Rigorous derivation of the nonlocal reaction-diffusion FitzHugh-Nagumo system

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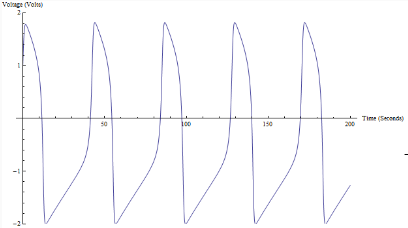
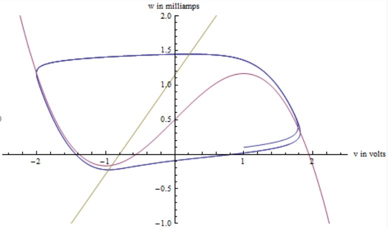
References

Hodgkin & Huxley '52 , FitzHugh '61 , Nagumo, Arimoto & Yoshizawa '62

We consider the membrane potential of the neuron $v(t) \in \mathbb{R}$ and an adaptation variable $w(t) \in \mathbb{R}$

$$\begin{cases} \dot{v} = N(v) - w + I_{\text{ext}}, \\ \\ \dot{w} = \tau\,(v + a - b\,w), \end{cases} \tag{1}$$

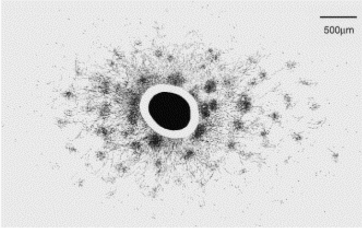


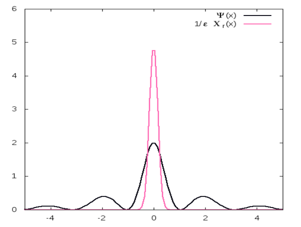


Neural network model

For $i \in \{1, ..., n\}$, we consider

$$\begin{cases} \dot{\mathbf{x}}_i &= 0, \\ \dot{v}_i &= N(v_i) - w_i - \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \Phi_{\varepsilon,r}(\mathbf{x}_i - \mathbf{x}_j)\,(v_i - v_j), \\ \dot{w}_i &= \tau\,(v_i + a - b\,w_i), \end{cases} \tag{2}$$





We choose $\Phi_{\varepsilon,r} := \Psi + \dfrac{1}{\varepsilon}\chi_r$, with

- Ψ models long-range excitatory interactions throughout the network,
- $\dfrac{1}{\varepsilon}\chi_r$ models short-range excitatory interactions with high intensity.

Regime of strong local interactions

Formally, we pass to the limit $r \to 0$ to study the regime of strong local interactions. The interaction kernel converges towards $\Psi + \frac{1}{\varepsilon}\delta_0$ and the limit density function f^ε satisfies the following nonlocal kinetic equation:

$$\begin{aligned} \partial_t f^\varepsilon \quad &+ \quad \partial_v \left[f^\varepsilon \left(N(v) - w - \mathcal{K}_\Psi[f^\varepsilon] - \frac{1}{\varepsilon}(\rho^\varepsilon v - j^\varepsilon) \right) \right] \\ &+ \quad \partial_w \left[\tau \left(v + a - b\, w \right) f^\varepsilon \right] = 0, \end{aligned} \tag{3}$$

where

$$\mathcal{K}_\Psi[f](t, \mathbf{x}, v) := \int \Psi(\mathbf{x} - \mathbf{x}')\,(v - v')\, f(t, \mathbf{x}', v', w')\, \mathrm{d}\mathbf{x}' \mathrm{d}v' \mathrm{d}w',$$

and where we define the macroscopic quantities:

$$\begin{cases} \rho^\varepsilon(t, \mathbf{x}) = \rho_0^\varepsilon(\mathbf{x}) := \displaystyle\int f_0^\varepsilon(\mathbf{x}, v, w)\, \mathrm{d}v\, \mathrm{d}w, \\ \rho_0^\varepsilon(\mathbf{x})\, V^\varepsilon(t, \mathbf{x}) = j^\varepsilon(t, \mathbf{x}) := \displaystyle\int f^\varepsilon(t, \mathbf{x}, v, w)\, v\, \mathrm{d}v\, \mathrm{d}w, \\ \rho_0^\varepsilon(\mathbf{x}) W^\varepsilon(t, \mathbf{x}) := \displaystyle\int f^\varepsilon(t, \mathbf{x}, v, w)\, w\, \mathrm{d}v\, \mathrm{d}w. \end{cases}$$

