Homogenization in non-convex Hencky plasticity and the limit of vanishing hardening in Hencky plasticity with non-convex potentials

Bernd Schmidt

Universität Augsburg, Germany

Topics in the Calculus of Variations: Recent Advances and New Trends

Banff, May 24^{th}, 2017

joint work with Martin Jesenko[1] (Augsburg, now Freiburg)

[1] responsible for the hard work

Overview

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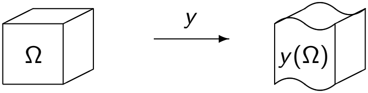
Modeling plastic deformations

Consider a solid material, occupying a region Ω, subject to a deformation $\Omega \ni x \mapsto x + u(x)$ with (small) displacement u, caused by some (small) loading.



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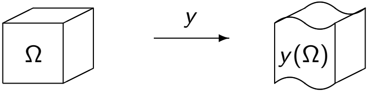

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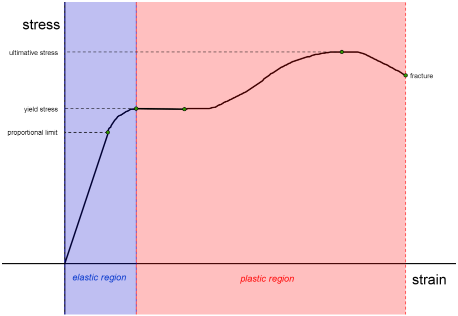
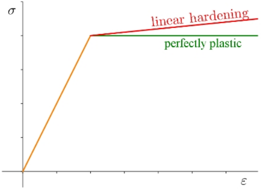


- For very small u, the body behaves elastically and will – after unloading – return to its original state $u = 0$.
- For larger values of u (small but not very small), the body behaves plastically. After unloading it is permanently deformed.

Atomistic explanation: Reorganized atomic bonds.

Hencky plasticity

Simplified static theory: Hencky plasticity (or ‘pseudoelasticity’).



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- perfectly plastic regime (Hencky plasticity),
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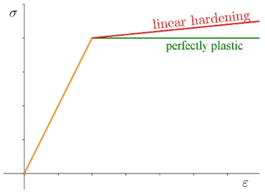

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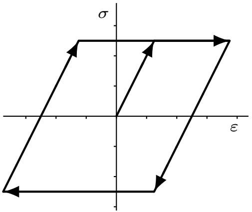
Draw back: Static theory does not keep track of the history.

- misses hysteresis effects
- can only apply to one-time loading.



Energy functionals and homogenization

Hencky energy functional at zero hardening

$$\mathcal{F}(u) = \int_\Omega f(\mathfrak{E}u(x))\,dx \quad \text{with} \quad f(X) = f_{\mathrm{dev}}(X_{\mathrm{dev}}) + \frac{\varkappa}{2}(\operatorname{tr} X)^2,$$

$\mathfrak{E}u(x) = \frac{1}{2}(\nabla u(x) + \nabla u(x)^T)$, $X_{\mathrm{dev}} = X - \frac{\operatorname{tr} X}{n} I$ and
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Note: Techniques completely different from [Demengel, Qi '90] who strongly use convex analysis tools (consider $f(\mu)$, $\mu \in M$).

Set-up: energy densities

Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary and $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\rm sym} \to \mathbb{R}$ a Carathéodory function that is $\mathbb{I}^n$-periodic, $\mathbb{I} = (0,1)$, in the first variable and satisfies

- the growth condition of Hencky plasticity

$$\alpha(|X_{\rm dev}| + (\operatorname{tr} X)^2) \le f(x,X) \le \beta(|X_{\rm dev}| + (\operatorname{tr} X)^2 + 1)$$

  for suitable $\alpha, \beta > 0$ for a.e. $x \in \Omega$ and every $X \in \mathbb{R}^{n\times n}_{\rm sym}$ and

- an asymptotic convexity assumption: $\forall\, \eta > 0$ there is $\beta_\eta > 0$ and a Carathéodory function $c : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\rm sym} \to \mathbb{R}$ that is $\mathbb{I}^n$-periodic in the first variable and convex in the second such that for a.e. $x \in \mathbb{R}^n$ and all $X \in \mathbb{R}^{n\times n}_{\rm sym}$

$$|f(x,X) - c^\eta(x,X)| \le \eta(|X_{\rm dev}| + (\operatorname{tr} X)^2) + \beta_\eta.$$

