Regularity (and Higher Regularity) for the Thin Obstacle Problem

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(joint work with Herbert Koch and Wenhui Shi)

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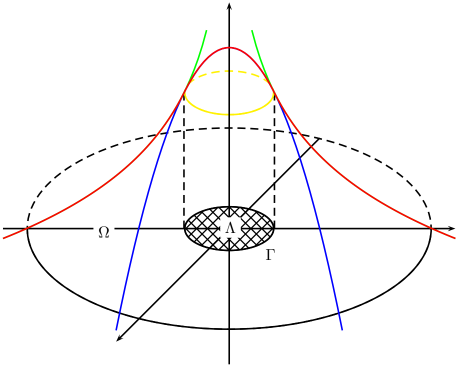
The Classical Obstacle Problem

Minimize

$$J(w) := \int_{B_1} |\nabla w|^2 dx \text{ in } \mathcal{K} := \{w \in H^1_g(B_1) : w \geq \phi \text{ in } B_1\}.$$



Questions:

- Existence and uniqueness,
- Regularity,
- Regularity of the free boundary.

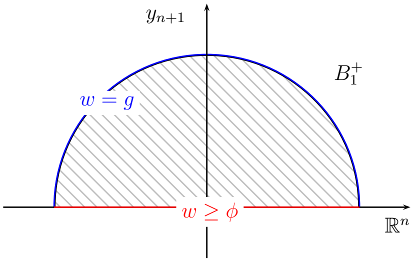
Literature:

- Caffarelli: The obstacle problem revisited.

The Thin Obstacle Problem

Minimize

$$J(w) := \int_{B_1^+} |\nabla w|^2 dx \text{ in } \mathcal{K} := \{w \in H_g^1(B_1^+) : w \geq \phi \text{ in } B_1'\}.$$



Applications:

- Osmosis,
- American options,
- Signorini problem for elastic bodies.

The Thin Obstacle Problem

Minimize

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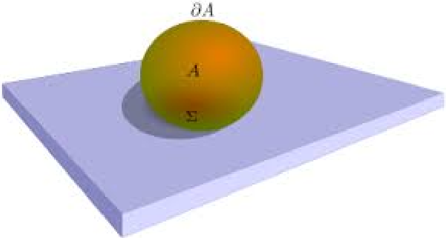

Applications:

- Osmosis,
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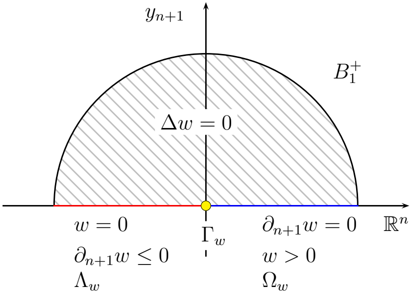
- Signorini problem for elastic bodies.

The Thin Obstacle Problem

Minimize $J(w) := \int_{B_1^+} |\nabla w|^2 dx$ in $\mathcal{K} := \{w \in H^1_g(B_1^+) : w \geq 0 \text{ in } B_1'\}$.

$$\begin{aligned} \Delta w &= 0 \text{ in } B_1^+, \\ w &\geq 0 \text{ on } B_1', \\ \partial_{n+1} w &\leq 0 \text{ on } B_1', \\ w\partial_{n+1} w &= 0 \text{ on } B_1'. \end{aligned}$$

- contact set Λ_w,
- non-coincidence set Ω_w,
- free boundary Γ_w.

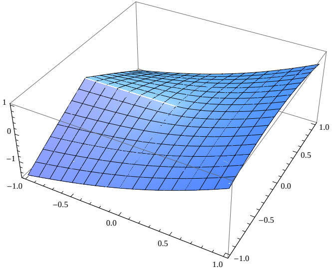


Properties of the Thin Obstacle Problem

Theorem (Athanasopoulos-Caffarelli-Salsa)

Let $w : B_1^+ \to \mathbb{R}$ be a solution of the thin obstacle problem. Then,

- $w \in C^{1,1/2}$,
- $\Gamma_w = \Gamma_{3/2}(w) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(w)$,
- $\Gamma_{3/2}(w) \in C^{1,\alpha}$.

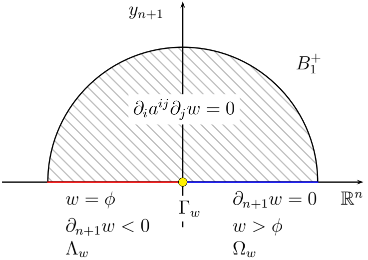


Literature:

- Lewy (1972), Richardson (1978),
- Caffarelli (1979), Kinderlehrer (1981), Uraltseva (1987),
- Athanasopoulos, Caffarelli, Salsa, Silvestre (2006-2008),
- Focardi, Spadaro (2015, 2017); Colombo, Spolaor, Velchikov (2017).