Neural network model

References

Baladron, Fasoli, Faugeras & Touboul '12:

- Mean-field limit of Hodgkin-Huxley and FitzHugh-Nagumo systems with noise an a conductance-based connectivity kernel,

Luçon & Stannat '14:

- Mean-field limit of FitzHugh-Nagumo-like equations with noise and a compactly supported singular connectivity kernel.

Mischler, Quiñinao & Touboul '15:

- Existence and stability of a stationary state of the FitzHugh-Nagumo system.

Our framework

- We neglect the noise from the environment, so our model is deterministic,
- the connectivity between neurons is weighted only by the distance,
- the support of the connectivity kernel can be unbounded.

Towards a macroscopic model

The macroscopic quantities derived from f^ε satisfy the following system:

$$\begin{cases} \rho_0^\varepsilon \left[\partial_t V^\varepsilon - \mathcal{L}_{\rho_0^\varepsilon}(V^\varepsilon)\right] = \rho_0^\varepsilon \left[N(V^\varepsilon) - W^\varepsilon\right] + \mathcal{E}(f^\varepsilon), \\ \rho_0^\varepsilon \, \partial_t W^\varepsilon = \tau \rho_0^\varepsilon \left[V^\varepsilon + a - b\, W^\varepsilon\right], \end{cases}$$

with

$$\mathcal{L}_{\rho_0}(V)(t, \mathbf{x}) := \int \Psi(\mathbf{x} - \mathbf{x}') \left(V(t, \mathbf{x}') - V(t, \mathbf{x})\right) \rho_0(\mathbf{x}') \, \mathrm{d}\mathbf{x}'.$$

and the error term is

$$\mathcal{E}(f^\varepsilon) := \int f^\varepsilon \left(N(v) - N(V^\varepsilon)\right) \mathrm{d}v \, \mathrm{d}w.$$

- The non local operator $\mathcal{L}_{\rho_0}(V)$ plays the role of diffusion in this system
- We want to prove that $\mathcal{E}(f^\varepsilon) \to 0$, when $\varepsilon \to 0$ and get the macroscopic FitzHugh-Nagumo model. This system is studied in the reaction diffusion community : propagation of front, pattern formation.
- This equation is not well-defined for $\mathbf{x} \in \mathbb{R}^d$ such that $\rho_0(x) = 0$!

Main result : link between kinetic and macroscopic models

Existence and uniquness for the macroscopic model

We first consider the nonlocal reaction-diffusion system

$$\begin{cases} \partial_t V \; - \; \mathcal{L}_{\rho_0}(V) = N(V) - W, \\ \\ \partial_t W = \tau \; (V + a - b\, W)\,, \end{cases} \tag{4}$$

Proposition

We choose $\Psi \in L^1(\mathbf{R}^d)$ to be non-negative, symmetric, we also suppose that ρ_0 and the initial data (V_0, W_0) satisfies $\rho_0 \geq 0$,

$$\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d), \quad V_0, W_0 \in L^\infty(\mathbb{R}^d). \tag{5}$$

Then for any $T > 0$, there exists a unique classical solution $(V, W) \in \mathscr{C}^1([0, T], L^\infty(\mathbb{R}^d))$ to the nonlocal reaction-diffusion system (4).

