Existence of ground states for aggregation-diffusion equations

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Motivation

Given an **interaction potential** $W : \mathbb{R}^d \to (-\infty, \infty]$, an **entropy function** $U : [0, \infty) \to \mathbb{R}$, and a **temperature** $\varepsilon \geqslant 0$, we consider the continuity equation

$$\partial_t \rho = \nabla \cdot ((\nabla W * \rho)\rho) + \varepsilon \nabla \cdot \left(\nabla U'(\rho)\rho\right), \qquad \rho \in \mathcal{P}(\mathbb{R}^d), \quad t > 0. \tag{1}$$

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We study the minimizers of the associated 2-**Wasserstein energy** to (1),

$$E_\varepsilon(\rho) = \mathcal{W}(\rho) + \varepsilon \mathcal{E}_U(\rho), \quad \rho \in \mathcal{P}(\mathbb{R}^d),$$

where

$$\mathcal{W}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y)$$

and

$$\mathcal{E}_U(\rho) = \begin{cases} \int_{\mathbb{R}^d} U(\rho(x)) \, \mathrm{d}x & \text{if } \rho \in \mathcal{P}_{\mathrm{ac}}(\mathbb{R}^d), \\ +\infty & \text{if } \rho \in \mathcal{P}(\mathbb{R}^d) \setminus \mathcal{P}_{\mathrm{ac}}(\mathbb{R}^d). \end{cases}$$

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- When $\varepsilon = 0$ we know minimizers exist for a wide class of interaction potentials [Carrillo–Cañizo–P. (2015), Simione–Slepčev–Topaloglu (2015), Choksi–Fetecau–Topaloglu (2014)].

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Motivation

Typical interaction potentials. For a given $\beta > -d$ the **power-law** interaction potential is defined by

$$W_\beta(x) = \begin{cases} \frac{|x|^\beta}{\beta} & \text{if } \beta \neq 0, \\ \log|x| & \text{if } \beta = 0, \end{cases} \quad \text{for all } x \in \mathbb{R}^d.$$

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General hypotheses.

- $W : \mathbb{R}^d \to (-\infty, \infty]$ is locally integrable lower semicontinuous and even.
- $U : [0,\infty) \to \mathbb{R}$ is continuous, of class C^2 on $(0,\infty)$ and convex, and $U(0) = 0$.

Questions.

- How **stable** are the (local) minimizers with zero diffusion $\varepsilon = 0$ when small noise ε is switched on? Can we relate this to metastability? [Evers–Kolokolnikov (2016)]

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Answers.

- We show for bounded-at-infinity, attraction-repulsion interaction potentials and $m \leqslant 1$ that **no minimizers** (local or global) of the energy exist as soon as $\varepsilon > 0$, no matter how small ε.
- We get a sufficient condition on general interaction potentials and diffusion for the unboundeness from below of the energy. The result is **sharp** for $U(r) = \frac{r^m}{m-1}$, with $m \geqslant 1$. The result is **not** sharp for $U(r) = r^m$, with $m < 1$ [Calvez–Carrillo–Hoffmann (2017)].

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Theorem 1. Let $U(r) = r\log(r)$, and let W be positive and such that $W \in L^\infty(\mathbb{R}^d \setminus B_\delta)$ for any $\delta > 0$. Then E_ε does **not admit** any W_p-local minimizer for any $p \in [1, \infty]$ in $\mathcal{P}(\mathbb{R}^d)$. Moreover, if W is Lipschitz continuous, then there are no critical points of E_ε in $\mathcal{P}_{\mathrm{ac}}(\mathbb{R}^d)$.

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$$\|\rho\|_{L^\infty(\mathbb{R}^d)} \le |\Omega|^{-1} e^{\frac{\|W\|_{L^\infty} - \inf_\Omega W}{\varepsilon}}.$$

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  So the larger the domain, the smaller the L^∞-norm of any steady state.
- We can extend the theorem to any U which is convex with u (the McCann's scaling function $u(r) = r^d U(r^{-d})$) nonincreasing and $\lim_{r\to 0} U'(r) = -\infty$.

