Domain patterns in thin ferromagnetic films

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(joint work with C. Muratov, F. Nolte)

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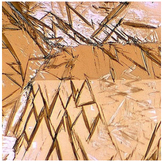
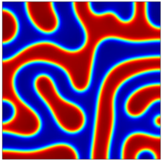
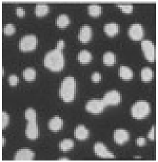
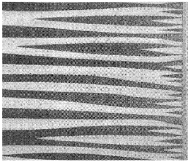
Jan 2018

Well deserved rest after the climb...

Models with perimeter + volume term

Domain formation for models with perimeter + volume term:

- Elasticity/Plasticity
- Diblock-Copolymers
- Raft formation in biomembranes (FHLZ '16)
- ...

Ohta-Kawasaki energy (diblock-copolymers)

$$E[u] = \int_Q \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx + \beta \int_Q \int_\Omega (u(x) - \overline{u}_\Omega) G(x,y)(u(y) - \overline{u}_\Omega) \, dxdy$$

for $u(x) \in (-1, 1)$ with prescribed volume fraction $\int_\Omega u \, dx = \lambda$

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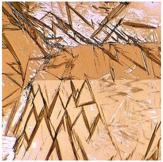
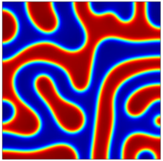
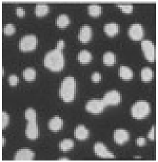
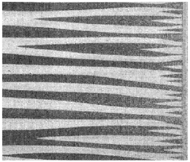

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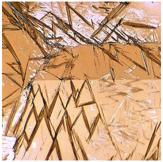
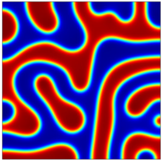
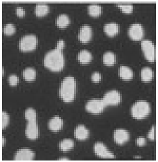
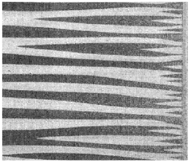

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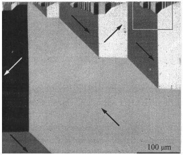
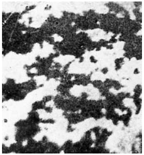
Continuum micromagnetic model

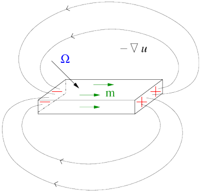
Images: Hubert & Schäfer, Magnetic Domains

$$\mathcal{E}[m] = \underbrace{\int_\Omega |\nabla m|^2 \, \mathrm{d}^3 x}_{\text{exchange}} + \underbrace{Q \int_\Omega (m_1^2 + m_2^2) \, \mathrm{d}^3 x}_{\text{anisotropy}} + \underbrace{\int_{\mathbb{R}^3} |h|^2 \, \mathrm{d}^3 x}_{\text{stray field}}$$

- Magnetization $m : \Omega \subset \mathbb{R}^3 \to \mathbb{S}^2$
- Stray field $h : \mathbb{R}^3 \to \mathbb{R}^3$ determined by

$$h = -\nabla \varphi, \quad -\Delta \varphi = -\nabla \cdot (m \chi_\Omega).$$

- DeSimone, Kohn, Müller, Otto, Serfaty, ...



Multiple length scales:

atoms — vortices & walls — magnetic domains — macro. description

"domain theory"

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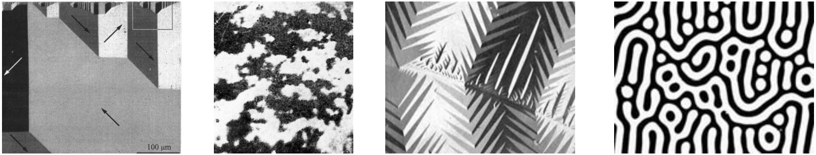

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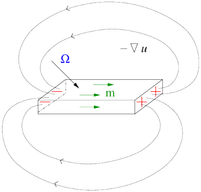

