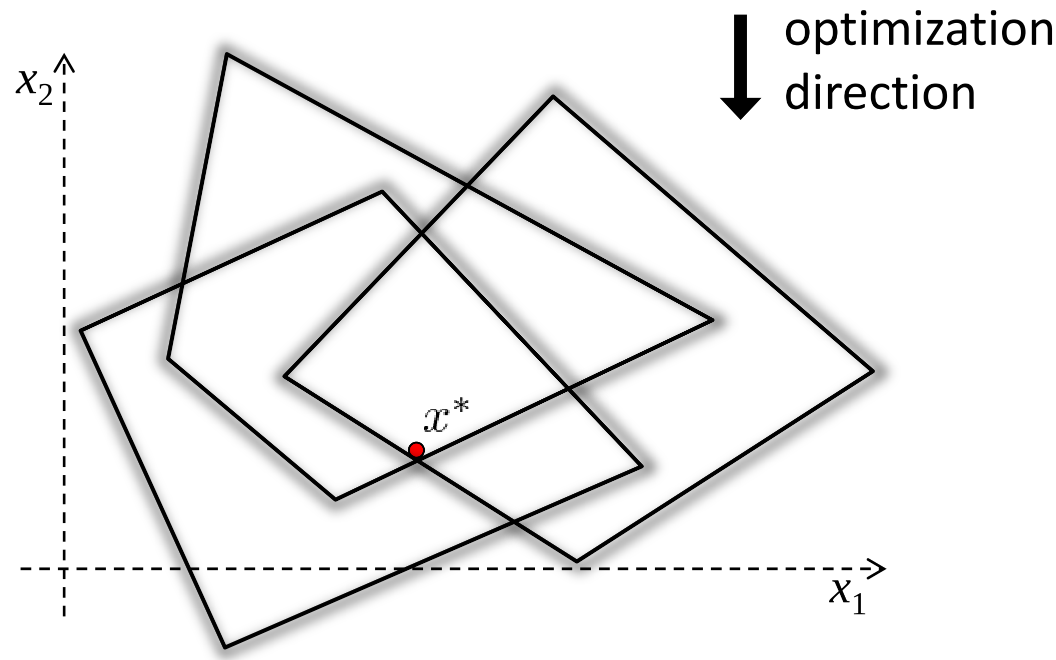

solution: x^*

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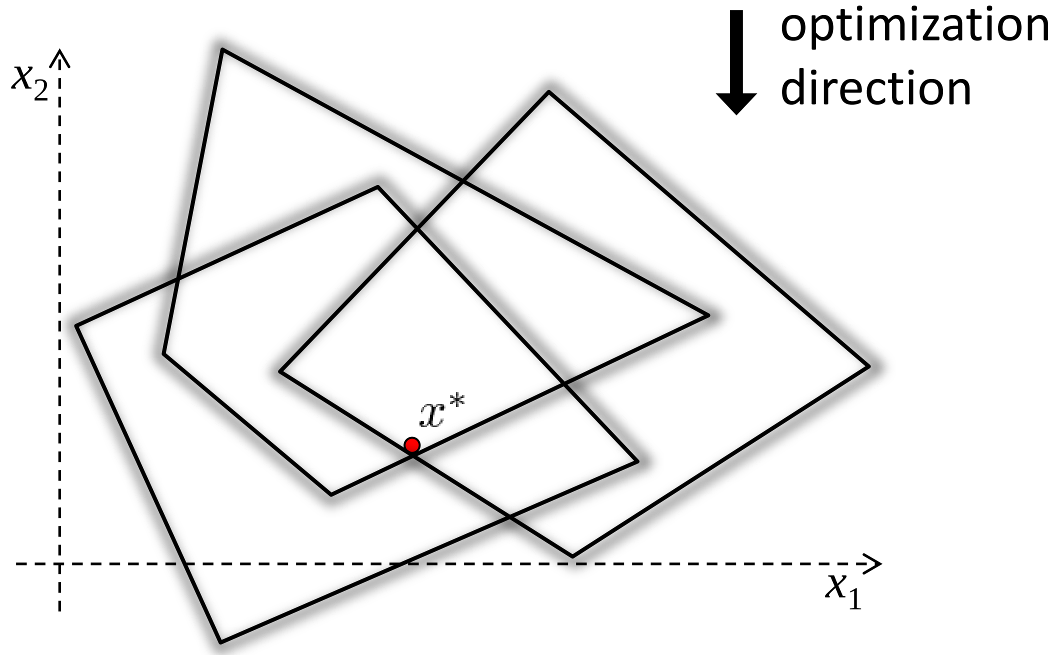

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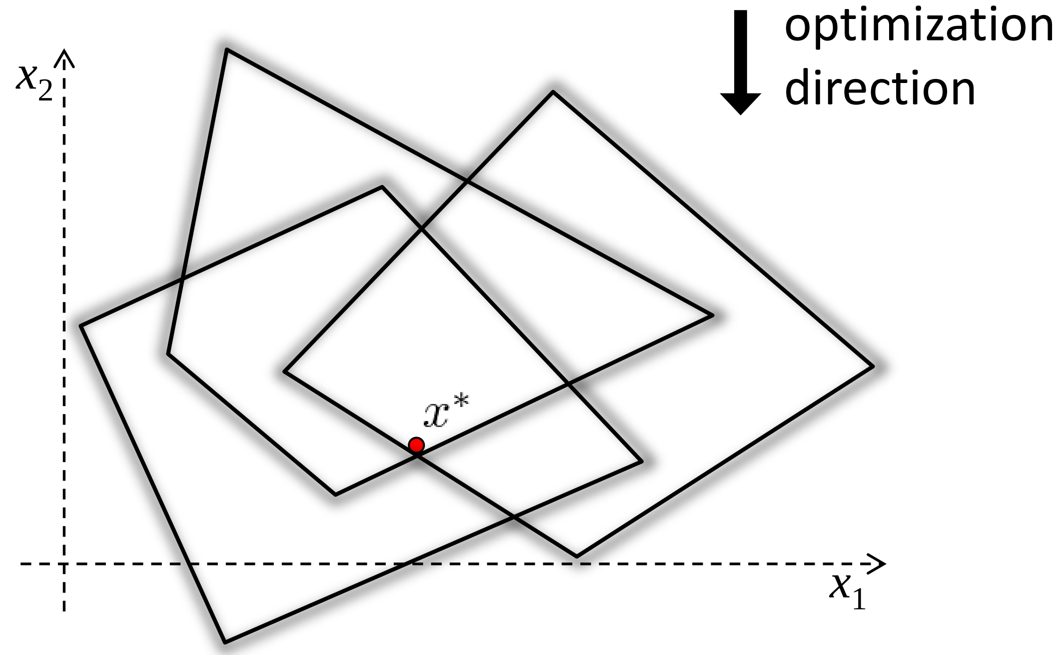

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performance $c(x^*)$ known

risk $V(x^*)$ can be tightly estimated from s^*

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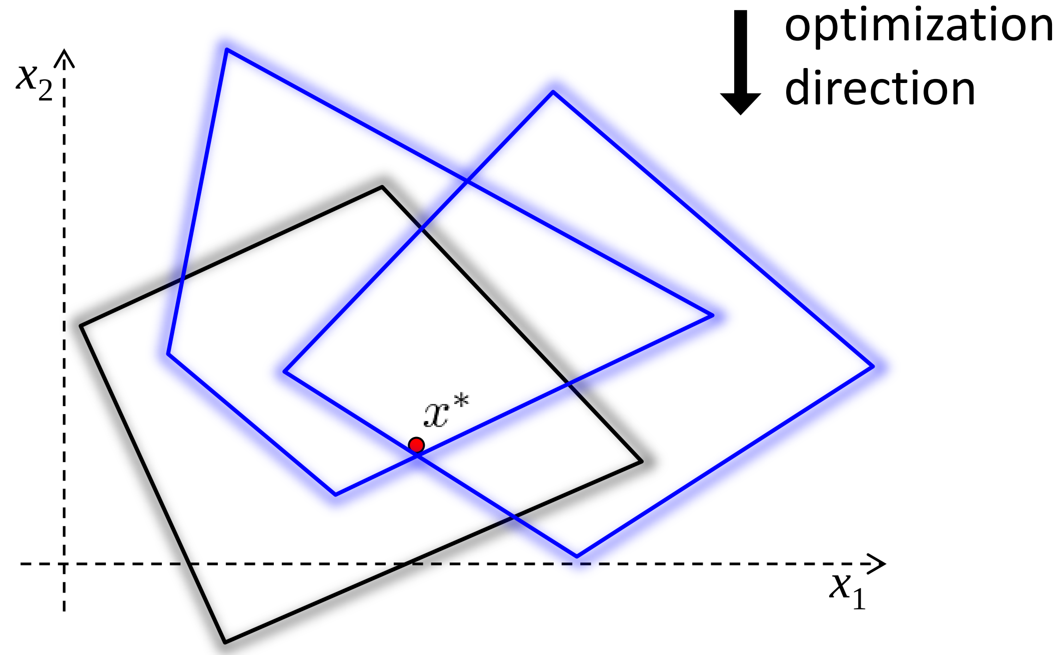