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Example. Periodic mixtures of shape memory alloys subject to elastic and plastic deformation. Common (geometrically linear) model:

$$W(x,X) = \frac{1}{2} \min_{i=1,\dots,N} Q\big(x, X - X_i(x)\big).$$

Main results (informal)

Let $f^{(\delta)}(x,X) = f(x,X) + \delta|X|^2$. We consider (each for suitable u):

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f\left(\tfrac{x}{\varepsilon}, \mathfrak{E}u(x)\right)\, dx \quad \text{(perfect plasticity)},$$

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$$\mathcal{F}_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}\left(\mathfrak{E}u(x)\right)\, dx + (f_{\mathrm{hom}})^{\#}\left(\text{sing. part of } \tfrac{(Du)^T + Du}{2}\right).$$

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Theorem B. The following diagrams commute:

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Requirements for the domain of $\mathcal{F}_\varepsilon$: $Eu = \frac{(Du)^T + Du}{2}$ must satisfy

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Hencky plasticity setting: Pass from $LU(\Omega)$ to

$$U(\Omega) = \{u \in BD(\Omega) : \mathrm{div}\, u \in L^2(\Omega)\},$$

$\|u\|_U = \|u\|_{BD} + \|\mathrm{div}\, u\|_{L^2}$. (Clearly, $E^s u = E^s_{\mathrm{dev}} u$.)

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Definition. ([Demengel, Temam '84], [Temam '85]) We say $u_j \xrightarrow{c} u$ ('c-strictly') in U if

- $u_j \to u$ in $L^1(\Omega; \mathbb{R}^n)$,
- $|Eu_j|(\Omega) \to |Eu|(\Omega)$,
- $\operatorname{div} u_j \to \operatorname{div} u$ in $L^2(\Omega)$.
- $\int_\Omega c(E_{\mathrm{dev}} u_j) \to \int_\Omega c(E_{\mathrm{dev}} u)$,
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Theorem. (Density, [Temam '83]) $\forall\, u \in U(\Omega)$ $\exists\, (u_j)_{j \in \mathbb{N}} \subset C^\infty(\Omega; \mathbb{R}^n) \cap LU(\Omega)$ such that

$$u_j|_{\partial\Omega} = u|_{\partial\Omega} \quad \text{and} \quad u_j \xrightarrow{c} u \text{ in } U(\Omega).$$

A technical but useful auxiliary result: improved integrability

Lemma. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$-boundary and let $(u_j)_{j\in\mathbb{N}}$ be a bounded sequence in $U(\Omega)$. There exist a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ and a sequence $(\tilde{u}_k)_{k\in\mathbb{N}} \subset U(\Omega)$ such that

- $((\mathrm{div}\, \tilde{u}_k)^2)_{k\in\mathbb{N}}$ is equiintegrable,
- $(u_{j_k} - \tilde{u}_k)_{k\in\mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^n)$ and therefore $E^s u_{j_k} = E^s \tilde{u}_k$,
- $\lim_{k\to\infty} \left|\{\nabla(\tilde{u}_k - u_{j_k}) \neq 0\} \cup \{\tilde{u}_k \neq u_{j_k}\}\right| = 0$.

Moreover, if $\{u_j\}_{j\in\mathbb{N}}$ converges weakly or c-strictly to u in $U(\Omega)$, then the $\tilde{u}_k$ can be chosen in such a way that $\tilde{u}_k|_{\partial\Omega} = u|_{\partial\Omega}$ and $\{\tilde{u}_k\}_{k\in\mathbb{N}}$ converges to u in $U(\Omega)$ in the same manner.

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Proof uses a Helmholtz decomposition in U:

$$U(\Omega) = (\ker \mathrm{div}) \oplus (\mathrm{im}\, \nabla)$$

and the corresponding improved integrability result on $W^{1,2}$, cf [Fonseca, Müller, Pedregal '98].

