Optimal regularity and structure of the free boundary for minimizers in cohesive zone models

Joint work with Luis Caffarelli and Alessio Figalli

Filippo Cagnetti - University of Sussex - Brighton, UK

Topics in the Calculus of Variations:
Recent Advances and New Trends,
Banff, 24 May 2018

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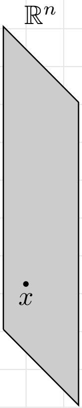

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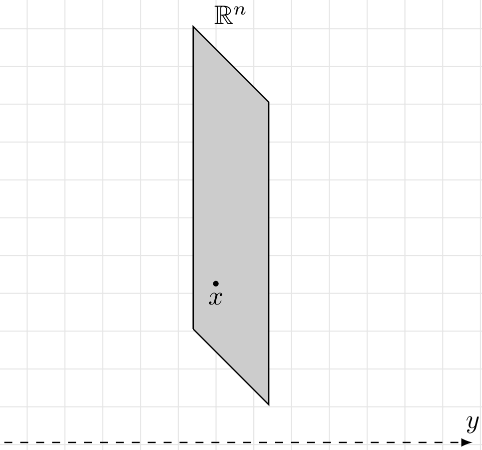

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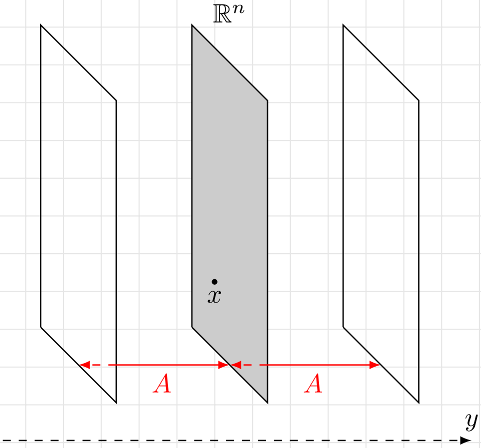

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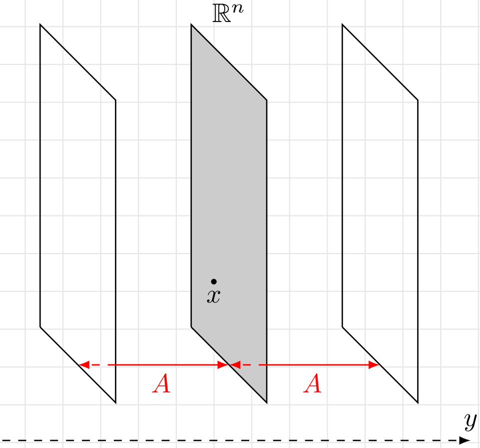

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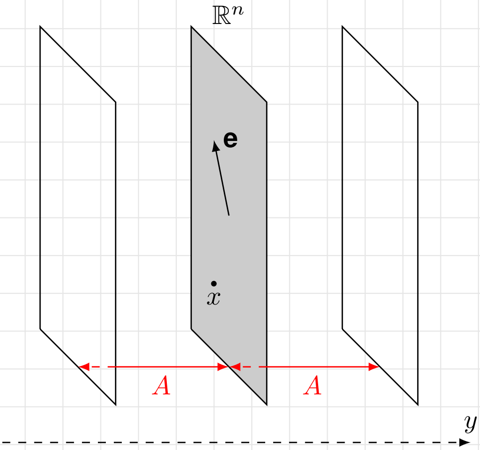

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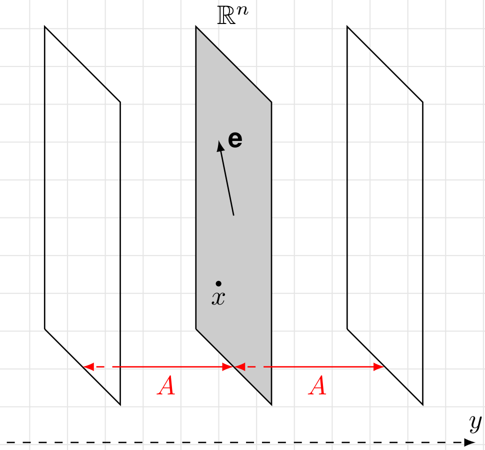

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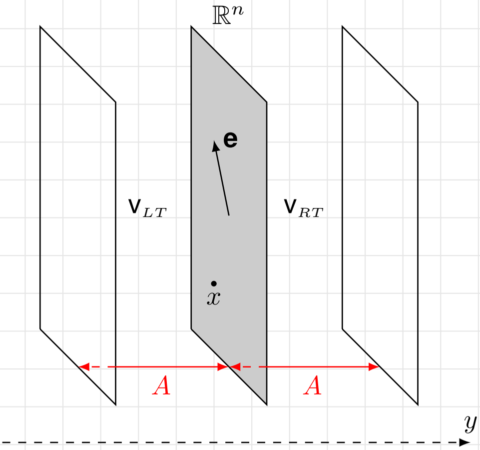