solution: x^*

performance $c(x^*)$ known

complexity

risk $V(x^*)$ can be tightly estimated from s^*

Quality ceritification

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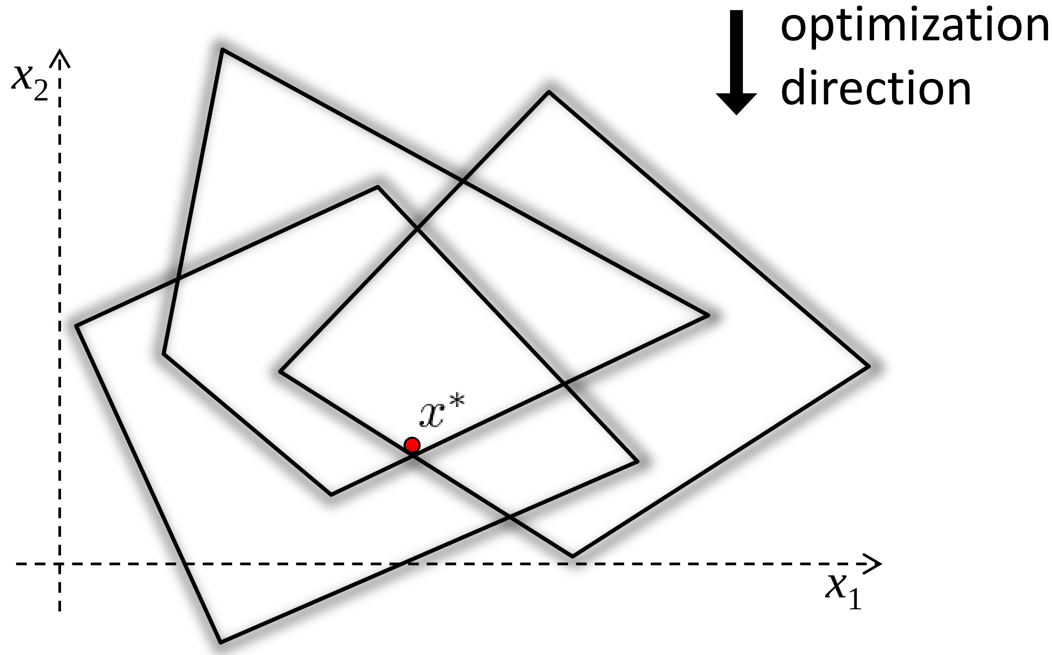

solution quality assessed via

$c(x^*)$, $\widehat{V}(s^*)$

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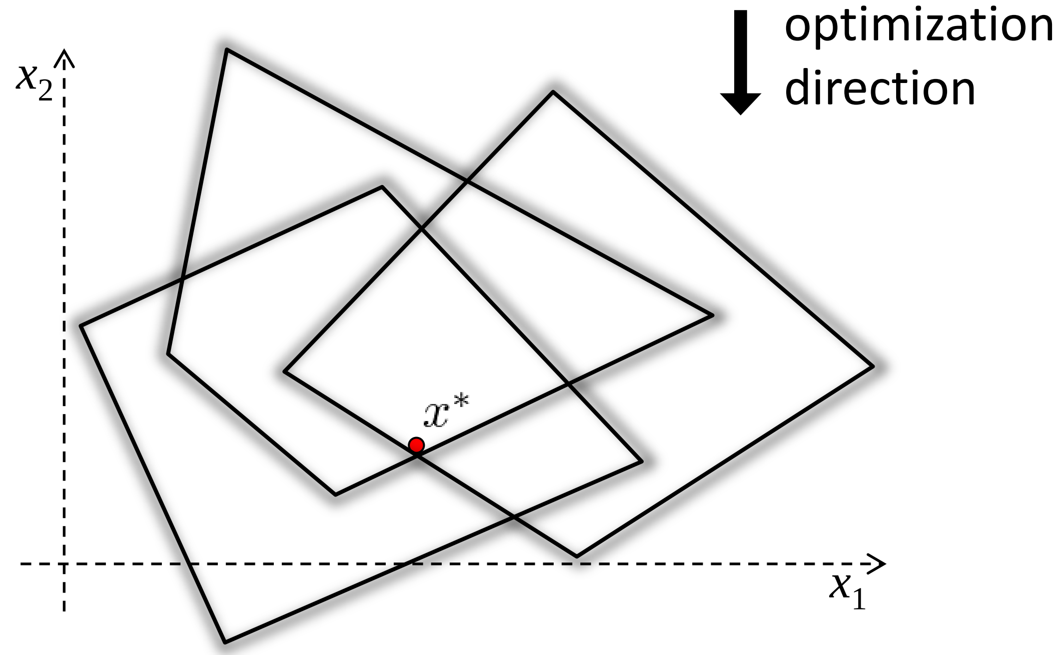

solution quality assessed via

$c(x^*)$, $\widehat{V}(s^*)$

solution quality may be not satisfactory:

$c(x^*)$ **is too big**

Take-home message

- It is possible to introduce many alternative scenario-based schemes → several "solutions" $x_1^*, x_2^*, x_3^*, \ldots$ each attaining a different performance
- Scenario theory as a tool that allows one to evaluate the risk of each solution
- Quantitative comparison in terms of performance (known) and risk (estimated) → select the "best" solution for the problem at hand

Mathematical tool: scenario decision-making (1/2)

uncertainty domain $(\Delta, \mathcal{F}, \mathbb{P})$

⇒ scenarios $(\delta_1, \delta_2, \ldots, \delta_N)$

decision space $\mathcal{Z}$

scenario-based decision $M_N : (\delta_1, \delta_2, \ldots, \delta_N) \to z^*$

support set: $(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_k})$ such that

i. $M_k(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_k}) = M_N(\delta_1, \delta_2, \ldots, \delta_N)$ ii. smallest

s^* = size of the support set = complexity

Mathematical tool: scenario decision-making (2/2)

$\delta \rightarrow \mathcal{Z}_\delta \subseteq \mathcal{Z}$ ⇒ $V(z^*) = \mathbb{P}\{\delta \in \Delta : \ z^* \notin \mathcal{Z}_\delta\}$

risk

scenario theory:

$V(z^*)$ can be estimated from s^*

$V(z^*) \leq \widehat{V}(s^*)$ with confidence $1-\beta$ where

$\widehat{V}(k)$, $k = 0, 1, \ldots$, is the solution to equation

$$\frac{\beta}{N+1}\sum_{m=k}^{N}\binom{m}{k}(1-v)^{m-k} - \binom{N}{k}(1-v)^{N-k} = 0$$