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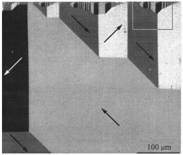

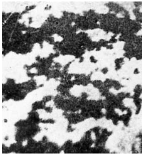

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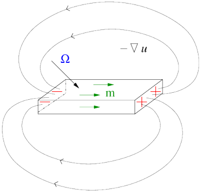

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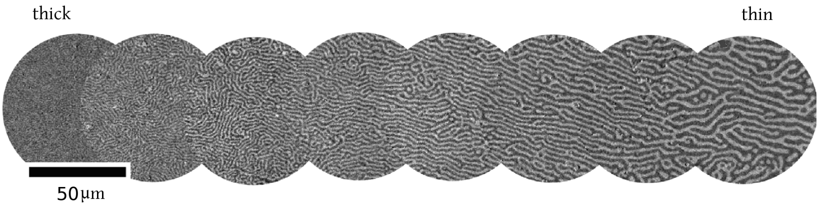
Experimental observations for thin ferromagnetic films



- Length scale of domains increases extremely as the thickness decreases, (Ibach et al. '95, . . .)
- Ansatz-based stripe patterns computations (Kaplan & Gehring '93) suggests that the typical width of domains scales like

$$\text{domain width} \quad \sim \quad s := \frac{1}{Q-1} \exp\left(\frac{2\pi\sqrt{Q-1}}{t}\right)$$

t = film thickness

Question: Rigorous justification of scaling law?

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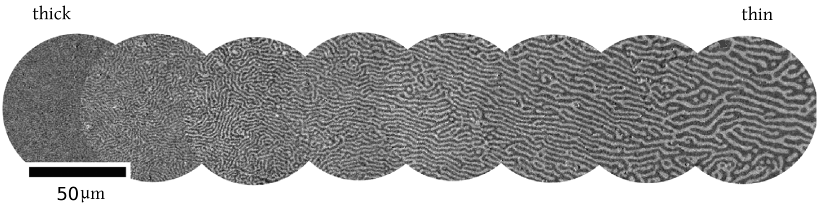

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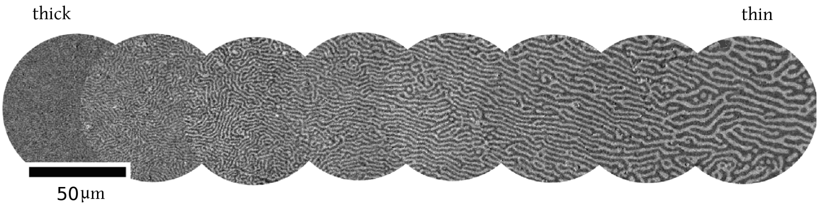

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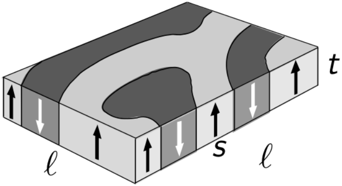
Setting

We consider

$$\mathcal{E}(m) = \ell_{ex}^2 \int_{[0,\ell)^2\times(0,t)} |\nabla m|^2 \, \mathrm{d}^3x + Q \int_{[0,\ell)^2\times(0,t)} (m_1^2 + m_2^2) \, \mathrm{d}^3x + \int_{[0,\ell)^2\times\mathbb{R}} |h|^2 \; \mathrm{d}^3x$$

Assumptions:

- high anisotropy materials ($Q > 1$)
- m is ℓ-periodic in both in–plane variables. Consider energy per periodicity cell



Goals:

- Identify scaling of minimal energy and domain size $s = s(Q, \ell, t)$.
- Identify critical scaling domain size $\ell = \ell(Q, t)$ where the phase transition between single- and multi-domain states occurs.