Overview

Nonzero hardening

For any $\delta \geq 0$ we set $f^{(\delta)}(x, X) := f(x, X) + \delta |X_{\rm dev}|^2$ and let

$$\mathcal{F}_\varepsilon^{(\delta)}(u) := \begin{cases} \int_\Omega f^{(\delta)}(\frac{x}{\varepsilon}, \mathfrak{E}u(x))\, dx, & u \in W^{1,2}(\Omega; \mathbb{R}^n), \\ \infty, & \text{else.} \end{cases}$$

For $\delta > 0$ the densities have a quadratic growth in $|X_{\rm sym}|$. With the help of Korn's inequality and standard homogenization results [Braides '85, Müller '87] we obtain

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon^{(\delta)} = \mathcal{F}_{\rm hom}^{(\delta)},$$

where $\mathcal{F}_{\rm hom}^{(\delta)}$ has domain $W^{1,2}(\Omega; \mathbb{R}^n)$ and density

$$f_{\rm hom}^{(\delta)}(X) = \inf_{k \in \mathbb{N}} \inf_{\varphi \in W_0^{1,2}(k\mathbb{I}^n; \mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f^{(\delta)}(x, X + \mathfrak{E}\varphi(x))\, dx.$$

Vanishing hardening

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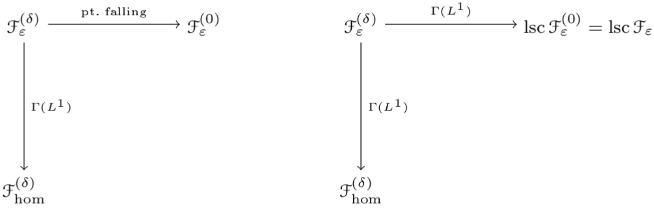

Let

$$f_{\hom}(X) := \inf_{k \in \mathbb{N}} \inf_{\varphi \in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f\big(x, X + \mathfrak{E}\varphi(x)\big)\, dx.$$

Clearly

$$f_{\hom}(X) = \inf_{\delta > 0} f_{\hom}^{(\delta)}(X).$$

Define

$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\hom}\big(\mathfrak{E}u(x)\big), & u \in LU(\Omega;\mathbb{R}^n) \cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

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$$
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\
\Big\downarrow \Gamma(L^1) & & \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\text{pt. falling}]{} & \mathcal{G}^{(0)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \mathrm{lsc}\,\mathcal{F}_\varepsilon^{(0)} = \mathrm{lsc}\,\mathcal{F}_\varepsilon \\
\Big\downarrow \Gamma(L^1) & & \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma(L^1)]{} & \mathrm{lsc}\,\mathcal{G}^{(0)} = \mathrm{lsc}\,\mathcal{G}
\end{array}
$$

Let

$$f_{\mathrm{hom}}(X) := \inf_{k\in\mathbb{N}} \inf_{\varphi\in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n}\int_{k\mathbb{I}^n} f\big(x, X + \mathfrak{E}\varphi(x)\big)\, dx.$$

Clearly

$$f_{\mathrm{hom}}(X) = \inf_{\delta>0} f_{\mathrm{hom}}^{(\delta)}(X).$$

Define

$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\big(\mathfrak{E}u(x)\big), & u \in LU(\Omega;\mathbb{R}^n)\cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

Vanishing hardening

$$\begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\ \downarrow{\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\hom}^{(\delta)} & \xrightarrow[\text{pt. falling}]{} & \mathcal{G}^{(0)} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma(L^1)} & \operatorname{lsc} \mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc} \mathcal{F}_\varepsilon \\ \downarrow{\scriptstyle \Gamma(L^1)} & & \\ \mathcal{F}_{\hom}^{(\delta)} & \xrightarrow[\Gamma(L^1)]{} & \operatorname{lsc} \mathcal{G}^{(0)} = \operatorname{lsc} \mathcal{G} \end{array}$$

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$$f_{\hom}(X) := \inf_{k \in \mathbb{N}} \inf_{\varphi \in C_c^\infty(k\mathbb{I}^n;\mathbb{R}^n)} \frac{1}{k^n} \int_{k\mathbb{I}^n} f\left(x, X + \mathfrak{E}\varphi(x)\right) \, dx.$$

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$$\mathcal{G}^{(0)}(u) := \begin{cases} \int_\Omega f_{\hom}(\mathfrak{E}u(x)), & u \in LU(\Omega;\mathbb{R}^n) \cap W^{1,2}(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

Notice

$$\mathcal{F}_{\hom}^{(\delta)} \geq \Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon \quad \text{and therefore} \quad \operatorname{lsc} \mathcal{G} \geq \Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon.$$