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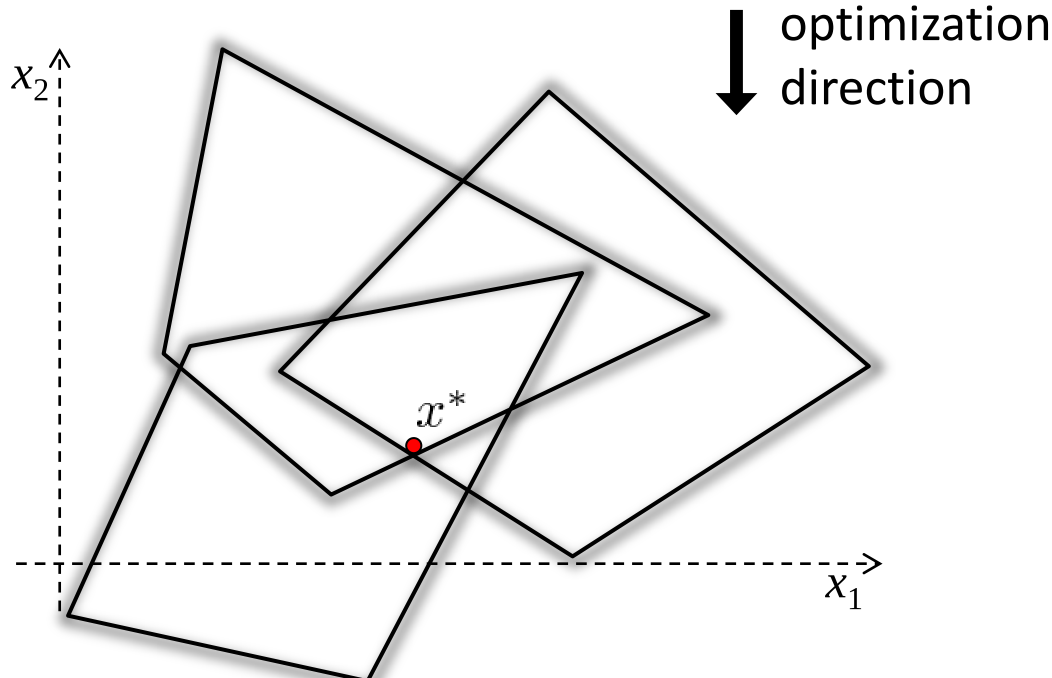
same theorem as in Marco's presentation

Scenario optimization with constraints relaxation

$$\min_{x \in \mathbb{R}^d} \quad c(x)$$

$$\text{s.t.} \quad f(x, \delta_i) \leq 0, \; i = 1, \ldots, N$$

solution: x^*

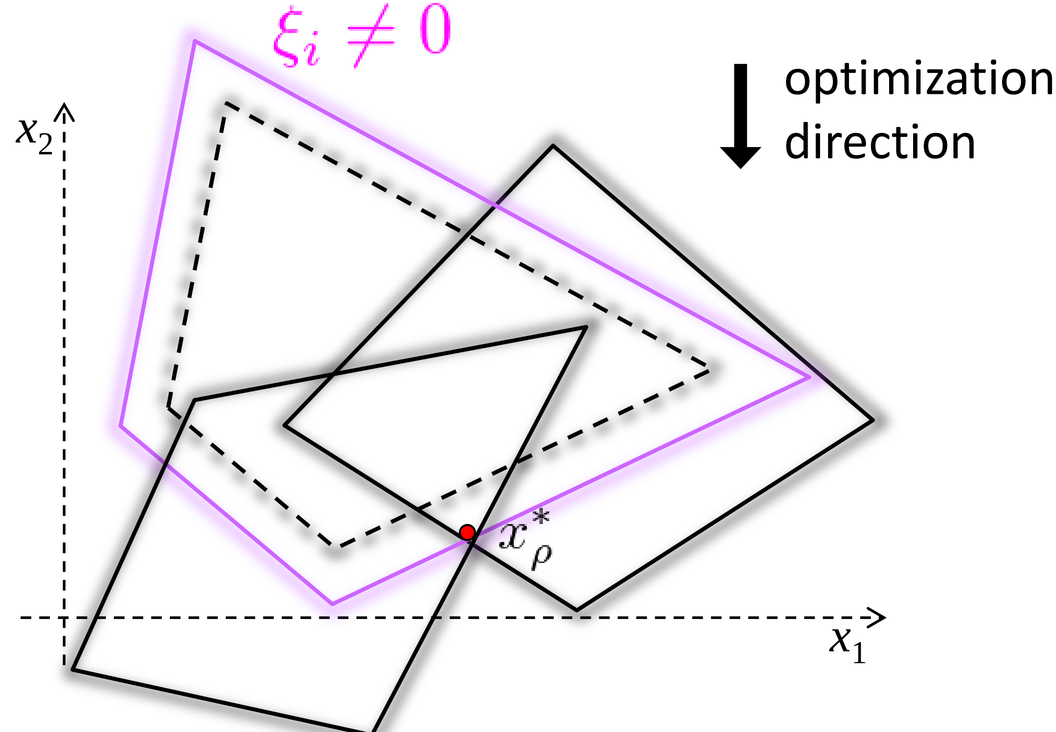


Scenario optimization with constraints relaxation

$$\min_{x \in \mathbb{R}^d, \xi \geq 0} \quad c(x) + \rho \sum_{i=1}^{N} \xi_i$$

$$\text{s.t.} \quad f(x, \delta_i) \leq \xi_i, \; i = 1, \ldots, N$$

decision: x^*_ρ , $\{\xi^*_i : \xi^*_i \neq 0\}$



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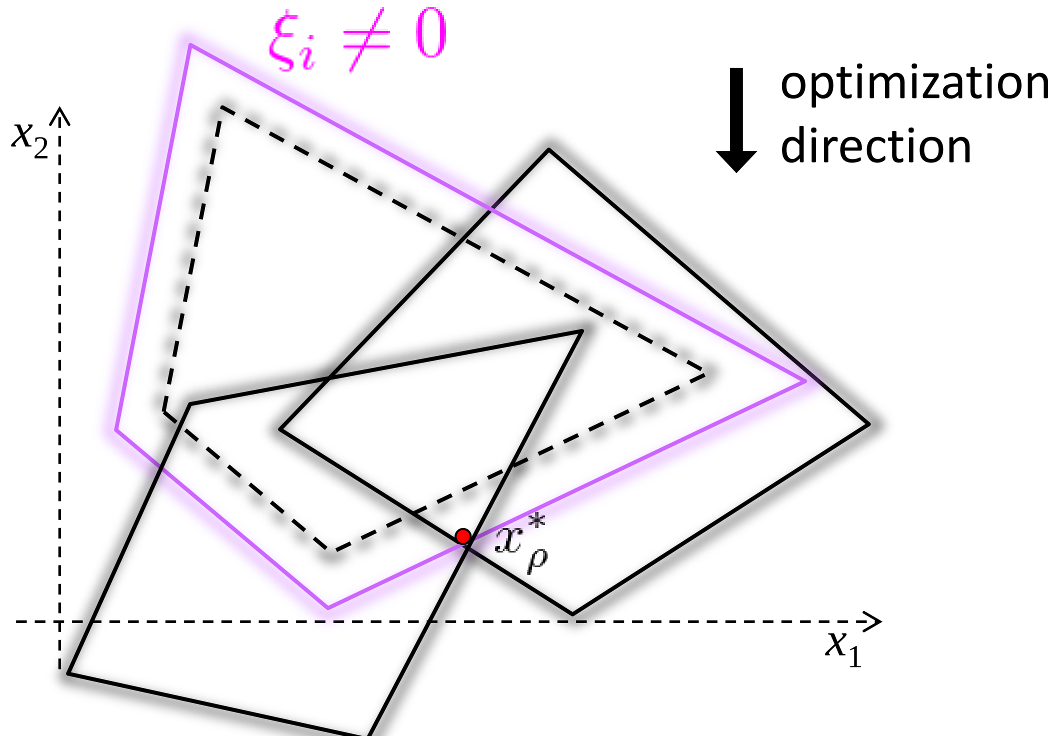


