# Asymptotic stability of the gradient flow of nonlocal energies

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Topics in the Calculus of Variations:

Recent Advances and New Trends

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$$P(E) + \text{ Volume term (nonlocal)}$$

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Mullins (1957,1958,1960), Davì-Gurtin (1990)

Evolution of a two phase interface controlled by mass diffusion within the surface

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- Surface diffusion is volume preserving

$$\frac{d}{d_t}|F_t| = \int_{\partial F_t} V_t \, d\mathcal{H}^{n-1} = \int_{\partial F_t} \Delta_{\Gamma_t} H_t \, d\mathcal{H}^{n-1} = 0$$

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- Surface diffusion (and mean curvature flow) reduce the perimeter

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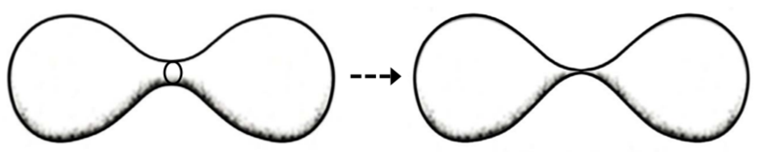
- Surface diffusion does not preserve convexity

  Mean curvature flow preserves convexity and shrinks a convex set to a point in finite time, so that by rescaling the evolving sets to the original volume, they converge to a ball (Huisken, 1984)

Singularities may appear in finite time even in 2-D (Giga-Ito, 1998)



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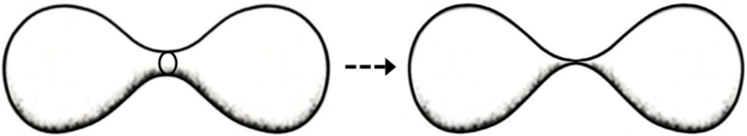

- Existence for small times (Escher-Mayer-Simonett, 1998)

$$F_o \in C^{2,\alpha} \Longrightarrow h \in C^0([0,T); C^{2,\alpha}(\Gamma_o)) \cap C^\infty((0,T); C^\infty(\Gamma_o))$$

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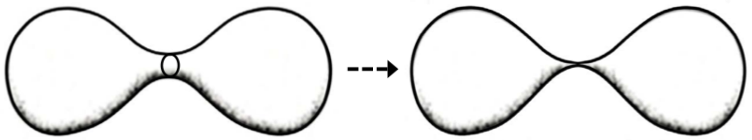

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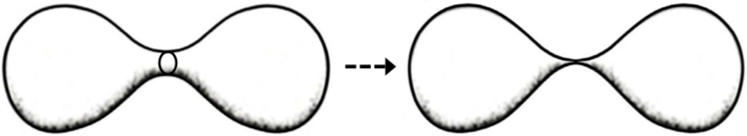

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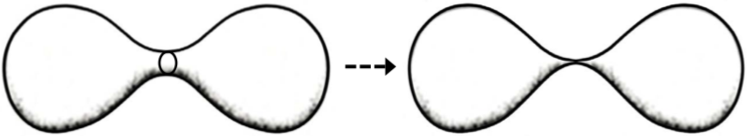

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- $n = 3$

  $F_o$ close to an infinite cylinder (LeCrone, Simonett, 2016)

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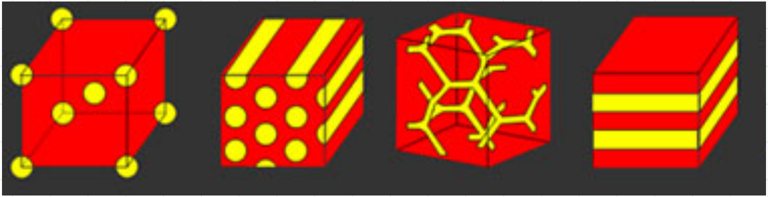

