Super Ricci flows for weighted graphs and Markov chains

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April 13, 2018

Classical (super-) Ricci flows

A Riemannian manifold (M, g_t) with time-dependent metric is a Ricci flow if

$$-\frac{1}{2}\partial_t g_t = \mathrm{Ric}_{g_t} \ .$$

Examples:

- 'Solitons': let $\mathrm{Ric}_{g_0} \geq \kappa g_0$ for $\kappa \in \mathbb{R}$, then

  $$g_t = (1 - 2\kappa t)g_0 \ ,$$

  is a super Ricci flow
  $\kappa = 0$: steady, $\kappa < 0$: expanding, $\kappa > 0$: shrinking (here $t < \frac{1}{2\kappa}$)
- Neck pinch: dim$= 3$

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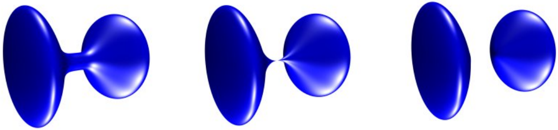

Robust approaches and characterizations I

Strong interest in robust descriptions of (super) Ricci flows in presence of singularities, many recent developments:

- [Bamler, Kleiner '17], [Kleiner, Lott '14]:
  'canonical' Ricci flow through singularities, limit of flow with surgery
- [Haslhofer, Naber '15]:
  characterization via functional inequalities on path space
- [Topping, McCann '08]:
  characterization via optimal transport
- [Sturm '16], [Kopfer, Sturm '16]:
  synthetic notion of super Ricci flow on metric measure spaces

Robust approaches and characterizations II

(M, g_t) is a super Ricci flow, i.e. $-\frac{1}{2}\partial_t g_t \leq \mathrm{Ric}_{g_t}$ iff any of the following hold

- Bochner inequality:

$$\Gamma_{2,t}(\psi) \geq \frac{1}{2}\partial_t \Gamma_t(\psi) \quad \text{for all smooth } \psi : M \to \mathbb{R}$$

Note that

$$\Gamma_{2,t}(\psi) := \frac{1}{2}\Delta|\nabla\psi|^2 - \langle \nabla\psi, \Delta\nabla\psi\rangle = \mathrm{Ric}_{g_t}(\nabla\psi) + \|\mathrm{Hess}\psi\|^2_{\mathrm{HS}}$$
$$\geq + \frac{1}{2}\partial_t|\nabla\psi|^2 = \frac{1}{2}\partial_t\Gamma_t(\psi)$$

- gradient estimates:

$$\Gamma_t(P_{t,s}\bar{\psi}) \leq P_{t,s}\Gamma_s(\bar{\psi}) \quad \text{for all } s \leq t\ ,$$

where $P_{t,s}\bar{\psi}$ is the solution to heat equation

$$\partial_t\psi(t,\cdot) = \Delta_t\psi(t,\cdot), \quad \psi(s,\cdot) = \bar{\psi}$$

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$$W_{2,s}(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_{2,t}(\mu,\nu) \quad \text{for all } \mu,\nu \in \mathcal{P}(M),\ s \leq t\ ,$$

  where $\hat{P}_{t,s}$ is the dual heat flow on measures and $W_{2,t}$ is the L^2 Kantorovich distance w.r.t. Riemannian distance d_t

$$W_{2,t}^2(\mu,\nu) := \inf_{q \in \mathsf{Cpl}(\mu,\nu)} \int d_t(x,y)^2 \mathrm{d}q(x,y)\ .$$

- dynamic convexity of entropy: for all $W_{2,t}$-geodesics $(\mu^a)_{a\in[0,1]}$

$$\partial_a\Big|_{a=1-} \mathrm{Ent}(\mu^a) - \partial_a\Big|_{a=0+} \mathrm{Ent}(\mu^a) \geq -\frac{1}{2}\partial_t W_{2,t-}^2(\mu^0,\mu^1)\ ,$$

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Robust approaches and characterizations III

The previous equivalent properties are meaningfull also in a non-smooth setting, i.e.