$c(x^*_\rho)$ known

$V(x^*_\rho)$ =

$\mathbb{P}\{\delta \in \Delta : \; f(x^*_\rho, \delta) > 0\}$

Scenario optimization with constraints relaxation

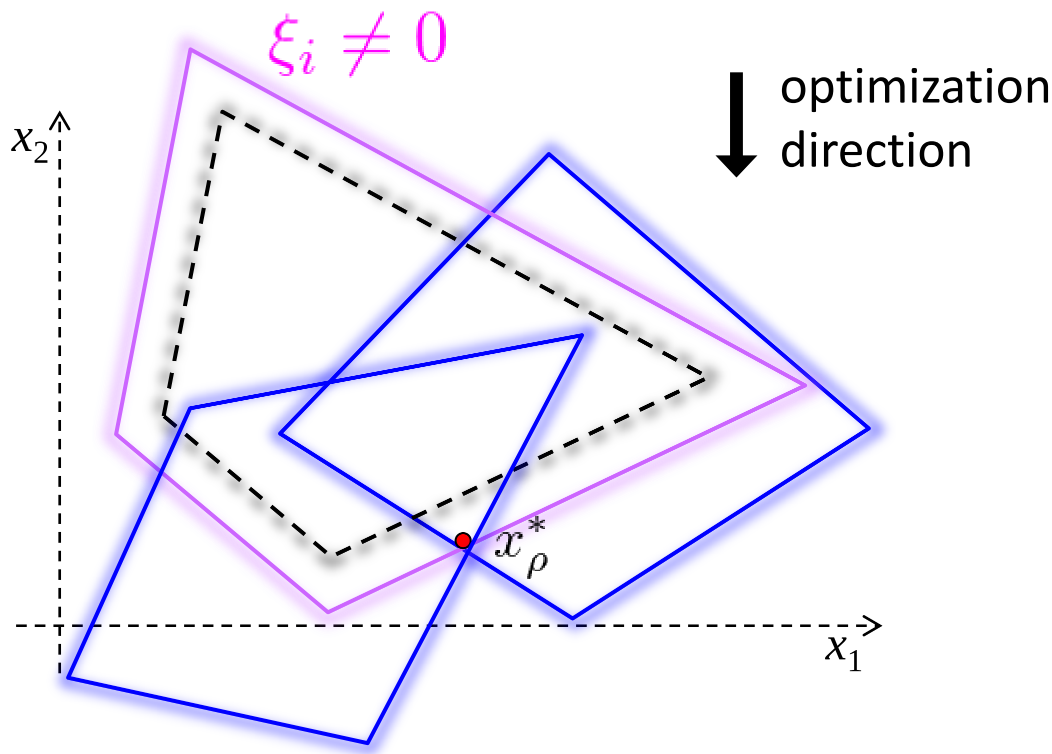
$$\min_{x\in\mathbb{R}^d,\xi\geq 0} \quad c(x) + \rho\sum_{i=1}^{N}\xi_i$$

$$\text{s.t.} \quad f(x,\delta_i) \leq \xi_i, \; i = 1,\ldots,N$$

decision: x^*_ρ , $\{\xi^*_i : \xi^*_i \neq 0\}$

complexity: s^*_ρ

active



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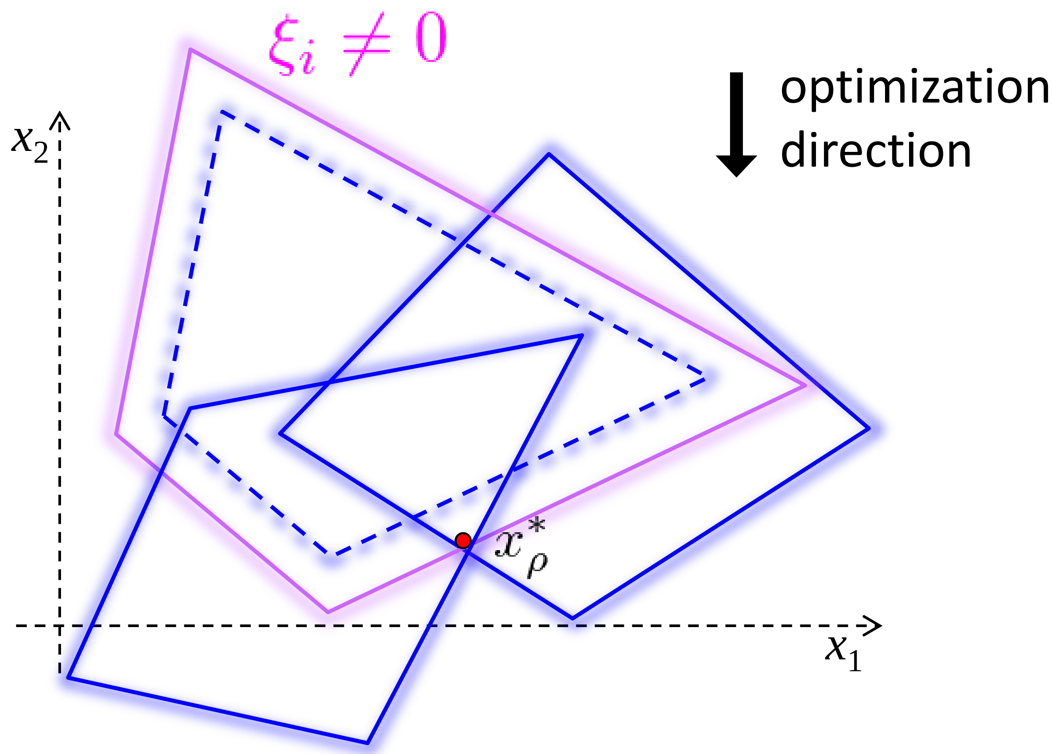
decision: x^*_ρ , $\{\xi^*_i : \; \xi^*_i \neq 0\}$

complexity: s^*_ρ

active + violated



Main theorem

N = no. of scenarios

choose $\beta \in (0,1)$ (confidence parameter)

let $\widehat{V}(k)$, $k = 0, 1, \ldots$, be the solution in $(0,1)$ to equation

$$\frac{\beta}{N+1} \sum_{m=k}^{N} \binom{m}{k} (1-v)^{m-k} - \binom{N}{k} (1-v)^{N-k} = 0$$

with confidence $1-\beta$ it holds that $V(x^*_\rho) \leq \widehat{V}(s^*_\rho)$

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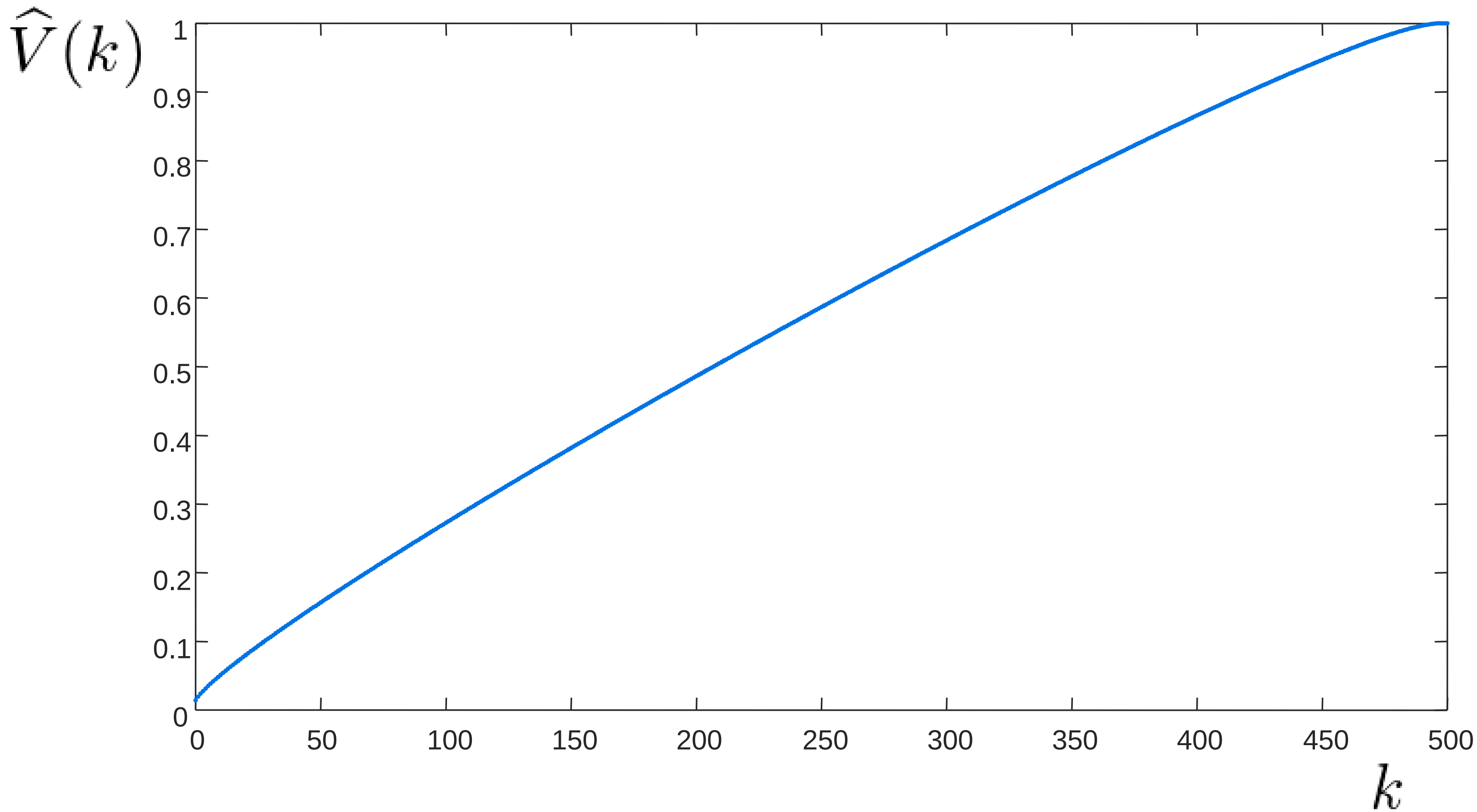
with confidence $1-\beta$ it holds that $V(x_\rho^*) \leq \widehat{V}(s_\rho^*)$