Variable Coefficients

$$J(w) := \int_{B_1^+} a^{ij}(\partial_i w)(\partial_j w)dx \text{ in } \mathcal{K} := \{w \in H^1_g(B_1^+) : w \geq 0 \text{ in } B_1'\}.$$



$$\begin{aligned} \partial_i a^{ij} \partial_j w &= 0 \text{ in } B_1^+, \\ w &\geq 0 \text{ on } B_1', \\ a^{n+1,j}\partial_j w &\leq 0 \text{ on } B_1', \\ w a^{n+1,j}\partial_j w &= 0 \text{ on } B_1'. \end{aligned}$$

Assumptions:

- Guillen (2009): $a^{ij} \in C^{1,\epsilon}$,
- Garofalo-Petrosyan-Smit Vega Garcia (2014, 2015): $a^{ij} \in W^{1,\infty}$,
- Koch-R.-Shi (AIM 2016, AIHPC 2016): $a^{ij} \in W^{1,p}$, $p > n+1$,
- R.-Shi (Calc. Var. PDE 2017): $a^{ij} \in C^{0,\alpha}$.

Variable $W^{1,p}$ Coefficients, $p > n+1$

Solution:

- $C^{1,\min\{\gamma,1/2\}}$ regularity of the solution, where $\gamma = 1 - \frac{n+1}{p}$ (Carleman + free boundary analysis).
- Identification of asymptotic expansion of solutions at the regular free boundary.

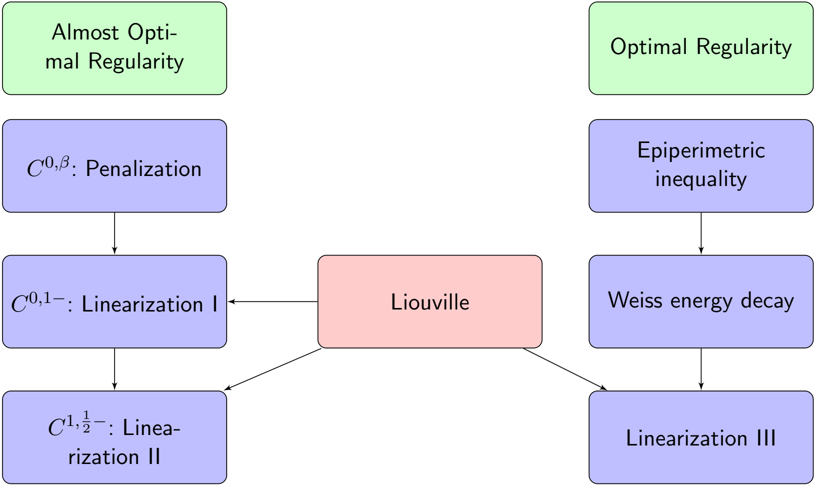
Free Boundary:

- Bound on Hausdorff dimension of the free boundary (similar as in [Simon-Wickramasekera '10]): $\dim_{\mathcal{H}}(\Gamma_w) \leq n-1$.
- Separation of the free boundary into the regular free boundary and a remainder:

$$\Gamma_w = \Gamma_{3/2}(w) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(w).$$

- Analysis of the regular free boundary: $C^{1,\alpha}$ regularity for some $\alpha \in (0,1)$.

Variable $C^{0,\alpha}$ Coefficients, $\alpha \in (0,1)$: "Linearization"



Liouville

Lemma (Liouville)

Let $w \in W^{1,2}_{loc}(\mathbb{R}^{n+1})$ *be a global solution to the thin obstacle problem for* $a^{ij} = \delta^{ij}$*, i.e., suppose that for any* $R \geq 1$*,*

$$\int_{B_R} \nabla w \cdot \nabla (v - w) dx \geq 0,$$

$$\text{for all } v \in \{v \in W^{1,2}(B_R) : v \geq 0 \text{ on } B_R',\ v = w \text{ on } \partial B_R\}.$$

Assume that $\sup_{B_R} |w| \leq C R^{\beta}$ *for* $R \to \infty$ *and some* $\beta \in (0, 2)$*. Then,*

$$w(x) = c1 + b x_{n+1} + d \mathrm{Re}(Qx' + i|x_{n+1}|)^{3/2}, \quad b, c, d \in \mathbb{R},\ Q \in SO(n).$$

Idea: Use Friedland-Hayman inequality to deduce a sign for $\partial_i w \rightsquigarrow$ two-dimensional dependence of w.