Furthermore, we construct one solution to $\mathcal{Z} = (\rho_0, \rho_0 V, \rho_0 W)$

$$\begin{cases} \partial_t \rho_0 V \; - \; \rho_0 \mathcal{L}_{\rho_0}(V) = \rho_0\, N(V) - \rho_0 W, \\ \\ \partial_t \rho_0 W = \tau \; (\rho_0 V + a\, \rho_0 - b\, \rho_0 W)\,, \end{cases}$$

Main result

Theorem: Hydrodynamic limit

Assume that $(f_0^\varepsilon)_\varepsilon$ is smooth and there exists a positive constant C such that for all $\varepsilon > 0$:

$$\int \left(1 + |\mathbf{x}|^4 + |v|^4 + |w|^4\right)\, f_0^\varepsilon(\mathbf{x}, v, w)\, \mathrm{d}\mathbf{x}\, \mathrm{d}v\, \mathrm{d}w \le C.$$

We also choose initial data (ρ_0, V_0, W_0) such that

$$\rho_0 \ge 0, \quad \rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d), \quad V_0,\ W_0 \in L^\infty(\mathbb{R}^d).$$

Furthermore, we have

$$\|\rho_0^\varepsilon - \rho_0\|_{L^2}^2 + \int \rho_0^\varepsilon(\mathbf{x}) \left[|V_0^\varepsilon(\mathbf{x}) - V_0(\mathbf{x})|^2 + |W_0^\varepsilon(\mathbf{x}) - W_0(\mathbf{x})|^2\right] \mathrm{d}\mathbf{x} \le C\, \varepsilon^{1/(d+6)}.$$

Then there exists a positive constant C_T such that for all $t \in [0; T]$:

$$\int \rho_0^\varepsilon(\mathbf{x})\, \frac{|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2}{2}(t, \mathbf{x})\, \mathrm{d}\mathbf{x} \le C_T\, \varepsilon^{1/(d+6)},$$

where (V, W) is the solution to the macroscopic reaction-diffusion system, and $(\rho_0^\varepsilon, V^\varepsilon, W^\varepsilon)$ are the macroscopic quantities computed from the solution f^ε of the kinetic equation.

Strategy of the proof

References

- Relative entropy method for hyperbolic conservation laws: Di Perna '79, Dafermos '79
- Hydrodynamic limit of Vlasov-type equations under strong local alignment regime: Kang & Vasseur '15
- Hydrodynamic limit of the kinetic Cucker-Smale system under strong local alignment regime: Karper, Mellet & Trivisa '12, Figalli & Kang '17

Key arguments for the FitzHugh-Nagumo model

- There is no transport term in $\mathbf{x}$: good and not good.
- The difficulty comes from the nonlinearity $N(v) = v - v^3$: we have to control moments of f^ε to estimate the error term

$$\int_0^T \int (V^\varepsilon(t) - V(t)) \left(\int f^\varepsilon(t) \left[N(v) - N(V^\varepsilon(t)) \right] \mathrm{d}v \, \mathrm{d}w \right) \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

Arguments of the proof

Define the kinetic dissipation $\mathcal{D}$ with:

$$\mathcal{D}(t) := \int_{\mathbb{R}^{d+2}} f^\varepsilon(t)\, v\, (v - V^\varepsilon(t,\mathbf{x}))\ \rho_0^\varepsilon(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}v\, \mathrm{d}w$$
$$= \int_{\mathbb{R}^{d+2}} f^\varepsilon(t)\, |v - V^\varepsilon(t,\mathbf{x})|^2\ \rho_0^\varepsilon(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{d}v\, \mathrm{d}w \geq 0.$$

Entropy equality: we set $\mathbf{z} = (v, w, \mathbf{x})$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{d+2}} \left(|v|^2 + |w|^2\right) f^\varepsilon(t)\, \mathrm{d}\mathbf{z} + \int_{\mathbb{R}^{d+2}} |v|^4 f^\varepsilon(t)\, \mathrm{d}\mathbf{z} + \frac{1}{\varepsilon}\mathcal{D}(t)$$
$$\leq C\left(\int_{\mathbb{R}^{d+2}} \left(|v|^2 + |w|^2\right) f^\varepsilon(t)\, \mathrm{d}\mathbf{z} + 1\right).$$

Writing the same entropy equality with moments of order 4,

$$\begin{cases} \sup_{t\in[0;T]} \int_{\mathbb{R}^{d+2}} \left(|\mathbf{x}|^4 + |v|^4 + |w|^4\right) f^\varepsilon(t)\, \mathrm{d}\mathbf{z} \leq C_T, \\ \int_0^T \int_{\mathbb{R}^{d+2}} |v|^6 f^\varepsilon(t)\, \mathrm{d}\mathbf{z}\, \mathrm{d}t \leq C_T. \end{cases}$$