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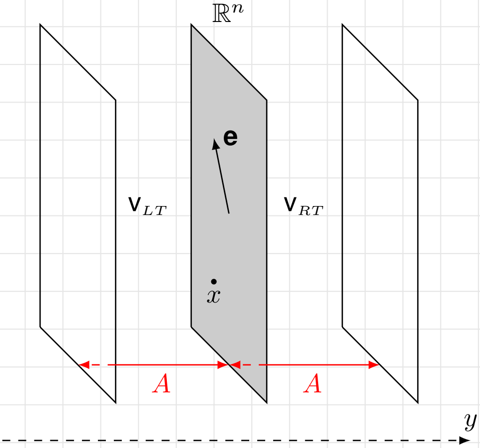

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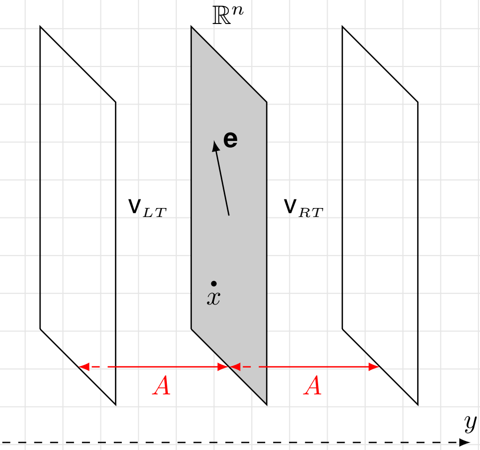

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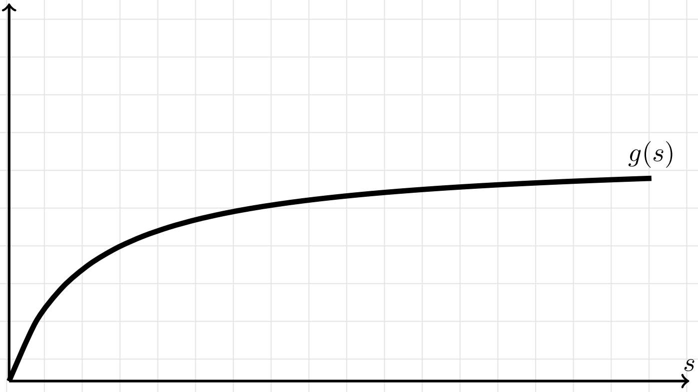

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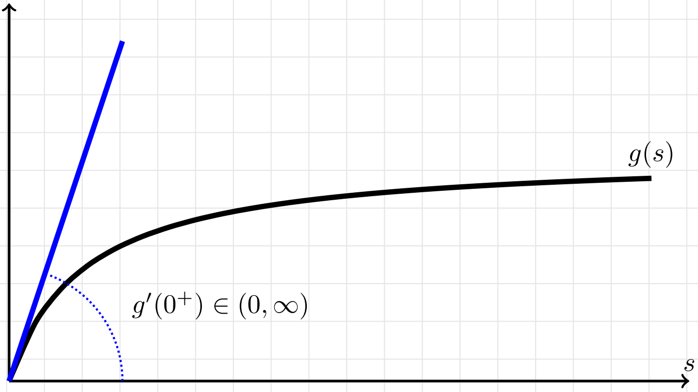

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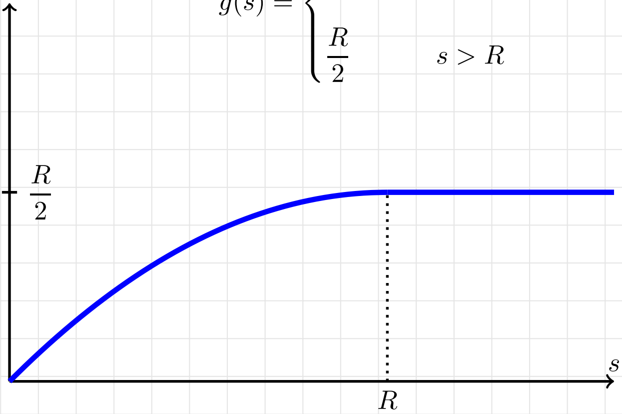

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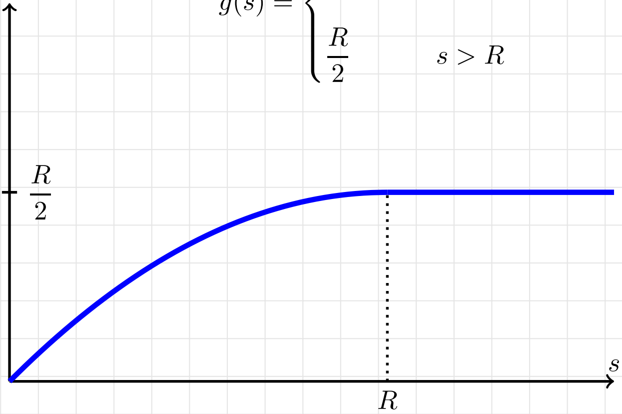