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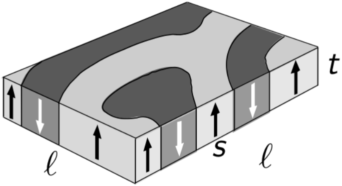

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Results – The subcritical regime

Introduce rescaled energy

$$E_{\varepsilon.\lambda}[m] := \frac{\mathcal{E}[m(\ell\cdot,\ell\cdot,t\cdot)] - \ell^2 t}{2\ell t\sqrt{Q-1}}$$

- $\varepsilon := \frac{1}{\ell\sqrt{Q-1}}$ (width of transition layer)
- $\lambda := \frac{t\ \ln\left(\ell\sqrt{Q-1}\right)}{4\sqrt{Q-1}}$ (rescaled thickness)

Theorem 1 (Subcritical regime $\lambda < \frac{\pi}{2}$)

1. **Γ–convergence:** *For $\lambda < \lambda_c := \frac{\pi}{2}$, we have as $\varepsilon \to 0$*

$$E_{\varepsilon,\lambda} \xrightarrow{\Gamma} \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_3| dx & \text{if } m \in BV(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty & \text{else.} \end{cases}$$

2. **Compactness:** *Every sequence m_ε in $H^1(\mathbb{T}^2;\mathbb{S}^2)$ with $\limsup_{\varepsilon\to 0} E_{\varepsilon,\lambda}[m_\varepsilon] < \infty$ converges in $L^1(\mathbb{T}^2)$ towards some in $m \in BV(\mathbb{T}^2; \{\pm e_3\})$ (up to subsequence).*

Related result on raft formation by Fonseca, Leoni, Hayrapetyan, Zwicknagl

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The supercritical regime $\lambda > \lambda_c$

Theorem 2 (Supercritical regime $\lambda > \frac{\pi}{2}$)

For $0 < \varepsilon < 1$ *and* $\lambda > \lambda_c = \frac{\pi}{2}$*, we have*

1. **Scaling of minimal energy:**

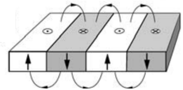
$$-C\frac{\lambda\varepsilon^{-\frac{\lambda-\lambda_c}{\lambda}}}{|\ln\varepsilon|} \leq \min_{m\in H^1(\mathbb{T}^2,\mathbb{S}^2)} E_{\varepsilon,\lambda} \leq -c\frac{\lambda\varepsilon^{-\frac{\lambda-\lambda_c}{\lambda}}}{|\ln\varepsilon|}. \qquad (1)$$

2. **Scaling of BV-norm:** *For any* $m \in H^1(\mathbb{T}^2;\mathbb{S}^2)$ *satisfying* (1)*, we have*

$$c\varepsilon^{-\frac{\lambda-\lambda_c}{\lambda}} \leq \int_{\mathbb{T}^2} |\nabla m_3|\,dx \leq C\varepsilon^{-\frac{\lambda-\lambda_c}{\lambda}}$$

Remark: The theorem confirms

- the optimal scaling of 1d Ansatz
- the expected wall distance for minimizing sequences
- Further results: Domain wall energy and nonlocal energy cancel in highest order



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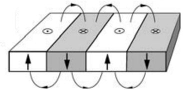


The critical regime $\lambda \approx \lambda_c$

Theorem 3 (critical regime)

Let $\lambda = \lambda_c$.

1. **Γ-convergence:** For $\varepsilon \to 0$, we have

$$F_{\varepsilon,\lambda_c} \quad \xrightarrow{\Gamma} \quad F_{*,\lambda_c}(m) = \begin{cases} 0, & \text{if } m \in L^1(\mathbb{T}^2; \{\pm e_3\}) \\ +\infty & \text{otherwise,} \end{cases}$$

2. **No compactness:** There is a sequence $(m_\varepsilon)_{\varepsilon>0}$ in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\lambda_c}[m_\varepsilon] \to 0 \qquad \text{and} \qquad (m_\varepsilon) \text{ is not precompact in } L^1.$$