Overview

Basic ingredient

Theorem. (Reshetnyak continuity theorem, cf. [Kristensen, Rindler '10]). Let $f \in \mathbf{E}(\Omega; \mathbb{R}^N)$, and

$$\mu_j \overset{*}{\rightharpoonup} \mu \quad \text{in } M(\Omega; \mathbb{R}^N) \quad \text{and} \quad \langle \mu_j \rangle(\Omega) \to \langle \mu \rangle(\Omega).$$

Then

$$\lim_{j\to\infty} \left[\int_\Omega f\left(x, \frac{d\mu_j^a}{d\mathcal{L}^n}(x)\right) \, dx + \int_\Omega f^\infty\left(x, \frac{d\mu_j^s}{d|\mu_j^s|}(x)\right) \, d|\mu_j^s|(x) \right] =$$
$$= \int_\Omega f\left(x, \frac{d\mu^a}{d\mathcal{L}^n}(x)\right) \, dx + \int_\Omega f^\infty\left(x, \frac{d\mu^s}{d|\mu^s|}(x)\right) \, d|\mu^s|(x).$$

Here

- $\mathbf{E} = \{\text{functions extendable to } \infty\}$ (with linear growth)
- $\langle A \rangle := \sqrt{1 + |A|^2}$
- $f^\infty(x_0, X_0) = \limsup_{X \to X_0,\ t \to \infty} \frac{f(x_0, tX)}{t} = \lim_{\substack{x \to x_0 \\ X \to X_0,\ t \to \infty}} \frac{f(x, tX)}{t}$

$\langle\cdot\rangle$-strict continuity

Theorem. Let $f : \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be a continuous function that

- is symmetric-rank-one-convex in the second variable,
- satisfies the Hencky growth condition.

Suppose that $(f_{\mathrm{dev}})^\infty(\cdot, P_0) = (f|_{\Omega\times\mathbb{R}^{n\times n}_{\mathrm{dev}}})^\infty(\cdot, P_0) = \limsup_{P\to P_0, t\to\infty} \frac{f_{\mathrm{dev}}(\cdot, tP)}{t}$ is continuous for every fixed $P_0 \in \mathbb{R}^{n\times n}_{\mathrm{dev}}$.

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$$\mathcal{F}(u)=\int_\Omega f\left(x,\mathfrak{E}u(x)\right)\,dx+\int_\Omega (f_{\mathrm{dev}})^\infty\left(x,\frac{dE^s u}{d|E^s u|}(x)\right)\,d|E^s u|(x)$$

is $\langle\cdot\rangle$-strictly continuous on $U(\Omega;\mathbb{R}^n)$.

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is $\langle\cdot\rangle$-strictly continuous on $U(\Omega; \mathbb{R}^n)$.

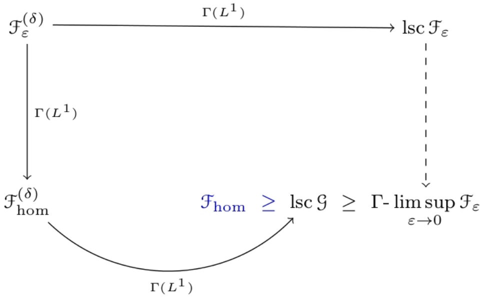
Ingredients of the proof:

- Careful Lipschitz estimate in the trace direction.
- Approximation of functions $\geq -\alpha(1 + |X|)$ by functions from $\mathbf{E}(\Omega; \mathbb{R}^N)$ [Alibert, Bouchitté '97].
- Rank-one theorem [De Philippis, Rindler '16]: *Let $u \in BD(\Omega; \mathbb{R}^n)$. Then, for $|E^s u|$-a.e. $x \in \Omega$, there exist $a(x), b(x) \in \mathbb{R}^n \setminus \{0\}$ such that*

$$\frac{dE^s u}{d|E^s u|} = a(x) \odot b(x) = \tfrac{1}{2}(a(x) \otimes b(x) + b(x) \otimes a(x)).$$