- for a Dirichlet space (X, m_t) with Dirichlet forms $\mathcal{E}_t$ and asociated Γ operators
- time-dependent metric measure spaces (X, d_t, m_t)

This allows for synthetic definition of super Ricci flow for mm-spaces, see [Sturm '16], [Kopfer, Sturm '16]

In particular, for a static mm-space recover synthetic definition of lower Ricci curvature bounds by [Lott,Villani '09], [Sturm '06]:

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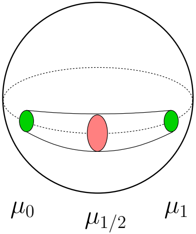

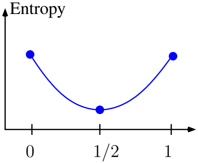

Setting

- $\mathcal{X}$ finite set
- $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ transition rates
- Δ generator of continuous time Markov chain

$$\Delta\psi(x) = \sum_y \big(\psi(y) - \psi(x)\big) Q_{xy}$$

- π reversible probability measure on $\mathcal{X}$

$$\forall x, y : Q_{xy}\pi(x) = Q_{yx}\pi(y)$$

- probability measures on $\mathcal{X}$

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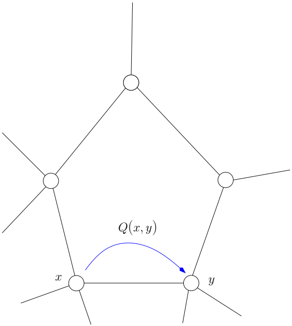

Problem:

L^2-Kantorovich distance W_2 is degenerate (for any choice of distance on $\mathcal{X}$) and does not admit non-trivial geodesics, gradient flows, ...

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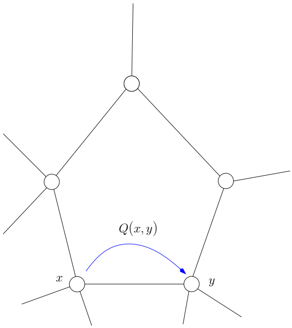

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Benamou–Brenier formula for W_2

$$W_2(\rho_0,\rho_1)^2 = \inf_{\rho,V}\left\{\int_0^1\int_{\mathbb{R}^n}\frac{|V(x)|^2}{\rho_t(x)}\mathrm{d}x\mathrm{d}t \ : \qquad \partial_t\rho+\nabla\cdot V=0\right\}$$

discrete transport distance

$$\mathcal{W}(\mu_0,\mu_1)^2 := \inf_{\mu,V}\left\{\frac{1}{2}\int_0^1\sum_{x,y\in\mathcal{X}}\frac{|V(x,y)|^2}{\Lambda(\mu_t(x)Q_{xy},\mu_t(y)Q_{yx})}\mathrm{d}t \ : \right.$$
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where $\Lambda(s,t) := \dfrac{s-t}{\log s-\log t}$ is the logarithmic mean

Theorem [Maas '11, E.-Maas '12]

$\mathcal{W}$ defines a geodesic distance on the set of probability measures $\mathcal{P}(\mathcal{X})$.
The law of the Markov chain evolves as the gradient flow of the entropy

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Entropic curvature bounds for Markov chains

In the spirit of the approach of [Lott–Villani '09, Sturm '06] we define in [E.-Maas'12]:

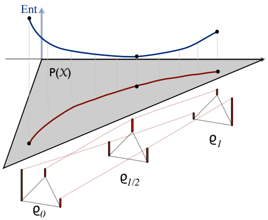
Definition:

Markov triple $(\mathcal{X}, Q, \pi)$ has Ricci curvature bounded below by $\kappa \in \mathbb{R}$ if the entropy is κ-convex along geodesics in $(\mathcal{P}(\mathcal{X}), \mathcal{W})$, i.e.

$$\mathcal{H}(\mu_t) \leq (1-t)\mathcal{H}(\mu_0) + t\mathcal{H}(\mu_1) - \frac{\kappa}{2} t(1-t)\mathcal{W}(\mu_0, \mu_1)^2 \ .$$

Alternative approaches to discrete Ricci curvature:

- [Ollivier '09] contractivity in W_1
- [Bonciocat–Sturm '09] approximate W_2 geodesics
- [Jost et al. '11] discrete Bakry–Émery condition
- [S.T. Yau et al. '15], [Münch '14], [Dier et al. '17] modified Bakry–Émery condition
- ...