Estimate of the kinetic dissipation

By integrating the entropy equality on time between 0 and T, we get:

Kinetic dissipation estimate

$$\int_0^T \mathcal{D}(t)\,\mathrm{d}t := \int_0^T \int_{\mathbb{R}^{d+2}} f^\varepsilon(t)\,\rho_0^\varepsilon(\mathbf{x})\,|v - V^\varepsilon(t,x)|^2\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le C_T\,\varepsilon.$$

This last estimate can be improved removing the weight ρ_0^ε: we use the moments estimates , and we divide $\mathbb{R}^d$ into three subsets:

$$\mathcal{A}_\varepsilon := \left\{\mathbf{x} \in \mathbb{R}^d \quad | \quad \rho_0^\varepsilon(\mathbf{x}) = 0\right\},$$

$$\mathcal{B}_\varepsilon^\eta := \left\{\mathbf{x} \in \mathbb{R}^d \quad | \quad \rho_0^\varepsilon(\mathbf{x}) > \eta\right\},$$

$$\mathcal{C}_\varepsilon^\eta := \left\{\mathbf{x} \in \mathbb{R}^d \quad | \quad 0 < \rho_0^\varepsilon(\mathbf{x}) \le \eta\right\},$$

for some $\eta > 0$ to be adjusted.

Improved kinetic dissipation estimate

$$\int_0^T \int f^\varepsilon(t)\,|v - V^\varepsilon(t,x)|^2\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le C_T\,\varepsilon^{2/(d+6)}.$$

Relative entropy estimate

we define the entropy $\eta(\widetilde{\mathcal{Z}})$ by

$$\eta(\widetilde{\mathcal{Z}}) := \widetilde{\rho}\, \frac{|\widetilde{V}|^2 + |\widetilde{W}|^2}{2}.$$

Proposition 3

In the same framework as in the Theorem, consider the macroscopic quantities

$$\mathcal{Z}^\varepsilon := (\rho_0^\varepsilon, \rho_0^\varepsilon\, V^\varepsilon, \rho_0^\varepsilon\, W^\varepsilon),$$

computed from the solution f^ε of the kinetic equation. Also consider

$$\mathcal{Z} := (\rho_0, \rho_0\, V, \rho_0\, W),$$

where (V, W) is the solution of the nonlocal reaction-diffusion equation. Then, $\mathcal{Z}^\varepsilon$ and $\mathcal{Z}$ satisfy for all $t \in [0; T]$:

$$\int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z})(t, \mathbf{x})\, \mathrm{d}\mathbf{x} := \int \rho_0^\varepsilon(\mathbf{x})\, \frac{|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2}{2}(t, \mathbf{x})\, \mathrm{d}\mathbf{x}$$
$$\leq C_T\, \varepsilon^{1/(d+6)}.$$

Relative entropy estimate

First of all, we define $\mathcal{F}$ the application such that

$$\partial_t \mathcal{Z} = \mathcal{F}(\mathcal{Z}).$$

Then, the solution of the kinetic equation satisfy:

$$\partial_t \mathcal{Z}^\varepsilon = \mathcal{F}(\mathcal{Z}^\varepsilon) + \mathcal{E}(f^\varepsilon),$$

where

$$\mathcal{E}(f^\varepsilon)(t,\mathbf{x}) = \int f^\varepsilon(t,\mathbf{x},v,w)\,[N(v) - N(V^\varepsilon(t,\mathbf{x}))]\,\mathrm{d}v\,\mathrm{d}w$$

is an error term. Therefore, we get the following equality:

Variation of entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\mathcal{Z}^\varepsilon)\,\mathrm{d}\mathbf{x} + \mathcal{S}(\mathcal{Z}^\varepsilon) = \int V^\varepsilon\,\mathcal{E}(f^\varepsilon(t))\,\mathrm{d}\mathbf{x},$$

where $\mathcal{S}(\mathcal{Z}^\varepsilon)$ gathers some local and nonlocal source terms.