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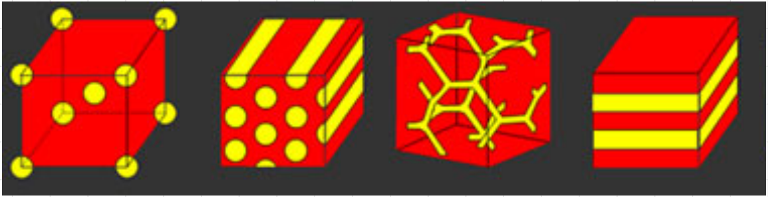

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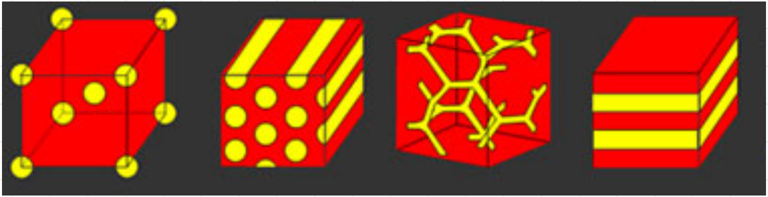

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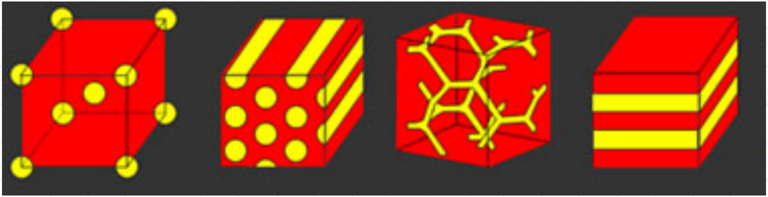

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Recall that for a critical point $F$ and for $\varphi \in H^1(\partial F)$ we have

$$\partial^2 J(F)[\varphi] = \int_{\partial F} \left( |\nabla \varphi|^2 - |B_{\partial F}|^2 \varphi^2 \right) d\mathcal{H}^{n-1}$$

$$\widetilde{H}^1(\partial F) := \left\{ \varphi \in H^1(\partial F) : \underbrace{\int_{\partial F} \varphi = 0}_{\text{volume pres.}}, \quad \underbrace{\int_{\partial F} \varphi \nu_F = 0}_{\text{translation inv.}} \right\}$$

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Then we say that a $C^2$ critical point $F$ is strictly stable if

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in \widetilde{H}^1(\partial F) \setminus \{0\}$$

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## Theorem (Acerbi-F.-Morini 2013)

*Let $F$ be a strictly stable $C^2$ critical configuration.*

*Then, $F$ is a strict local minimizer, i.e., there exists $\delta, C_0 > 0$, s.t. if $\min_\tau |F \Delta (\tau + G)| < \delta$, then*

$$J(G) \geq J(F) + C_0 \min_\tau |F \Delta (\tau + G)|^2$$

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The local minimality w.r.t. $L^\infty$ perturbations (B.White, 1994)
or w.r.t. $L^1$ perturbations ($n \leq 7$, Morgan-Ros, 2010)
In both cases there was no quantitative estimate

## Theorem (Acerbi, F., Julin, Morini, JDG to appear)

*Let $G \subset \mathbb{T}^3$ be a smooth strictly stable critical set. For every $M > 0$ there exists $\delta > 0$ s.t.:*

*If* $\partial F_o = \{x + h_o(x)\nu_G : \ x \in \partial G, \ \|h_o\|_{H^3(\partial G)} \le M\}$,

$$|F_o| = |G|\,, \qquad |F_o \Delta G| \le \delta\,, \qquad \text{and} \qquad \int_{\partial F_o} |\nabla H_{\partial F_o}|^2 \, d\mathcal{H}^2 \le \delta\,,$$

*then the unique classical solution $(F_t)_t$ to the surface diffusion flow with initial datum $F_o$ exists for all $t > 0$.*

*Moreover, $F_t \to G + \sigma$ in $H^3$ as $t \to +\infty$, for some $\sigma \in \mathbb{R}^3$.*

*The convergence is exponentially fast, i.e., there exist $\eta$, $c_G > 0$ such that for all $t > 0$, writing*

$$\partial F_t = \{x + \psi_{\sigma,t}(x)\nu_{G+\sigma}(x) : \ x \in \partial G + \sigma\}\,,$$