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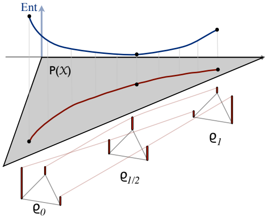
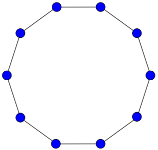


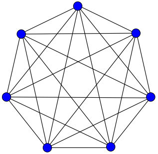
First examples

- **Circle:** Ric ≥ 0



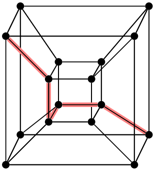
$$\mathcal{X} = \{1, \ldots, n\}, \quad Q(i, i \pm 1) = 1$$

- **Complete graph:** Ric $\geq n/2$



$$\mathcal{X} = \{1, \ldots, n\}, \quad Q(i, j) = 1$$

- **Discrete cube:** Ric ≥ 2



$$\mathcal{Q}_n = \{0, 1\}^n, \quad Q(x, y) = 1 \text{ for all } x \sim y$$

A first definition of super Ricci flow

Definition:

The time-dependent Markov triple $(\mathcal{X}, Q_t, \pi_t)$ is a super Ricci flow iff the entropy is dynamically convex in $(\mathcal{P}(\mathcal{X}), \mathcal{W}_t)$, i.e.

$$\partial_a\Big|_{a=1-}\mathrm{Ent}_t(\mu^a) - \partial_a\Big|_{a=0+}\mathrm{Ent}_t(\mu^a) \geq -\frac{1}{2}\partial_t \mathcal{W}_t^2(\mu^0, \mu^1)\ .$$

A first example:

Let $\mathrm{Ric}(\mathcal{X}, Q_0, \pi_0) \geq \kappa$, then $(\mathcal{X}, Q_t, \pi_0)$ is a super Ricci flow for

$$Q_t = \frac{1}{1-2\kappa t} Q_0$$

If $\kappa > 0$, collapse at $t_* = 1/2\kappa$

$$Q_t \to +\infty \text{ as } t \uparrow t_*$$

Goal: Give meaning to flow past singularities!

Idea: Use heat flow!

A first definition of super Ricci flow

Definition:

The time-dependent Markov triple $(\mathcal{X}, Q_t, \pi_t)$ is a super Ricci flow iff the entropy is dynamically convex in $(\mathcal{P}(\mathcal{X}), \mathcal{W}_t)$, i.e.

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A first example:

Let $\mathrm{Ric}(\mathcal{X}, Q_0, \pi_0) \geq \kappa$, then $(\mathcal{X}, Q_t, \pi_0)$ is a super Ricci flow for

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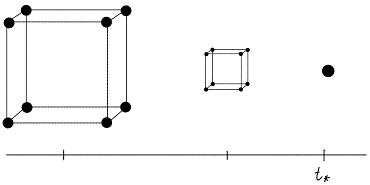

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Heat flow on singular space times

Goal: solve heat equation

$$\partial_t \psi(t,x) = \Delta_t \psi(t,x) = \sum_{y \in \mathcal{X}_t} \big[\psi(t,y) - \psi(t,x)\big] Q_t(x,y)$$

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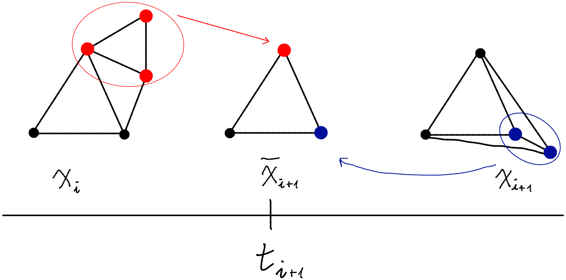

Heat flow on singular space times – setting

Setting: Let $(\mathcal{X}_t, Q_t, \pi_t)_{t\in[0,T]}$ satisfy the following

- ex. partition $0 = t_0 < t_1 < \cdots < t_n = T$ and sets $\mathcal{X}_i, \bar{\mathcal{X}}_i$ such that $\mathcal{X}_{t_i} = \bar{\mathcal{X}}_i$, $\mathcal{X}_t = \mathcal{X}_i$ for $t \in I_i = (t_i, t_{i+1})$ and surjective $c_i : \mathcal{X}_i \to \bar{\mathcal{X}}_{i+1}$, $s_{i+1} : \mathcal{X}_{i+1} \to \bar{\mathcal{X}}_i$
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$$Q_{t_{i+1}}(z,z') = \frac{1}{\pi_{t_{i+1}}(z)} \sum_{\substack{x\in c_i^{-1}(z)\\ x'\in c_i^{-1}(z')}} \lim_{t\uparrow t_{i+1}} Q_t(x,x')\pi_t(x)$$