For all problems in the world,

risk $V(x_\rho^*)$ can be assessed through $\widehat{V}(s_\rho^*)$

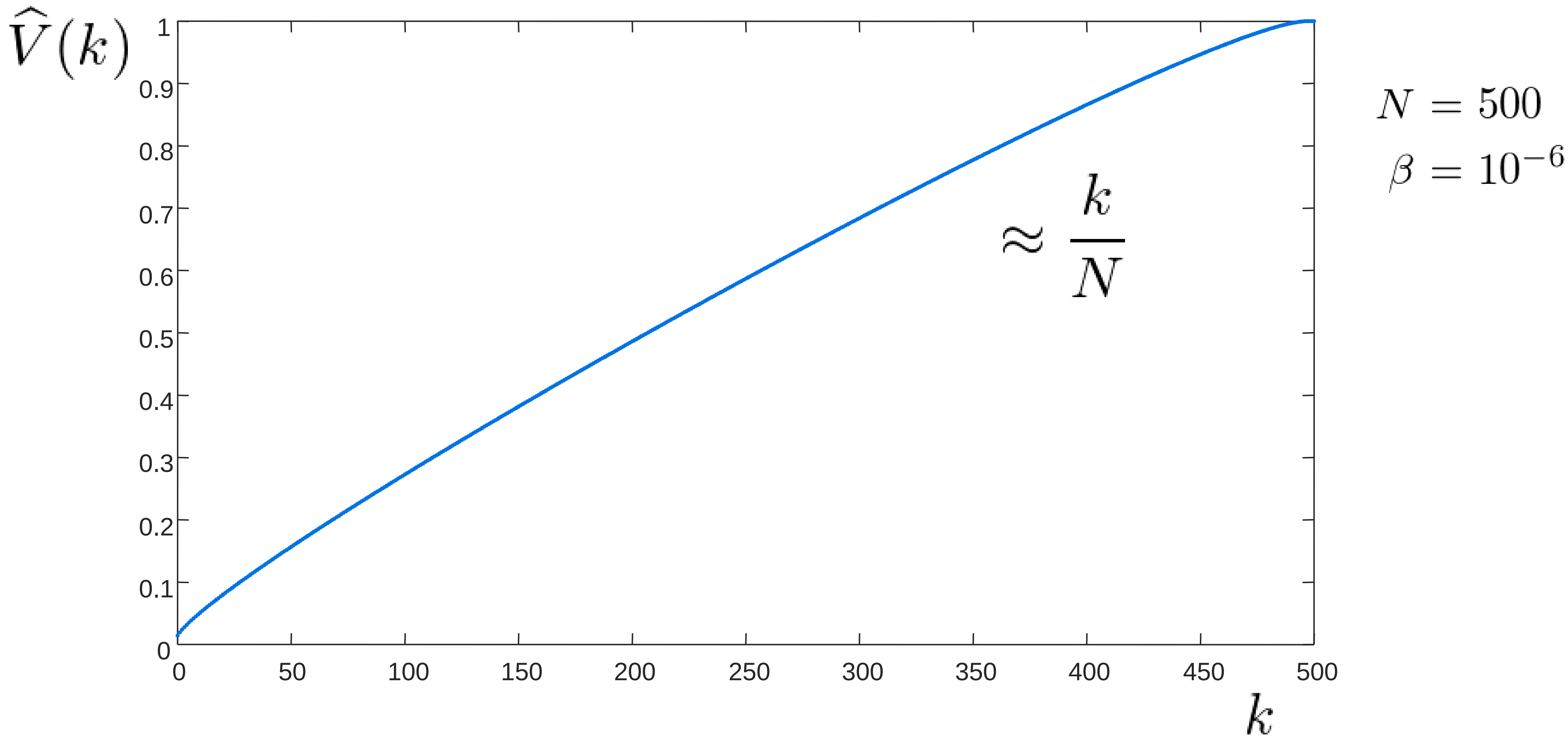
Main theorem (cont'd)



$N = 500$

$\beta = 10^{-6}$

Main theorem (cont'd)



$$\widehat{V}(s^*_\rho) \approx \frac{s^*_\rho}{N} \approx \frac{\text{active}}{N} + \frac{\text{violated}}{N}$$

empirical violation

Risk vs. performance tradeoff

$$\min_{x \in \mathbb{R}^d, \xi \geq 0} \quad c(x) + \rho \sum_{i=1}^{N} \xi_i$$

$$\text{s.t.} \quad f(x, \delta_i) \leq \xi_i, \; i = 1, \ldots, N$$

solution: x^*_ρ ⟵ tunable parameter

as $\rho \to 0$

cost $c(x^*_\rho)$ decreasing

risk $\widehat{V}(s^*_\rho)$ increasing (trend)

} risk vs. performance tradeoff

Risk vs. performance tradeoff

$$\begin{array}{cccc} \rho_1 & \rho_2 & \rho_3 & \cdots \\ \downarrow & \downarrow & \downarrow & \cdots \\ x^*_{\rho_1}, s^*_{\rho_1} & x^*_{\rho_2}, s^*_{\rho_2} & x^*_{\rho_3}, s^*_{\rho_3} & \cdots \end{array}$$