*we have*

$$\|\psi_{\sigma,t}\|_{H^3(\partial G+\sigma)} \le \eta e^{-c_G t}\,.$$

*Both $|\sigma|$ and $\eta$ vanish as $\delta \to 0^+$.*

# Idea of the proof

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t} |\nabla_\tau H_t|^2\, dx\right) = -\,\partial^2 J(F_t)\,[\Delta_\tau H_t] - \int_{\partial F_t} B_t\,[\nabla_\tau H_t]\,\Delta_\tau H_t\, d\mathcal{H}^2 + \frac{1}{2}\int_{\partial F_t} H_t|\nabla_\tau H_t|^2 \Delta_\tau H_t\, d\mathcal{H}^2\,,$$

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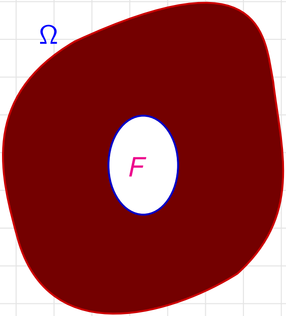
$$\int_{\partial F_t} |\nabla_\tau H_t|^2\, d\mathcal{H}^2 \leq \mathrm{e}^{-c_1 t}\int_{\partial F_0} |\nabla_\tau H_{E_0}|^2\, d\mathcal{H}^2 = C_0 \mathrm{e}^{-c_1 t}$$

# Evolution of material voids

Material void inside a stressed elastic material
(Siegel-Miksis-Voorhees 2004)



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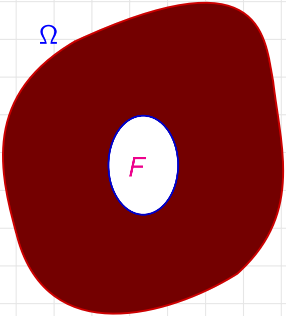


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$\Omega \setminus F =$ the region occupied by the material

$F =$ the void

$u_F : \Omega \setminus F \mapsto \mathbb{R}^2 =$ the elastic equilibrium

$$u_F = \operatorname{argmin}\left\{ \int_{\Omega \setminus F} W(E(u))\, dx : \; u = u_o \text{ on } \partial\Omega \right\}$$

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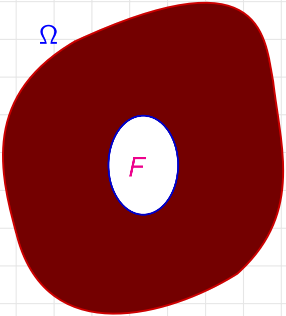


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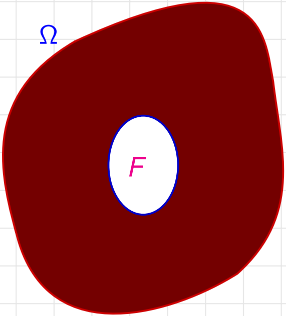


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Note $\quad u_o = 0 \implies J(F) = \int_{\partial F} \varphi(\nu_F)$

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Existence and regularity in 2D (Fonseca-F-Leoni-Millot, 2011)

# Morphology evolution: surface diffusion

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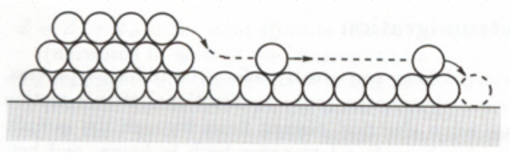

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Einstein-Nernst law: surface flux of atoms $\propto \nabla_{\Gamma_t}\mu$

$\mu$= chemical potential $\quad\rightsquigarrow\quad V_t = \kappa\,\Delta_{\Gamma_t}\mu$

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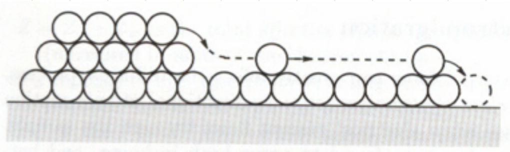