Example from ***C., Math. Models Methods Appl. Sci. (2008)***

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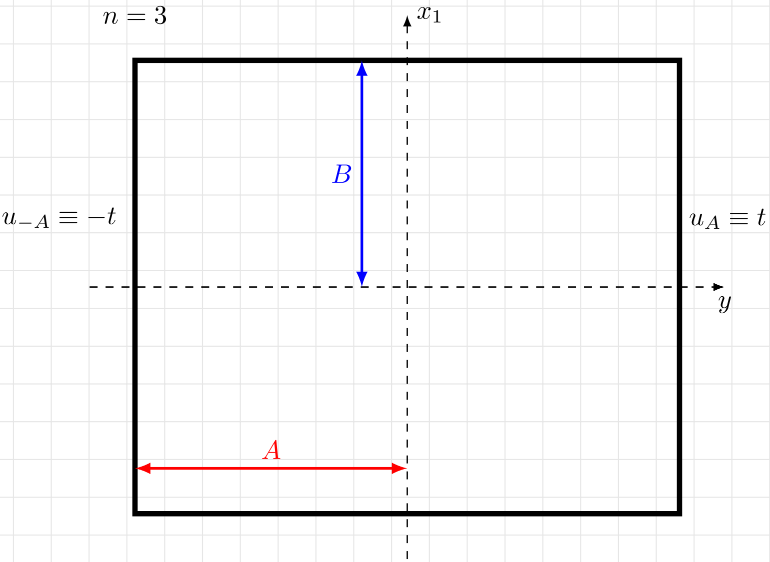

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$$u_3(t) := \begin{cases} t & y > 0 \\ -t & y < 0 \end{cases}$$

Energy graph for $A > R/2$

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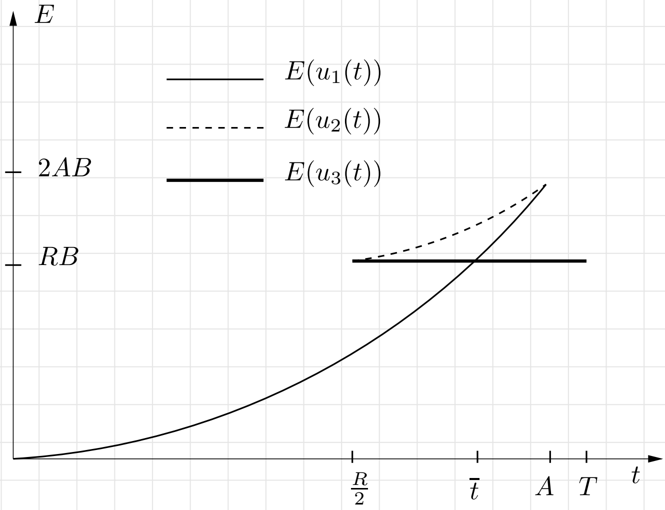

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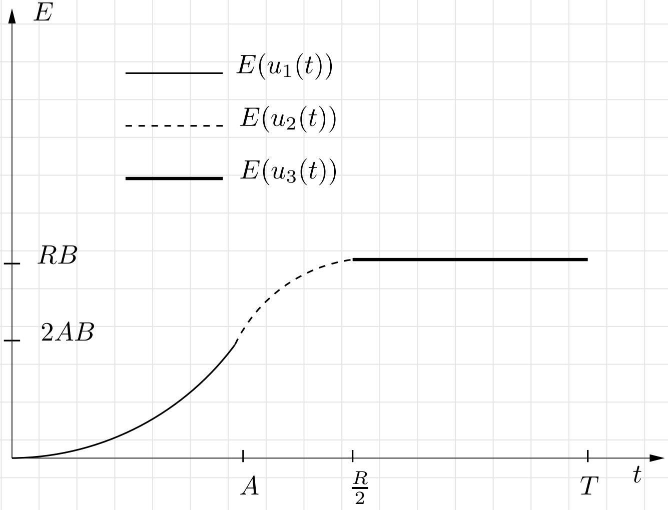

Uniqueness

Lemma

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Then, there exists a unique solution u*. In particular, there is a unique critical point of the energy, that coincides with the global minimizer.*

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for every $(x,y) \in \mathbb{R}^n \times [0,A]$, *and* $h \in \mathbb{R}^n$.

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Sketch of the proof

Suppose, by contradiction, that

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- Then, if $x \in \{u < 0\}$ and $x \to 0$ we have $|\nabla_x u(x,0)| \to \infty$

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$$v(x,y) := \begin{cases} u(x,y) - g'(0^+)y & \text{for every } (x,y) \in \mathbb{R}^n \times (0,A), \\ v(x,-y) & \text{for every } (x,y) \in \mathbb{R}^n \times (-A,0). \end{cases}$$

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NOTE: this is a “perturbation” of Signorini Problem:

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THANK YOU!

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Now send $r \to 0^+$ and use

Athanasopoulos-Caffarelli-Salsa, Amer. J. Math. (2008)

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$$v_\infty(x, y) = \rho^{3/2} \cos \frac{3}{2}\theta,$$

where $\rho^2 = x_n^2 + y^2$ *and* $\tan\theta = y/x_n$*.*

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THANK YOU!