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- The Euler–Lagrange conditions for ρ give

$$\begin{cases} \varepsilon \log(\rho) + W * \rho = C_i & \text{on } A_i, \\ \varepsilon \log(\rho) + W * \rho \geqslant C_i & \text{on } \mathbb{R}^d, \end{cases}$$

where A_i is any closed, connected component of $\operatorname{supp}(\rho)$, for some $C_i \in \mathbb{R}$.

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- Since $\|W\|_{L^\infty(\mathbb{R}^d)} < \infty$, the Euler–Lagrange inequality implies that ρ cannot vanish on $\mathbb{R}^d$, otherwise we would have a point $x \in \mathbb{R}^d$ such that

$$-\infty = \varepsilon \log(0) = \varepsilon \log(\rho(x)) \geqslant C_1 - \|W\|_{L^\infty(\mathbb{R}^d)} > -\infty.$$

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- Then, for all $x \in \mathbb{R}^d$, the Euler–Lagrange equation gives

$$\rho(x) = \exp\left(\frac{C_1 - W * \rho(x)}{\varepsilon}\right) \geqslant \exp\left(\frac{C_1 - \|W\|_{L^\infty(\mathbb{R}^d)}}{\varepsilon}\right),$$

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If now we assume ρ is a **critical point**, then we can show, using a bootstrap argument, $\rho \in C^\alpha(\mathbb{R}^d)$, $\alpha > 1$ [Carrillo–Hittmeir–Volzone–Yao (2016)].

Theorem 2. Suppose that the interaction potential W is differentiable away from the origin, and suppose that U is such that u is nonincreasing. If

$$\limsup_{r\to\infty}\left(\frac{1}{2}\sup_{z\in B_{2r}}(\nabla W(z)\cdot z)-\varepsilon v(r\omega_d^{1/d})\right)<0$$

or

$$\liminf_{r\to 0}\left(\frac{1}{2}\inf_{z\in B_{2r}}(\nabla W(z)\cdot z)-\varepsilon v(r\omega_d^{1/d})\right)>0,$$

where $v(r)=-ru'(r)$, then E_ε is **not** bounded below.

Main results

When we consider the **power** cases for U and W, the theorem's conditions become

$$\lim_{r\to\infty}\left(2^{\beta-1}r^{\beta}-\varepsilon d\omega_d^{1-m}r^{(1-m)d}\right)<0 \quad \text{if } \beta\geqslant 0,$$

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- $m=1$ and $2\varepsilon d\neq 1$.

Proof of Theorem 2. Considering $\rho_r = r^{-d}\omega_d^{-1}\chi_{B_r}$, we claim that if either condition in the theorem holds, then $\lim_{r\to\infty} E_\varepsilon(\rho_r) = -\infty$ or $\lim_{r\to 0} E_\varepsilon(\rho_r) = -\infty$.

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- We proceed analogously for the second condition.

Theorem 3 (sharpness for $m = 1$). Suppose that the entropy function is given by $U(r) = r \log r$ and that W is positive. If

$$\limsup_{|x|\to\infty} \nabla W(x) \cdot x < 2d\varepsilon,$$

then E_ε is **not bounded** below. Alternatively, if

$$\liminf_{|x|\to\infty} \nabla W(x) \cdot x > 2d\varepsilon,$$

then E_ε is **bounded** below; more precisely, there exists $\rho_\infty \in \mathcal{P}(\mathbb{R}^d)$ such that

$$E_\varepsilon(\rho_\infty) = \inf E_\varepsilon > -\infty.$$

Main results

Consider the energy functional

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log|x-y|\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y) + \varepsilon \int_{\mathbb{R}^d} \rho(x)\log\rho(x)\,\mathrm{d}x,$$

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corresponding to the Keller–Segel model. It is known that there is a **critical** value of the noise, $\varepsilon_c = 1/(2d)$, such that the energy functional is bounded from below if and only if $\varepsilon = \varepsilon_c$ [Dolbeault–Perthame (2004), Blanchet–Dolbeault-Perthame (2006), Blanchet–Carrillo–Laurençot (2009). Blanchet–Carlen–Carrillo (2012)].

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Similarly, our theorem shows that if W is bounded from below and

$$\lim_{|x|\to\infty} \nabla W(x)\cdot x = L > 0,$$

then there also exists a **critical** diffusion $\varepsilon_c = L/(2d)$ separating the boundedness from the unboundeness from below of the energy.

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THANK YOU!