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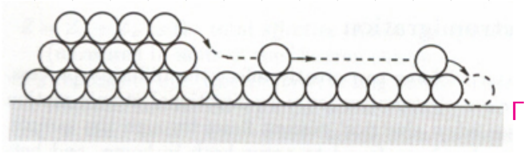
$\mu$= first variation of energy = $\mathrm{div}_{\Gamma_t} \nabla \varphi(\nu_t) - W(E(u_t)) + \lambda$

# Morphology evolution: surface diffusion

$$J(F) = \int_{\Omega \setminus F} W(E(u_F))\,dx + \int_{\partial F} \varphi(F)\,d\mathcal{H}^{n-1}$$



$\Gamma_t = \partial F_t$

Einstein-Nernst law: surface flux of atoms $\propto \nabla_{\Gamma_t}\mu$

$\mu$= chemical potential $\rightsquigarrow$ $V_t = \kappa\,\Delta_{\Gamma_t}\mu$

$\mu$= first variation of energy = $\mathrm{div}_{\Gamma_t}\nabla\varphi(\nu_t) - W(E(u_t)) + \lambda$

$\mathrm{div}_{\Gamma_t}\nabla\varphi(\nu_t) := H_{\varphi,t}$ = anisotropic curvature

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If $n = 2$, then

$$H_{\varphi,t} = g(\nu_t)k_t$$

where

$k_t =$ curvature of $\partial F_t$, $\quad g(\nu) = \langle D^2\varphi(\nu)\tau, \tau\rangle$ for all $\nu, \tau \in \mathbb{S}^1$, $\nu \perp \tau$

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The equation becomes

$$V_t = \partial_{\sigma\sigma}(g(\nu_t)k_t - W(E(u_t)))$$

## Theorem (F.-Julin-Morini, 2017)

*Let* $G \subset\subset \Omega \subset\subset \mathbb{R}^2$ *smooth. For every* $M > 0$ *there exist* $\delta > 0,\ T > 0$ *s.t. if*

$$\partial F_o = \{x + h_o(x)\nu_G : \ x \in \partial G,\ \|h_o\|_{H^3(\partial G)} \leq M\}, \qquad |G \Delta F_o| \leq \delta,$$

*then there exists a unique classical solution classical solution* $(F_t)_t$, $t \in (0, T)$. *More precisely*

$$\partial F_t = \{x + h(x,t)\nu_G(x) : \ x \in \partial G\}$$

*where for every* $\alpha \in (0, 1/2)$

$$h \in C([0, T]; C^{2,\alpha}(\partial G)) \cap C^\infty((0, T); C^\infty(\partial G))$$

# Long time existence

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*Let $G \subset\subset \Omega$ be a smooth strictly stable critical point and let $M > 0$.*

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$$|F_o \Delta G| < \delta, \qquad \int_{\partial F_o} \left|\partial_\sigma \left(g(\nu_{F_o})k_{F_o} - W(E(u_{F_o}))\right)\right|^2 d\mathcal{H}^1 < \delta,$$

*Then the unique solution $(F_t)_{t>0}$ of the flow with initial datum $F_o$ is defined for all times $t > 0$.*

*Moreover $F_t \to G$ $H^3$-exponentially fast.*

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But we can say more. . . . . .

Denote by $\Gamma_1, \ldots, \Gamma_m$ the connected components of $\partial F$

and by $\mathcal{O}_1, \ldots \mathcal{O}_m$ the open sets enclosed by the $\Gamma_i$

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$F$ is stationary if

$$g(\nu_F)k_F - W(E(u_F)) = \kappa_i \qquad \text{on } \Gamma_i, \ i = 1, \ldots, m$$

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then $\qquad F_t \to F_\infty \qquad$ in $H^3$

where $F_\infty$ is the only stationary point $H^3$-close to $G$ s.t.

$$|\mathcal{O}_{i,\infty}| = |\mathcal{O}_{i,o}| \qquad \forall i = 1, \ldots, m$$

THANK YOU FOR YOUR ATTENTION!