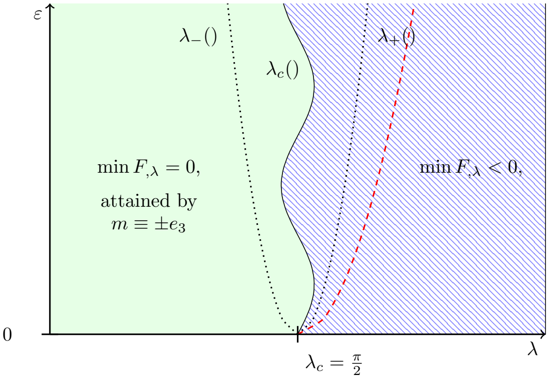
3. **Compactness upon rescaling:** Every sequence $(m_\varepsilon)_{\varepsilon>0}$ with $\limsup_{\varepsilon\to 0} |\ln(\varepsilon)| F_{\varepsilon,\lambda_c}(m_\varepsilon) < \infty$ is precompact in $L^1(\mathbb{T}^2)$ with limit in $BV(\mathbb{T}^2; \{\pm e_3\})$.

Phase diagram

Theorem 4 (Cross-over of global minimizers)

There is $0 < \beta_1 < 1 < \beta_2$ such that

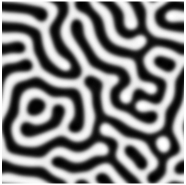
$$\min_{m \in H^1(\mathbb{T}^2, \mathbb{S}^2)} F_{\varepsilon,\lambda} \quad \begin{cases} = 0 & \text{for } \varepsilon > 0 \text{ and } \lambda \leq \lambda_c \left(1 - \frac{|\ln \beta_1|}{|\ln \varepsilon|}\right) \\ < 0 & \text{for } \varepsilon > 0 \text{ and } \lambda \geq \lambda_c \left(1 - \frac{|\ln \beta_2|}{|\ln \varepsilon|}\right) \end{cases}$$



Numerical Experiments

Scaling law numerically confirmed for similar energy by Condette '11.

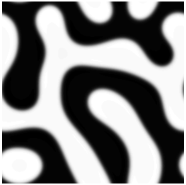
$\lambda = 28\lambda_*$



$\lambda = 17.5\lambda_*$

$\lambda = 7\lambda_*$

$\lambda = 4.4\lambda_*$



$\lambda = 2\lambda_*$

$\lambda = \lambda_*$

Images show apparent steady states of the gradient flow
Interfacial cost is approximately doubled from one image to the next.

Strategy for the proof

Reduction to two-dimensional problem

1. Exchange energy controls oscillation in thickness direction $m \neq m(x_3)$ in Ω
2. Suitable approximation for for stray field energy

$$\int_{[0,\ell)^2\times\mathbb{R}} |h|^2 = \frac{t}{\ell^2} \sum_{k\in\frac{2\pi}{\ell}\mathbb{Z}^2} \sigma(t|k|)|\widehat{m}_3|^2 + \frac{t}{\ell^2} \sum_{k\in\frac{2\pi}{\ell}\mathbb{Z}^2} (1-\sigma(t|k|))\frac{|k\cdot\widehat{m}'|^2}{|k|^2}$$

$$\approx t\int_{[0,\ell)^2} m_3^2 - \frac{t^2}{2}\int_{[0,\ell)^2} |\nabla^{1/2} m_3|^2.$$

In leading order we have

$$E_{\varepsilon,\lambda}(m) \approx F_{\varepsilon,\lambda}(m) := \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}\left(1-m_3^2\right)\right) - \frac{\lambda}{|\ln\varepsilon|}\int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2.$$

Estimate of two-dimensional energy $F_{\varepsilon,\lambda}$

- Upper bound by suitable constructions
- Lower bound is based on interpolation inequality

  The case $\lambda = 0$ is a classical result (Anzelloti, Baldo, Visintin '90)

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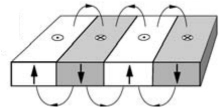
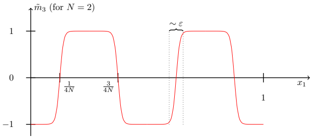
  The case $\lambda = 0$ is a classical result (Anzelloti, Baldo, Visintin '90)