quantitative comparison via $c(x^*_\rho)$ and $\widehat{V}(s^*_\rho)$

Risk vs. performance tradeoff

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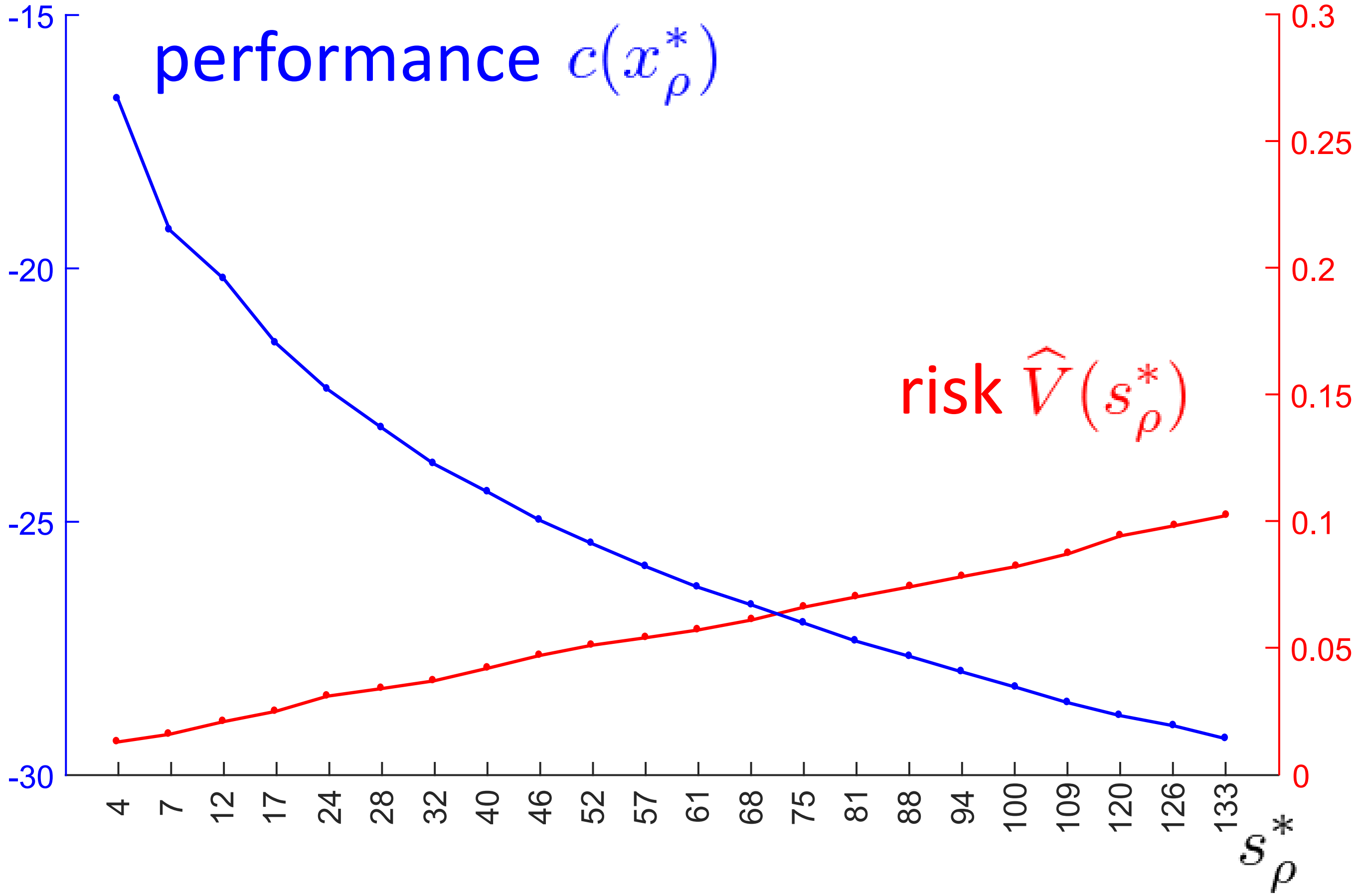

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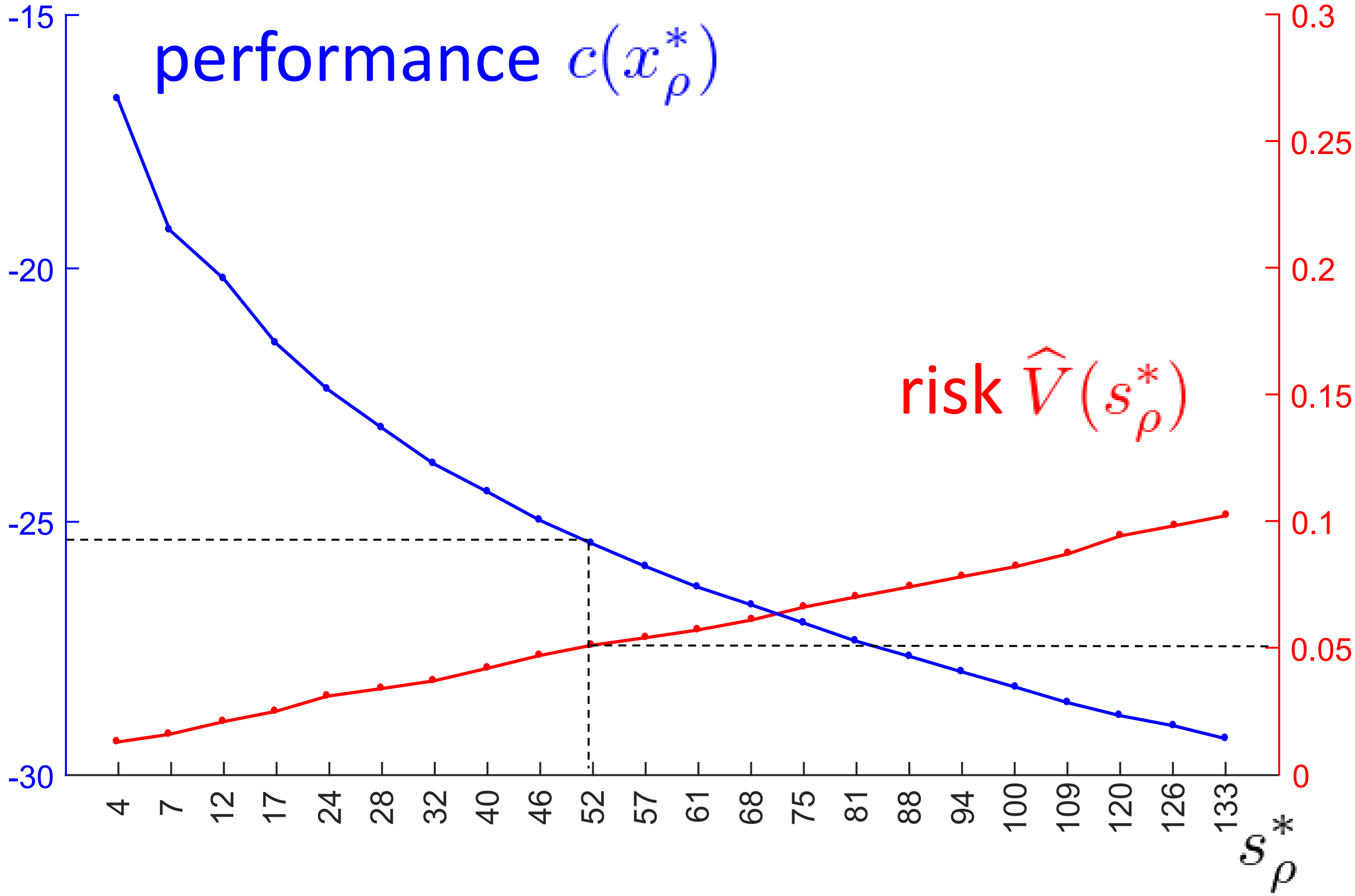

The other side of the coin

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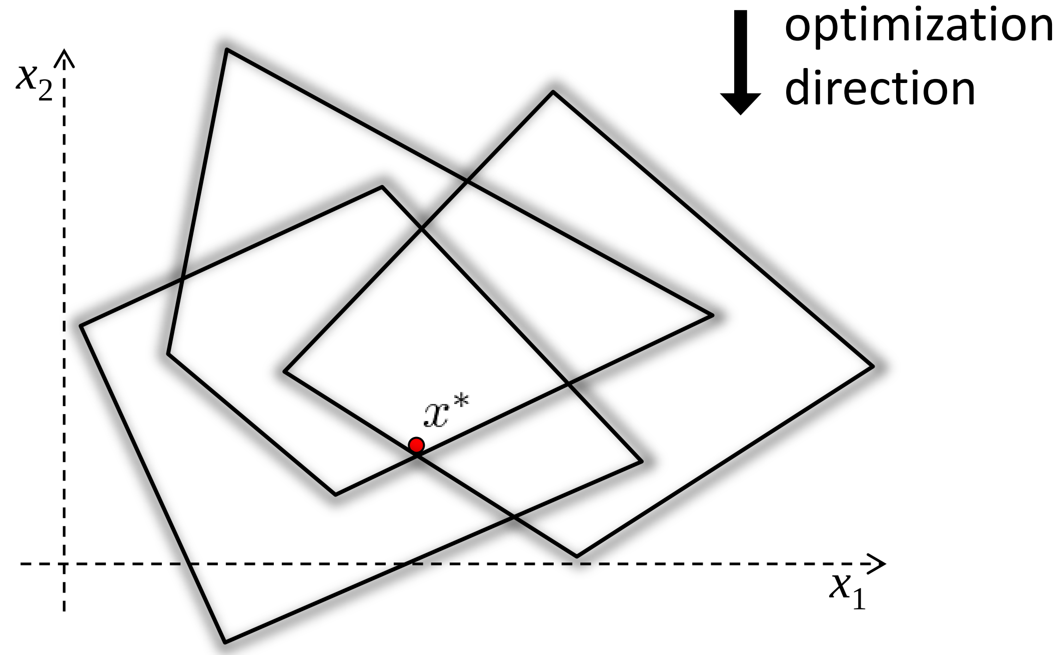
solution: x^*