Then, the relative entropy satisfies the equality:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z})\,\mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\mathcal{Z}^\varepsilon)\,\mathrm{d}\mathbf{x} - \int D\eta(\mathcal{Z})\left[\partial_t \mathcal{Z}^\varepsilon - \mathcal{F}(\mathcal{Z}^\varepsilon)\right]\,\mathrm{d}\mathbf{x} \\ &\quad + \mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z}) + \mathcal{S}(\mathcal{Z}^\varepsilon) \\ &= \mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z}) + \int (V - V^\varepsilon)\mathcal{E}(f^\varepsilon)(t,\mathbf{x})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

where $\mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z})$ gathers local and nonlocal relative terms.

Estimate of the relative terms

There exists a constant C_T such that:

$$\int_0^T \left|\mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z})\right|(s)\,\mathrm{d}s \leq C_T\left[\|\rho_0^\varepsilon - \rho_0\|_{L^2}^2 + \int_0^T\int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z})(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right]$$

Relative entropy estimate

It remains to estimate the error term:

$$\int (V - V^\varepsilon)\mathcal{E}(f^\varepsilon)(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = \int (V - V^\varepsilon)\left(\int f^\varepsilon(t)\,[N(v) - N(V^\varepsilon)]\,\mathrm{d}v\,\mathrm{d}w\right)\,\mathrm{d}\mathbf{x}$$
$$\leq \alpha(t)\left(\int |v - V^\varepsilon|^2 f^\varepsilon(t)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{x}\right)^{1/2},$$

where

$$\alpha(t) := \frac{3}{2}\left(\int \left[(V^\varepsilon(t))^2 + v^2\right]^2 [V^\varepsilon(t) - V(t)]^2\; f^\varepsilon(t)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\mathbf{x}\right)^{1/2}.$$

Using the moment estimate $\int_0^T \int |v|^6\, f^\varepsilon(t)\,\mathrm{d}\mathbf{z}\,\mathrm{d}t \leq C_T$, we get:

$$\int_0^T |\alpha(s)|^2\,\mathrm{d}s \leq C_T.$$

Finally, using the estimate of the kinetic dissipation, we have:

Estimate of the error term

$$\int_0^T \int (V - V^\varepsilon)\mathcal{E}(f^\varepsilon)(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \leq C_T\,\varepsilon^{1/(d+6)}.$$

Relative entropy estimate

We have:

- $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z})\,\mathrm{d}\mathbf{x} = \mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z}) + \int (V - V^\varepsilon)\mathcal{E}(f^\varepsilon)(t,\mathbf{x})\,\mathrm{d}\mathbf{x},$
- $\displaystyle\int_0^T |\mathcal{R}(\mathcal{Z}^\varepsilon|\mathcal{Z})|\,(s)\,\mathrm{d}s \le C_T\left[\varepsilon^{1/(d+6)} + \int_0^t\int \eta(\mathcal{Z}^\varepsilon|\mathcal{Z})(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\right],$
- $\displaystyle\int_0^T\int (V - V^\varepsilon)\mathcal{E}(f^\varepsilon)(s,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \le C_T\,\varepsilon^{1/(d+6)}.$

We conclude with Grönwall's lemma:

Estimate of the relative entropy

For all $t \in [0;T]$:

$$\int \rho_0^\varepsilon(\mathbf{x})\,\frac{|V - V^\varepsilon|^2 + |W - W^\varepsilon|^2}{2}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \le C_T\,\varepsilon^{1/(d+6)}.$$

Conclusion

Conclusion:

- We have rigorously established a link between the mean-field model of FitzHugh-Nagumo type towards a macroscopic nonlocal reaction-diffusion system, with an estimate of the error with respect to the parameter ε, using a relative entropy estimate.

Perspectives:

- Here, we have forced the local interactions. We would like to have a more regular kernel using a different scaling, $\varepsilon^{-(d+2)}\,\Psi(\frac{\cdot}{\varepsilon})$ for instance, to derive a reaction-diffusion system with a local diffusion term. We will need more regularity in space than before.
- Is it possible to observe Turing instabilities on the kinetic equation : numerical simulations and stability analysis of the kinetic model.