Approximation of functions with small jump sets and existence of strong minimizers of the Griffith's energy in dimension n

Flaviana Iurlano

Laboratoire Jacques-Louis Lions
Sorbonne Université (Paris 6)

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Setting of the problem

We aim to study the existence of minimizers to the following ("strong") problem

$$\min_{(\Gamma,u)} \int_{\Omega\setminus\Gamma} |\nabla^{sym} u|^2 dx + \mathcal{H}^{n-1}(\Gamma),$$

where $\Gamma \subset \Omega$ closed, $u \in C^1(\Omega \setminus \Gamma, \mathbb{R}^n)$, and some boundary or volume conditions are assumed.

Mechanical interest:

minimizers $\rightsquigarrow$ equilibria of the static Griffith's fracture energy
$\Gamma \rightsquigarrow$ crack,
$u \rightsquigarrow$ elastic displacement out of the crack

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Mathematical interest: lack of semicontinuity of the term $\mathcal{H}^{n-1}(\Gamma)$ with respect to the Hausdorff distance (the direct method of the calculus of variations doesn't work)

Alternative strategy

Alternative strategy:

- proving existence of minimizers for a suitable weak problem
- showing that such weak minimizers are in fact more regular
  ($\rightsquigarrow$ then strong minimizers)

Weak problem:

$$\min_u \int_\Omega |\nabla^{sym} u|^2 dx + \mathcal{H}^{n-1}(J_u),$$

where u belongs to a suitable space of "discontinuous" functions ($u \in SBD$) and J_u is its jump set (+ boundary or volume conditions).

The space SBD

A function $u \in L^1(\Omega, \mathbb{R}^n)$ is a Special function of Bounded Deformation if

$$Eu := (Du + Du^t)/2 \text{ is a bounded Radon measure and}$$

$$Eu = \underbrace{\nabla^{sym} u \; \mathcal{L}^n \lfloor (\Omega \setminus J_u)}_{\text{absolutely cont.}} + \underbrace{[u] \odot \nu_u \; \mathcal{H}^{n-1} \lfloor J_u}_{\text{singular}}$$

→ J_u jump set of u, it is $\mathcal{H}^{n-1}$-rectifiable (in general not closed)

→ $\nabla^{sym} u \in L^1(\Omega, \mathbb{R}^{n\times n})$ (in general $u \notin C^1(\Omega \setminus J_u, \mathbb{R}^n)$)

→ $[u]$ amplitude of the jump, ν_u normal vector to J_u

Existence and regularity for the weak problem

$$\exists \quad \min_{u \in (G)SBD} \int_\Omega |\nabla^{sym} u|^2 dx + \mathcal{H}^{n-1}(J_u)$$

under:

- a volume condition and a uniform bound [Bellettini–Coscia–Dal Maso '98]
- a volume condition [Dal Maso '13]
- Dirichlet conditions and $n = 2$ [Friedrich–Solombrino '18]
- Dirichlet conditions and $n > 2$ [Chambolle–Crismale '18]

Regularity issue: if J_u is closed and $u \in C^1(\Omega \setminus J_u, \mathbb{R}^n)$, then (J_u, u) is a competitor for the strong problem $\rightsquigarrow$ then strong minimizer!

Bibliography

$$\exists \quad \min_{(\Gamma,u)} \int_{\Omega\setminus\Gamma} |f(\nabla u)|^2 dx + \mathcal{H}^{n-1}(\Gamma)$$

(by proving that J_u is essentially closed)

- if $u : \Omega \to \mathbb{R}$ and $f(\nabla u) = |\nabla u|^2$ (scalar case)
  [De Giorgi–Carriero–Leaci '89, Dal Maso–Morel–Solimini '92, Solimini '97, Maddalena–Solimini '01]
- if $u : \Omega \to \mathbb{R}^n$ and $f(\nabla u) \sim |\nabla u|^p$ (full gradient case)
  [Carriero–Leaci '91, Fonseca–Fusco '97]
- if $u : \Omega \to \mathbb{R}^n$ and $f(\nabla u) \sim |\nabla^{sym} u|^p$ and $n = 2$
  [Conti–Focardi–F.I. '15]
- if $u : \Omega \to \mathbb{R}^n$ and $f(\nabla u) \sim |\nabla^{sym} u|^2$ and $n > 2$ (or $p \neq 2$, $n = 3$...)
  [Chambolle–Conti–F.I. '17]

Bibliography

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Crucial point

Given a sequence $u_k \in SBD$, we would like to say

$$\|\nabla^{sym} u_k\|_2 \le c, \qquad \mathcal{H}^{n-1}(J_{u_k}) \to 0 \qquad \text{(small jumps)}$$

$$\Downarrow ?$$

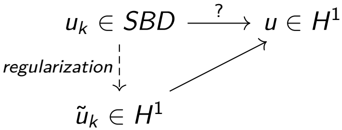
$$u_{k_j} - a_{k_j} \ \to\ u \in H^1 \qquad \text{(without jump)}$$

for some a_{k_j} skew-symmetric affine functions.