Recovery sequence

We now have



with

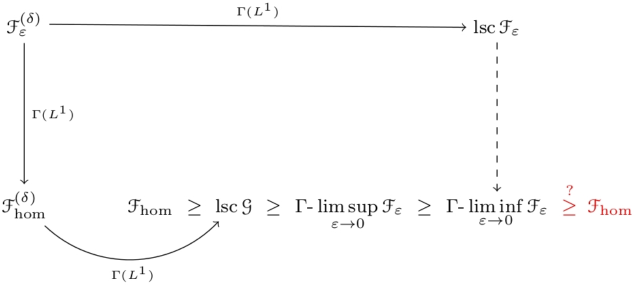
$$\mathcal{F}_{\text{hom}}(u) := \begin{cases} \int_\Omega f_{\text{hom}}\big(\mathfrak{E}u(x)\big)\, dx + \int_\Omega (f_{\text{hom}})^{\#}\Big(\frac{dE^s u}{d|E^s u|}(x)\Big)\, d|E^s u|(x), & u \in U(\Omega; \mathbb{R}^n), \\ \infty, & \text{else.} \end{cases}$$

and

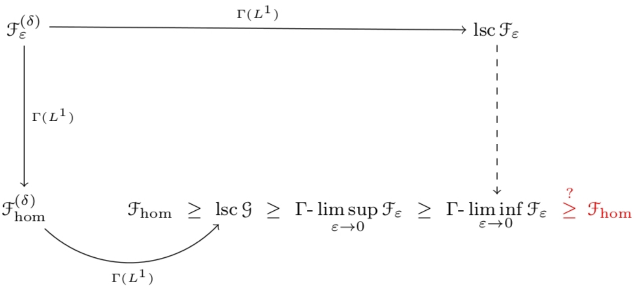
$$g^{\#}(X) := \limsup_{t \to \infty} \frac{g(tX)}{t}.$$

Overview

Regular points



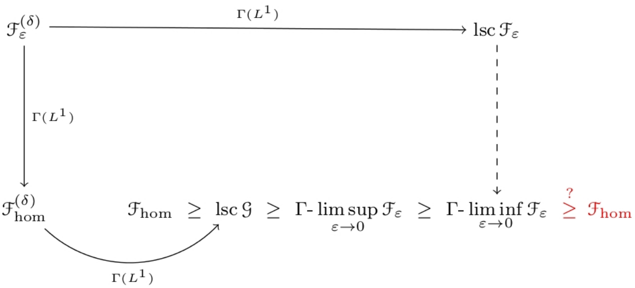
Regular points



Lemma. (regular points) If $u \in U(\Omega)$, $u_j \to u$ in L^1 and $\varepsilon_j \searrow 0$, then

$$\liminf_{j \to \infty} \mathcal{F}_{\varepsilon_j}(u_j) \geq \int_\Omega f_{\mathrm{hom}}(\mathfrak{E}u(x))\, dx.$$

Regular points



Lemma. (regular points) If $u \in U(\Omega)$, $u_j \to u$ in L^1 and $\varepsilon_j \searrow 0$, then

$$\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) \geq \int_\Omega f_{\text{hom}}(\mathfrak{E}u(x))\, dx.$$

Lemma. Every $u \in BD(\Omega; \mathbb{R}^n)$ is a.e. $L^{\frac{n}{n-1}}$-differentiable: for a.e. $x_0 \in \Omega$

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x_0)} \left| \frac{u(x) - u(x_0) - \nabla u(x_0)(x - x_0)}{r} \right|^{\frac{n}{n-1}} dx = 0.$$

(L^q-differentiability for $q < \frac{n}{n-1}$ shown by [Alberti, Bianchini, Crippa '14].)

Blow up

We may suppose $\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$. Let us fix some $1 < q < \frac{n}{n-1}$ and define the measures

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n.$$

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Extracting subsequences we get

- $\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j)$ equals the lim inf above with all $u_j \in LU(\Omega; \mathbb{R}^n)$,
- $u_j \to u$ in $L^q(\Omega; \mathbb{R}^n)$ due to the lower bound on f and since LU is compactly embedded in L^q,
- and $\mu_j \overset{*}{\rightharpoonup} \mu$ in $M(\Omega; \mathbb{R}^n)$.

Let

$$\mu = g\mathcal{L}^n + \mu^s$$

Goal: $g(x) \geq f_{\mathrm{hom}}(\mathfrak{E}u(x))$ for a.e. $x \in \Omega$.