Linearization: Almost Optimal Regularity

Lemma ($C^{1,\alpha} \Rightarrow C^{1,1/2-}$)

Let $a^{ij} \in C^{0,\alpha}$ *and* $w \in C^{1,\alpha}(B_1)$ *with* $\|w\|_{L^2(B_1)} = 1$ *be a solution of*

$$\partial_i a^{ij} \partial_j w = 0 \text{ in } B_1,$$
$$w \geq 0,\ \partial_{n+1} w \leq 0,\ w\partial_{n+1} w = 0.$$

For any $\beta \in (0, 3/2)$ *there exists a constant* $C = C(\beta, n, \|a^{ij}\|_{C^{0,\alpha}}) > 0$ *such that*

$$\sup_{B_r(x_0)} |w| \leq C r^{\beta} \text{ for all } r \in (0, 1/4),\ x_0 \in B_{1/2} \cap \Gamma_w.$$

Linearization: Ideas

- Contradiction: $\exists\, w^k$, $\|w^k\|_{L^2(B_1)} = 1$, $w^k(0) = 0 = \partial_{n+1} w^k(0)$, $r_k \in (0, 1/4)$ s.t.

$$\sup_{B_{r_k}} |w^k| = k r_k^{\beta} \text{ and } \sup_{B_R} |w^k| \leq k R^{\beta} \text{ for any } R \geq r_k.$$

- $C^{1,\alpha}$ regularity: $r_k \to 0$.
- Blow-up: Consider $\tilde{w}^k(x) = \frac{w^k(r_k x)}{k r_k^{\beta}}$

$$\sup_{B_{\tilde{R}}} |\tilde{w}^k| \leq \tilde{R}^{\beta} \text{ and } \sup_{B_1} |\tilde{w}^k| = 1 \text{ for any } \tilde{R} > 1.$$

- Convergence in $W^{1,2}_{loc}(\mathbb{R}^{n+1})$ and $C^{1,\alpha}_{loc}(\mathbb{R}^{n+1})$. Limit w_0 is a global solution to the constant coefficient thin obstacle problem with

$$\sup_{B_R} |w_0| \leq R^{\beta} \text{ and } \sup_{B_1} |w_0| = 1 \text{ for any } R > 1.$$

$\rightsquigarrow$ Liouville & $w_0(0) = 0 = \partial_{n+1} w_0(0)$.

Ingredients of Higher Order Linearization Arguments

- Weiss energy:

$$W_{3/2}(r,w) := \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 dx - \frac{3}{2}\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 d\mathcal{H}^n.$$

  Decay by means of epiperimetric inequality.

- Consider decay of

$$\|w - c\,\mathrm{Re}(Qx' + i|x_{n+1}|)^{3/2}\|_{\tilde{L}^2(\partial B_r)}$$

  $\rightsquigarrow$ compactness by decay of Weiss energy (no monotonicity available!).

- Regular free boundary regularity: $C^{1,\gamma}$, $(n-1)$-dimensional manifold.
  $\rightsquigarrow$ relies on precise control of asymptotics of solution.

Higher Regularity

Theorem (Koch-R.-Shi '16)

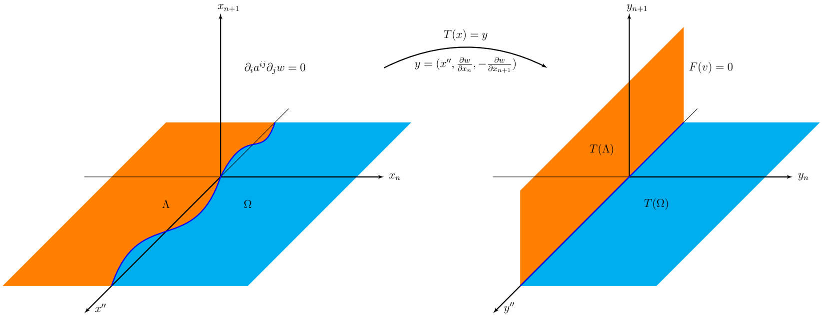
Let w be a solution of the thin obstacle problem with $W^{1,p}$, $p > n+1$, regular coefficients a^{ij} and $\phi = 0$. Then

- $\Gamma_{3/2}(w) \in C^{1,1-\frac{n+1}{p}}$.
- *If $a^{ij} \in C^{k,\gamma}$ mit $k \geq 1$, then $\Gamma_{3/2}(w) \in C^{k+1,\gamma}$.*
- *If a^{ij} is analytic, then $\Gamma_{3/2}(w)$ is analytic.*

Remarks:

- Tangential regularity result.
- Optimal dependence on a^{ij}.
- Similar results in the presence of non-trivial obstacles.
- Higher regularity for $a^{ij} = \delta^{ij}$: De Silva-Savin (2014-15), Koch-Petrosyan-Shi (2015).