Note: true if $u_k \in H^1$ by Korn's inequality

Crucial point: idea

Idea: replace u_k by a suitable Sobolev regularization $\tilde{u}_k$ and take the limit of $\tilde{u}_k$ (up to an affine skew-symmetric function)



Difficulties: differently from the gradient case, in the symmetric gradient case we have no chain rule, roughly

$$\nabla^{sym}(f(u)) \neq \nabla^{sym} f \nabla^{sym} u,$$

hence

- no truncations
- no coarea formula

The regularization result

Theorem [Chambolle–Conti–F.I. '17]

$\exists\ \eta, c > 0$ such that if $u \in SBD(Q)$ with

- $\|\nabla^{sym} u\|_2 \leq c$ (bounded)
- $\mathcal{H}^{n-1}(J_u) < \eta^n$ (small)

then, set $\delta := \mathcal{H}^{n-1}(J_u)^{1/n}$, there exist

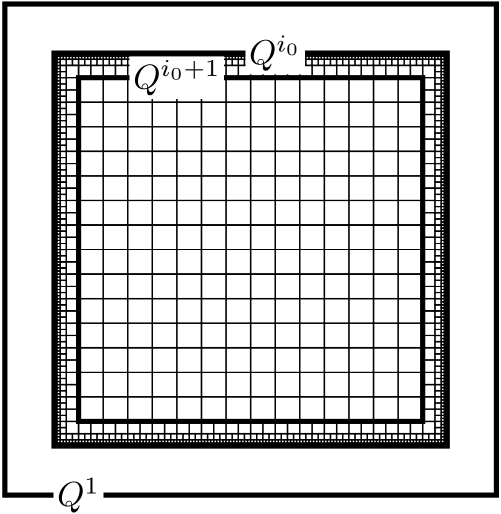
$$\tilde{u} \in C^\infty(Q_{1-\sqrt{\delta}}) \cap SBD(Q) \quad (\text{"regular"})$$

and an exceptional set $\tilde{\omega} \subset Q$, $|\tilde{\omega}| < c\sqrt{\delta}$, such that (close to u)

- $\|\nabla^{sym} \tilde{u}\|_{L^2} \leq \|\nabla^{sym} u\|_{L^2} + c\sqrt{\delta}$
- $\mathcal{H}^{n-1}(J_{\tilde{u}} \setminus J_u) < c\sqrt{\delta}$
- $\int_{Q \setminus \tilde{\omega}} |u - \tilde{u}|^2 dx \leq c\sqrt{\delta}$.

Idea of the proof

1. Covering most of Q with cubes of side $\delta := (\mathcal{H}^{n-1}(J_u))^{1/n} < \eta$, then with dyadic cubes



2. Identifying the cubes of the covering that still contain a small amount of jump

By assumption u has a small jump in the whole Q

$$\delta^n := \mathcal{H}^{n-1}(J_u) < \eta^n.$$

A cube q of the covering is said to be good if

$$\mathcal{H}^{n-1}(J_u \cap q) < \eta \delta_q^{n-1}$$

where δ_q denotes the side of q.

Note: The cubes of side δ (the biggest cubes) are all good.

3. Performing two different constructions in good and bad cubes

Bad cubes: u is left as it is

Good cubes: we will construct a C^∞ regularization in each cube and then we will take a partition of unity on the union of the good cubes

Note: since the cubes of side δ are all good, $\tilde{u}$ will be C^∞ on a big compact set of Q! ($\sim Q_{1-\sqrt{\delta}}$)

How to construct such regularization in each good q?

4. Construction in a good cube

If $J_u \cap q$ is small, that is

$$\mathcal{H}^{n-1}(J_u \cap q) < \eta \delta_q^{n-1},$$

a Korn–Poincaré-type inequality holds [Chambolle–Conti–Francfort '15]:

$$\exists\ a_q \text{ affine skew, } \omega_q \text{ with } \ |\omega_q| < \delta_q \mathcal{H}^{n-1}(J_u \cap q),$$

such that

$$\int_{q\setminus\omega_q} |u - a_q|^2 dx \le c\delta_q^2 \underbrace{\int_q |\nabla^{sym} u|^2 dx}_{\text{abs. cont. part only}}.$$

Note: ω_q has not finite perimeter $\rightsquigarrow$ no similar controls of $\nabla u - \nabla a_q$ by means of $\nabla^{sym} u$ (but if $n = 2$...)

4. Construction in a good cube

But, if one defines

$$\tilde{u} := (1_{q\setminus\omega_q} u + 1_{\omega_q} a_q) * \rho_q \in C^\infty(q, \mathbb{R}^n),$$

one can prove that

$$\int_q |\nabla^{sym}\tilde{u} - \nabla^{sym} u * \rho_q|^2 \leq c\delta^r,$$

with $r = r(n)$, then recovering the control of $\nabla^{sym}\tilde{u}$ with $\nabla^{sym} u$.

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Thank you for your attention!