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Let

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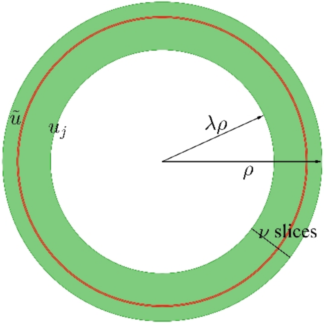
Goal: $g(x) \geq f_{\mathrm{hom}}(\mathfrak{E}u(x))$ for a.e. $x \in \Omega$.

Need to show: For a.e. $x_0 \in \Omega$

$$\lim_{\rho\to 0} \lim_{j\to\infty} \frac{\mu_j(B_\rho(x_0))}{|B_\rho(x_0)|} \geq f_{\mathrm{hom}}(\mathfrak{E}u(x_0)).$$

De Giorgi's slicing method and Bogovskii's operator



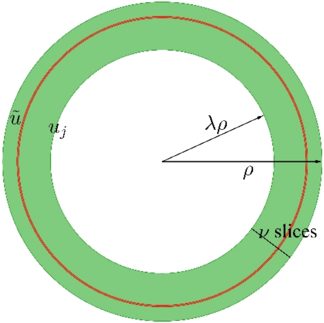
Fix any x_0 where u is approximately differentiable, let

$$\tilde{u}(x) := u(x_0) + \nabla u(x_0)\,(x - x_0).$$

Localize to suitable $B_1, \ldots, B_\nu$ with cut-off functions φ_i ($\equiv 1$ on B_{i-1}, $\equiv 0$ on B_i^c):

$$\tilde{u}_{j,i} := \tilde{u} + \varphi_i(u_j - \tilde{u}) \in L^1(\Omega; \mathbb{R}^n).$$

De Giorgi's slicing method and Bogovskii's operator



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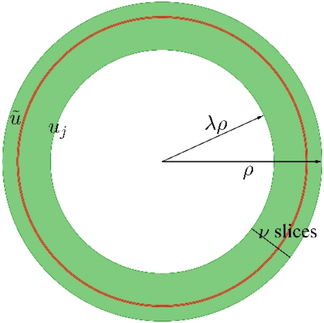
Problem: No L^2-control on the last term in

$$\operatorname{div} \tilde{u}_{j,i} = (1-\varphi_i)\operatorname{div} \tilde{u} + \varphi_i \operatorname{div} u_j + \\ + \nabla\varphi_i \cdot (u_j - \tilde{u})$$

De Giorgi's slicing method and Bogovskii's operator



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Let $\zeta_{j,i} :=$ average of $\nabla\varphi_i \cdot (u_j - \tilde{u})$ in $B_i \setminus B_{i-1}$. Bogovskii's operator yields $z_{j,i} \in W_0^{1,q}(B_i \setminus \overline{B_{i-1}})$ such that

$$\operatorname{div} z_{j,i} = -\nabla\varphi_i \cdot (u_j - \tilde{u}) + \zeta_{j,i}$$

with

$$\|z_{j,i}\|_{W^{1,q}(B_i \setminus \overline{B_{i-1}})} \leq \frac{C\nu}{(1-\lambda)\rho} \|u_j - \tilde{u}\|_{L^q(B_i \setminus \overline{B_{i-1}})}.$$

De Giorgi's slicing method and Bogovskii's operator

Now define $u_{j,i} := \tilde{u}_{j,i} + z_{j,i} \in LU(\Omega; \mathbb{R}^n)$. Notice that

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Then

$$\begin{aligned} f_{\mathrm{hom}}(\mathfrak{E}u(x_0)) &= \lim_{j\to\infty} \inf_{\varphi\in LU_0(B_\rho(x_0),\mathbb{R}^n)} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\left(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u(x_0) + \mathfrak{E}\varphi(x)\right)\,dx \\ &\leq \liminf_{j\to\infty} \frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} f\left(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\right)\,dx \end{aligned}$$

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Averaging: $$f_{\mathrm{hom}}(\mathfrak{E}u(x_0)) \le \liminf_{j\to\infty} \frac{1}{\nu} \sum_{i=1}^{\nu} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\left(\tfrac{x}{\varepsilon_j}, \mathfrak{E}u_{j,i}(x)\right) dx.$$

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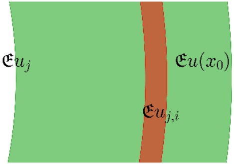