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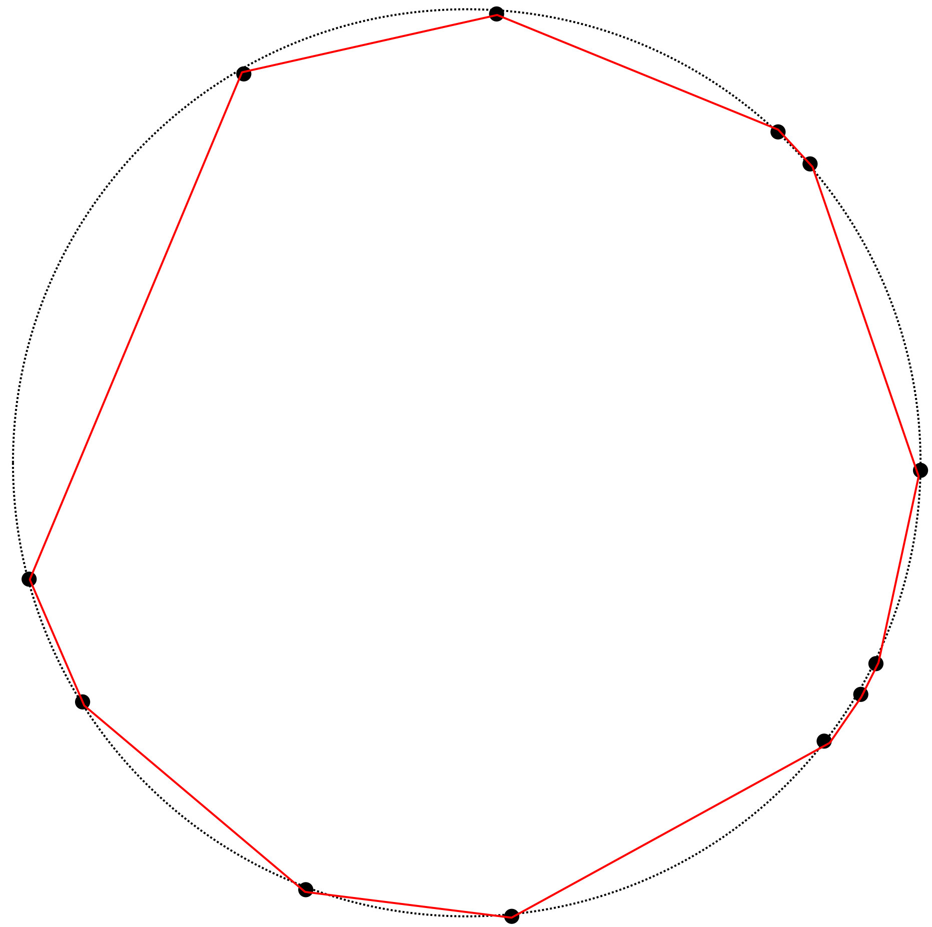
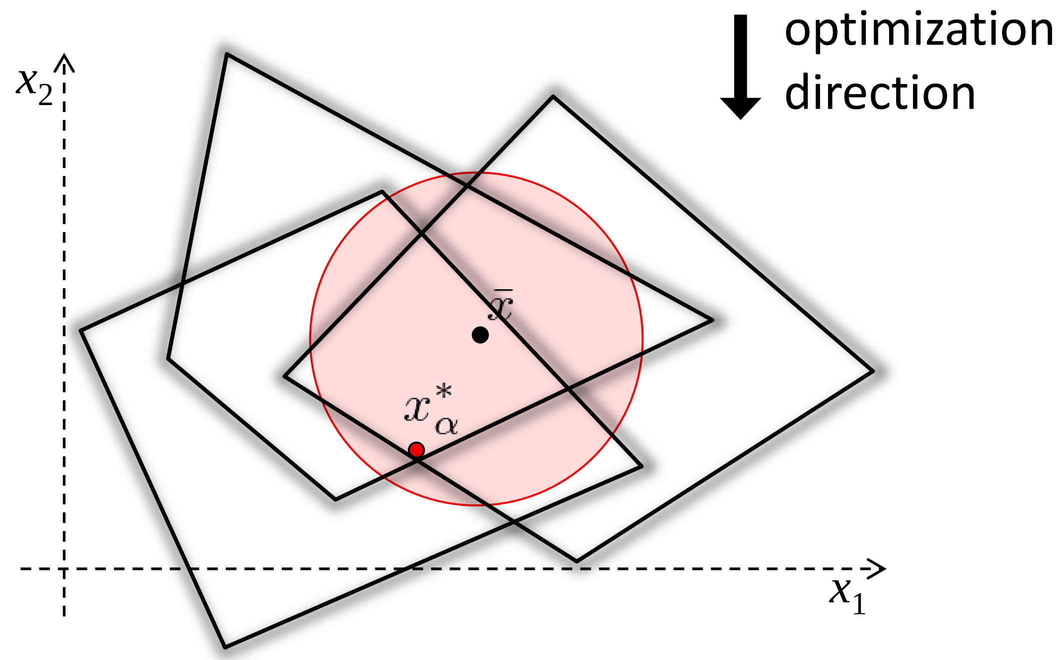
The other side of the coin (example)



Scenario optimization with regularization

$$\min_x \quad c(x)$$
$$\text{s.t.} \quad x \in \mathcal{X}_{\delta_i}$$
$$i = 1, \ldots, N$$
$$\|x - \bar{x}\| \leq \alpha$$



solution: x^*_α

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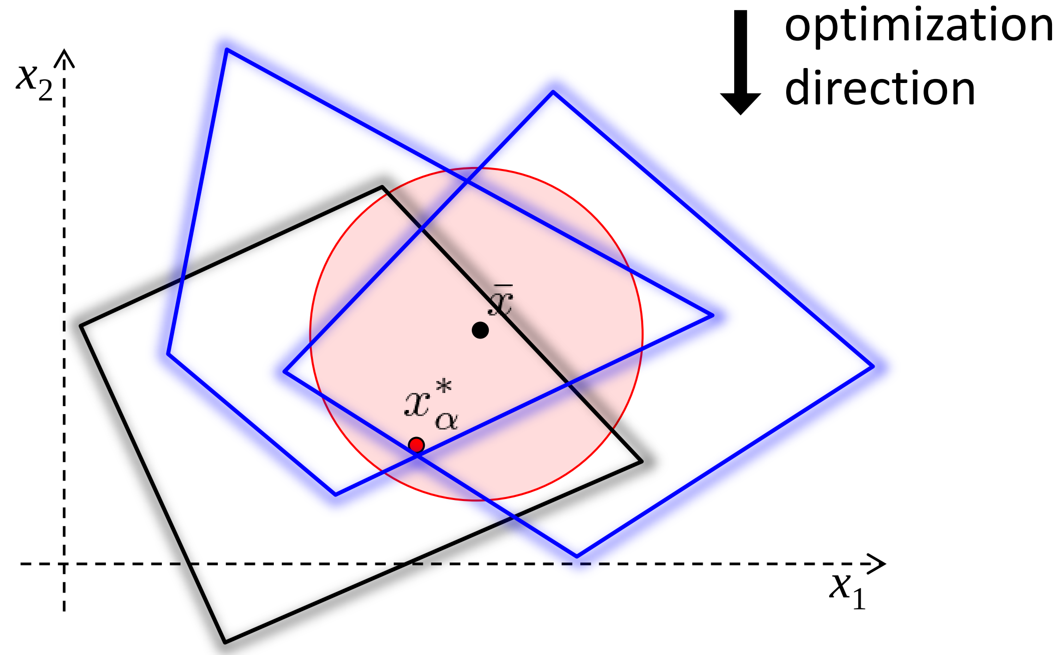

solution: x^*_α

complexity: s^*_α

(support set)