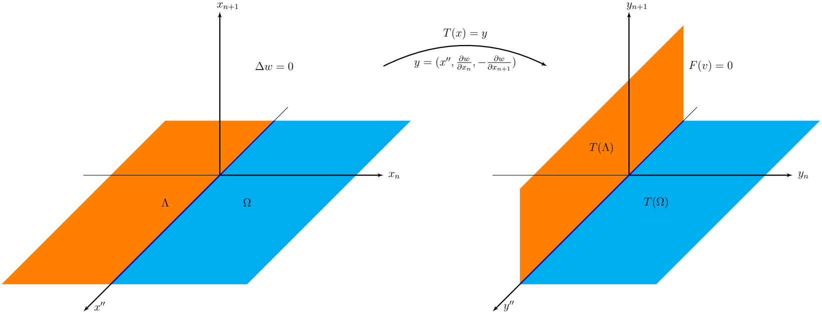
Higher Regularity: Ideas



- partial Legendre-Hodograph transformation.

$$
\boxed{\begin{gathered}
y = T(x) := (x'', \frac{\partial w(x)}{\partial x_n}, -\frac{\partial w(x)}{\partial x_{n+1}}), \\
v(y) = w(x) - x_n y_n + x_{n+1} y_{n+1}, \ x = T^{-1}(y), \\
\Gamma_{3/2}(w) = \{(x'', x_n, 0) : x_n = -\partial_{y_n} v(y)|_{y=(y'',0,0)}\}.
\end{gathered}}
$$

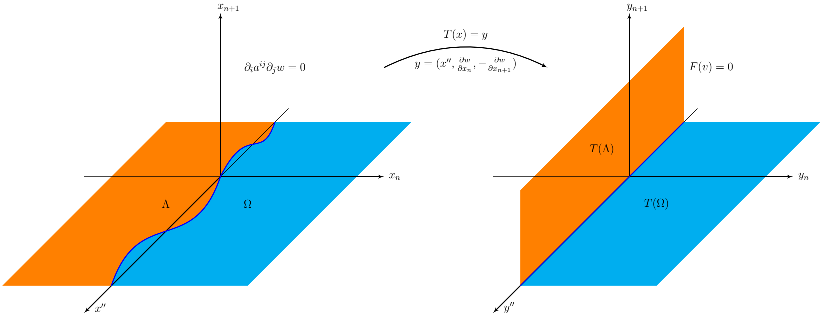
Higher Regularity: Ideas



- partial Legendre-Hodograph transformation.

$$\begin{aligned} y(x) &= (x'', \frac{\partial w}{\partial x_n}, -\frac{\partial w}{\partial x_{n+1}}) \\ &= (x'', \mathrm{Re}(x_n + i x_{n+1})^{1/2}, \mathrm{Im}(x_n + i x_{n+1})^{1/2}), \\ w(x) &= \mathrm{Re}(x_n + i x_{n+1})^{3/2} \leftrightsquigarrow v(y) = y_n^3 - 3 y_n y_{n+1}^2. \end{aligned}$$

Higher Regularity: Ideas



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$$\Gamma_{3/2}(w) = \{(x'', x_n, 0) : x_n = -\partial_{y_n} v(y)|_{y=(y'',0,0)}\}.$$

Higher Regularity: Ideas

- Identification of operator as fully nonlinear perturbation of Baouendi-Grushin Laplace $\Delta_G := (x_n^2 + x_{n+1}^2)\Delta'' + \partial_{nn}^2 + \partial_{n+1,n+1}^2$.
- Adapted Geometry und modified Hölder spaces.
- Implicit functions.
- Splitting technique for low regularity situation.

Generalizations

- Inhomogeneities and obstacles:

$a^{ij} \in W^{1,p}$, $f \in L^p$	$a^{ij} \in C^{k,\gamma}$, $f \in C^{k-1,\gamma}$	a^{ij}, f analytic
$C^{1,\frac{1}{2}-\frac{n+1}{p}}$	$C^{k+[\gamma+\frac{1}{2}],\gamma+\frac{1}{2}-[\gamma+\frac{1}{2}]}$	analytic

- degenerate elliptic problems, e.g., fractional thin obstacle problem (Koch-R.-Shi '16).

Summary and Outlook

Summary:

- Linearization strategy: robust strategy for regularity; compactness by means of epiperimetric inequality.
- Higher regularity: Hodograph-Legendre transform.

Questions:

- More information on possible blow-ups at free boundary (outside of regular free boundary)?
- Rougher coefficients?
- Vectorial problems, e.g. Lamé?