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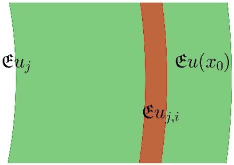

- First term: ✓

De Giorgi's slicing method and Bogovskii's operator

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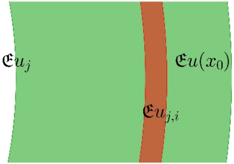

- First term: ✓
- Third term: $\lambda \nearrow 1$

De Giorgi's slicing method and Bogovskii's operator

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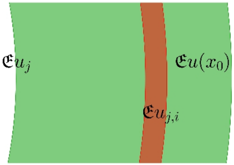

- First term: ✓
- Third term: $\lambda \nearrow 1$
- Second term: L^q-differentiability

Singular points: asymptotic convexity

Recall

$$\mu_j := f(\tfrac{\cdot}{\varepsilon_j}, \mathfrak{E}u_j(\cdot))\mathcal{L}^n \overset{*}{\rightharpoonup} \mu = g\mathcal{L}^n + \mu^s.$$

Need to show:

$$\mu^s \geq (f_{\mathrm{hom}})^{\#}(\tfrac{dE^s u}{d|E^s u|})|E^s u|.$$

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$$|f(x,X) - c^\eta(x,X)| \leq \eta(|X_{\mathrm{dev}}| + (\mathrm{tr}\, X)^2) + \beta_\eta.$$

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We may suppose $(|\mathfrak{E}_{\mathrm{dev}}\, u_j| + (\mathrm{div}\, u_j)^2)\mathcal{L}^n \overset{*}{\rightharpoonup} \sigma$ in $M(\Omega)$. Results in [Demengel, Qi 90] for the convex case and asymptotic convexity give

$$\begin{aligned}\mu^s &\ge \lim_{\eta\to 0}\Big[(c^\eta_{\mathrm{hom}})^{\#}(\tfrac{dE^s u}{d|E^s u|})|E^s u| - \eta\sigma^s\Big]\\ &= (f_{\mathrm{hom}})^{\#}(\tfrac{dE^s u}{d|E^s u|})|E^s u|.\end{aligned}$$

Overview

Homogenization

Theorem A. Suppose $f : \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a Carathéodory function which

- is $\mathbb{I}^n$-periodic in the first variable,
- has Hencky plasticity growth,
- and is asymptotically convex.

Let

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega f\left(\frac{x}{\varepsilon}, \mathfrak{E}u(x)\right)\,dx, & u \in LU(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}, \end{cases}$$

and

$$\mathcal{F}_{\mathrm{hom}}(u) := \begin{cases} \int_\Omega f_{\mathrm{hom}}\left(\mathfrak{E}u(x)\right)\,dx + \int_\Omega (f_{\mathrm{hom}})^{\#}\left(\frac{dE^s u}{d|E^s u|}(x)\right)\,d|E^s u|(x), & u \in U(\Omega;\mathbb{R}^n), \\ \infty, & \text{else}. \end{cases}$$

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Remark. Without the asymptotic convexity assumption we still have

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}_{\mathrm{hom}} \text{ on } LU(\Omega)$$

(and $\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon \le \mathcal{F}_{\mathrm{hom}}$ in general).

Homogenization and vanishing hardening

Theorem B. Under the same assumptions the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\text{pt. falling}} & \mathcal{F}_\varepsilon^{(0)} \\
\Big\downarrow \Gamma & & \Big\downarrow \Gamma \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma]{} & \mathcal{F}_{\mathrm{hom}}
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
\mathcal{F}_\varepsilon^{(\delta)} & \xrightarrow{\Gamma} & \operatorname{lsc} \mathcal{F}_\varepsilon^{(0)} = \operatorname{lsc} \mathcal{F}_\varepsilon \\
\Big\downarrow \Gamma & & \Big\downarrow \Gamma \\
\mathcal{F}_{\mathrm{hom}}^{(\delta)} & \xrightarrow[\Gamma]{} & \mathcal{F}_{\mathrm{hom}}
\end{array}
$$

(All Γ-limits are with respect to the L^1-norm.)

Thanks

Thank you for your attention!

References:

M. Jesenko, B. Schmidt:
Homogenization and the limit of vanishing hardening in Hencky plasticity with non-convex potentials,
Calc. Var. Partial Differential Equations **57** (2018), no. 1, Art. 2, 43 pp.