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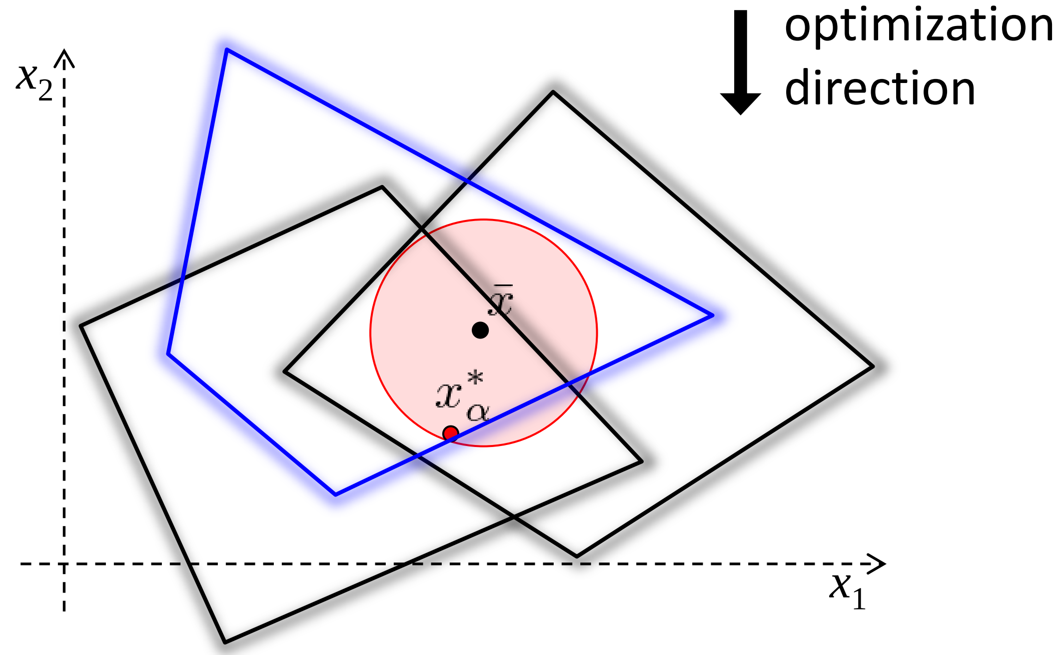

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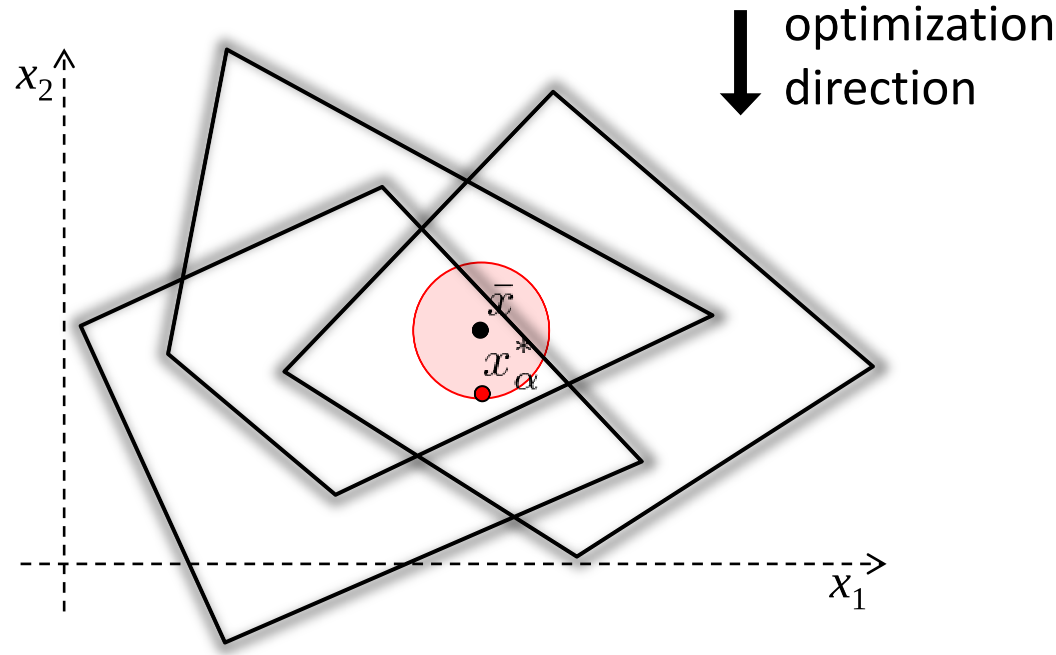

solution: x^*_α

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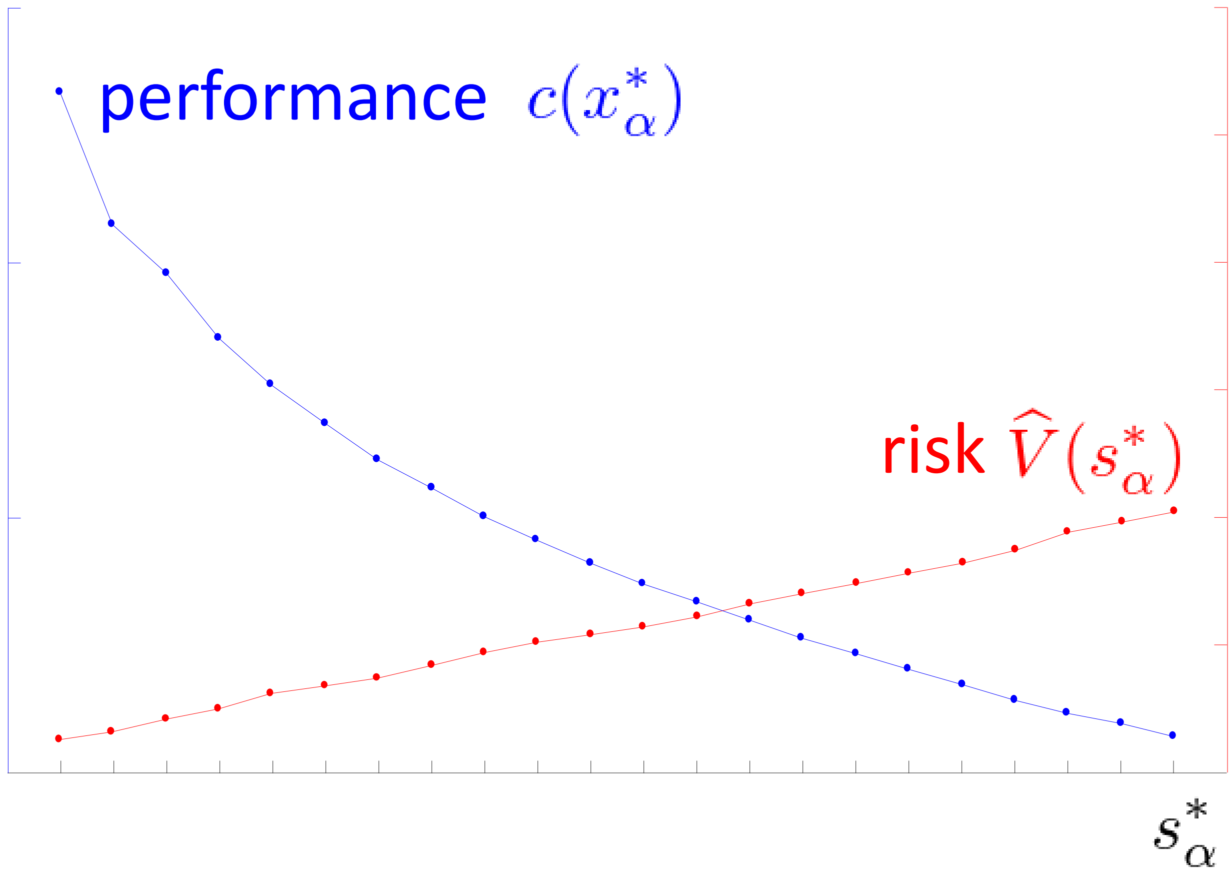
(support set)

as $\alpha \to 0$

cost $c(x^*_\alpha)$ increasing

risk $\widehat{V}(s^*_\alpha)$ decreasing

Scenario optimization with regularization



Conclusions

- Scenario optimization extended to scenario decision-making: a very general setup
- Alternative (tunable) schemes to obtain many alternative “solutions” (many other schemes exists, many others have to be discovered)
- Scenario theory: for each solution the risk (invisible) can be estimated from the complexity (visible)
- The risk estimate along with the performance allows the user to perform a quantitative comparison among the obtained solutions and to choose the one that is best suited for the problem at hand

Thank you !