Heuristics: One-dimensional configurations

We consider the energy

$$F_{\varepsilon,\lambda}(m) := \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} \left(1 - m_3^2\right) \right) - \frac{\lambda}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2.$$





A one-dimensional ansatz with N walls of distance $\frac{1}{N}$ yields $(\lambda_c = \frac{\pi}{2})$

$$F_{\varepsilon,\lambda}(m^*) \leq 2N\Big(\underbrace{1}_{\text{domain walls}} - \underbrace{\frac{\lambda}{|\ln \varepsilon|} \frac{2}{\pi} \ln |\frac{2\varepsilon N}{c_2}|}_{\text{interaction between neighboring domains}} \Big)$$

Crossover at $\lambda = \lambda_c := \frac{\pi}{2}$. Optimization in N yields

- For $\lambda \leq \lambda_c$, we get $\quad N = 0,\ F_\varepsilon = 0$
- For $\lambda > \lambda_c$, we get $\quad F_{\varepsilon,\lambda}(m^*) \lesssim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\ln \varepsilon|}, \quad N \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}.$

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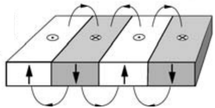

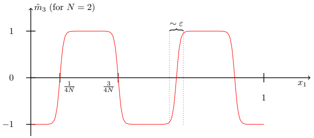

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domain walls — interaction between neighboring domains

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Key interpolation estimate

For the lower bound of both Theorem 1 & 2, we need an upper bound for $\frac{1}{|\ln \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2$ with optimal constant. It is sufficient to show

Lemma 1

There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $f \in C^\infty(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \leq \frac{2}{\pi} \ln\left(\frac{c}{\varepsilon} \min\left\{\frac{\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}x}, 1\right\}\right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x + \frac{\varepsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2$$

- Quantifies critically failing inequality:

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \, \mathrm{d}^2 x \not\lesssim C \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x$$

- Sharp leading order constant!
- Incorporates three length scales: sample size $(= 1)$, transition layer $(= \varepsilon)$, domain size $(s = \frac{1}{N} = \|f\|_\infty / \|f\|_{BV})$

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Sketch of Proof of Theorem 1 (1/2)

Claim: Let $\lambda < \lambda_c$. For any $m_\varepsilon \to m$ in L^1, we have $\liminf F_{\varepsilon,\lambda}(m_\varepsilon) \geq F_{*,\lambda}(m)$

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Together with the Modica-Mortola-type estimate

$$\int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon}(1 - m_3^2) = \int_{\mathbb{T}^2} \frac{\varepsilon}{2} \frac{|\nabla m_3|^2}{(1 - m_3^2)} + \frac{1}{2\varepsilon}(1 - m_3^2) \geq \int_{\mathbb{T}^2} |\nabla m_3|$$

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Claim: For $m \in BV$ there is $m_\varepsilon \to m$ in L^1 with $\limsup_{\varepsilon \to 0} F_{\varepsilon,\lambda}(m_\varepsilon) \leq F_{*,\lambda}(m) = (1 - \frac{\lambda}{\lambda_c})\|f\|_{BV}$

The 1d case:

- Define m_3 by gluing together optimal profiles $\xi_\varepsilon(x) = \tanh(x/\varepsilon)$. Leading order contribution per wall of the nonlocal part

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The rotation can be chosen in the wall plane.

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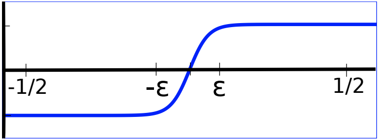

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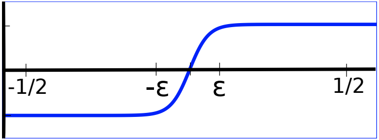

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Proof of Theorem 3 and 4 — sketch

We want to show

(i) $F_{\lambda_c,\varepsilon} \xrightarrow{\Gamma} 0$

(ii) Lack of compactness for $\lambda = \lambda_c$

(iii) compactness upon rescaling

(iv) Estimates on cross-over for $\varepsilon > 0$ and $\lambda \approx \lambda_c$

ad (i): Review proof of lower and upper bound

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Proof of Lemma 1

We want to show that:

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \leq \frac{1}{\lambda_c} \ln\left(\frac{c}{\varepsilon} \min\left\{\frac{4\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| \,\mathrm{d}x}, 1\right\}\right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \,\mathrm{d}^2x + \varepsilon \int_{\mathbb{T}^2} |\nabla f|^2$$

We use a real space representation of the $H^{1/2}$-Norm,

$$4\pi \int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 = \int_{\mathbb{R}^2} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} \,\mathrm{d}^2x \,\mathrm{d}^2z.$$

For $0 < \varepsilon < R$, we consider the decomposition

$$\begin{aligned} 4\pi \int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \,\mathrm{d}^2x &= \int_{B_\varepsilon} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} \,\mathrm{d}^2x \,\mathrm{d}^2z \\ &\quad + \int_{B_R \setminus B_\varepsilon} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} \,\mathrm{d}^2x \,\mathrm{d}^2z \\ &\quad + \int_{\mathbb{R}^2 \setminus B_R} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} \,\mathrm{d}^2x \,\mathrm{d}^2z \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

An elementary estimate yields

$$I_1 = \int_{B_\varepsilon} \int_{\mathbb{T}^2} |f(x+z) - f(x)|^2 \,\mathrm{d}^2x \frac{1}{|z|^3} \,\mathrm{d}^2z \leq 2\pi\varepsilon \int_{\mathbb{T}^2} |\nabla f|^2 \,\mathrm{d}^2x.$$

Proof of Lemma 1

We want to show that:

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \leq \frac{1}{\lambda_c} \ln\left(\frac{c}{\varepsilon} \min\left\{\frac{4\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}x}, 1\right\}\right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x + \varepsilon \int_{\mathbb{T}^2} |\nabla f|^2$$

We use a real space representation of the $H^{1/2}$-Norm,

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For the estimate of I_2, we use the calculation in polar coordinates,

$$\int_{B_R \setminus B_\varepsilon} \frac{|\nabla f(x) \cdot z|}{|z|^3} \, \mathrm{d}^2 z = \int_\varepsilon^R \int_0^{2\pi} \frac{|\nabla f(x)| \, |\cos \varphi|}{\rho} \, \mathrm{d}\varphi \, \mathrm{d}\rho = 4 \ln \left(\frac{R}{\varepsilon} \right) |\nabla f(x)|.$$

This yields

$$I_2 \leq 2\|f\|_\infty \int_{\mathbb{T}^2} \int_{B_R \setminus B_\varepsilon} \frac{|\nabla f(x) \cdot z|}{|z|^3} \, \mathrm{d}^2 x \, \mathrm{d}^2 z \leq 8 \ln \left(\frac{R}{\varepsilon} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x.$$

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$$\int_{\mathbb{T}^2} |f(x+z) - f(x)| \, \mathrm{d}^2 x \leq \min \left\{ 2\|f\|_\infty, \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x \right\} \quad \text{for all } z \in \mathbb{R}^2.$$

With this inequality, we may estimate I_3 by

$$I_3 \leq 2\|f\|_{L^\infty} \int_{\mathbb{R}^2 \setminus B_R} \int_{\mathbb{T}^2} |f(x+z) - f(x)| \, \mathrm{d}^2 x \frac{1}{|z|^3} \, \mathrm{d}^2 z \leq \frac{2\|f\|_\infty}{R} \min \left\{ 2\|f\|_\infty, \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x \right\}.$$

In total, we obtain

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \, \mathrm{d}^2 x \leq \frac{\varepsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \, \mathrm{d}^2 x + \left(\frac{2}{\pi} \ln \left(\frac{R}{\varepsilon} \right) + \frac{1}{R} \min \left\{ \frac{4\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}x}, 1 \right\} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, \mathrm{d}^2 x.$$

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Thank you!