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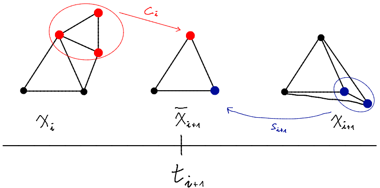

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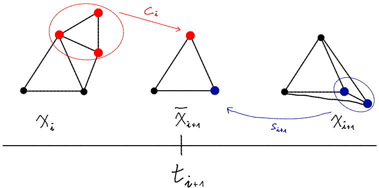

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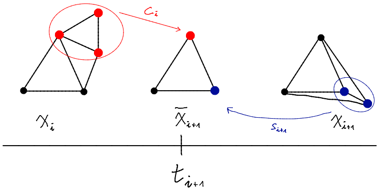

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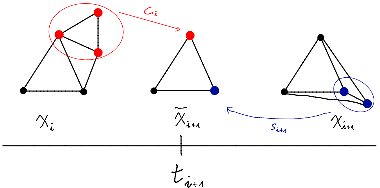

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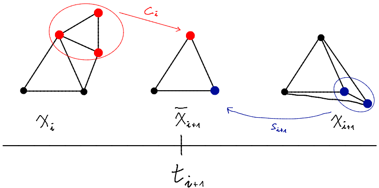

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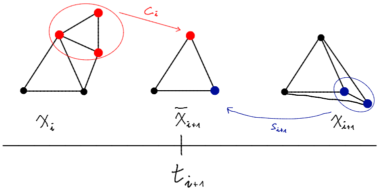

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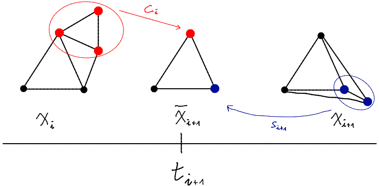

Heat flow on singular space-times – existence, uniqueness

Define the space-time $\mathcal{S}_{s,t} = \{(r,x) \ : \ r \in [s,t], \ x \in \mathcal{X}_r\}$

Theorem:

For $s \in [0,T)$ and $\bar{\psi} \in \mathcal{R}^{\mathcal{X}_s}$ exist unique $\psi : \mathcal{S}_{s,T} \to \mathbb{R}$ s.t. $\quad (\ \psi(t,\cdot) =: P_{t,s}\bar{\psi} \ )$

- $\partial_t \psi(t,\cdot) = \Delta_t \psi(t,\cdot)$ on $\mathcal{X}_i \times I_i$, $\psi(s,\cdot) = \bar{\psi}$,
- for $x \in c_i^{-1}(z)$ and $y \in s_{i+1}^{-1}(z)$ we have

$$\psi(t_{i+1}, z) = \lim_{t \uparrow t_{i+1}} \psi(t,x) = \lim_{t \downarrow t_{i+1}} \psi(t,y) \ .$$

For $t \in (0,T]$ and $\bar{\mu} \in \mathbb{R}^{\mathcal{X}_t}$ exist unique $\mu : \mathcal{S}_{0,t} \to \mathbb{R}$ s.t. $\quad (\ \mu(s,\cdot) =: \hat{P}_{t,s}\bar{\mu} \ )$

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We have adjointness: $\langle P_{t,s}\psi, \mu \rangle = \langle \psi, \hat{P}_{t,s}\mu \rangle$

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Discrete caré du champs operators

For $\mu \in \mathcal{P}(\mathcal{X}_t)$ and $\psi \in \mathbb{R}^{\mathcal{X}_t}$ we define the integrated Γ-operators

$$\Gamma_t(\mu, \psi) := \langle \nabla\psi, \nabla\psi \Lambda_t(\mu) \rangle$$

where $\Lambda_t(\mu)(x,y) = \Lambda\big(\mu(x)Q_t(x,y), \mu(y)Q_t(y,x)\big)$, and

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These are discrete analogues of

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Note that

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Characterizations of discrete super Ricci flows

Theorem:

For a time-dependent Markov triple $(\mathcal{X}_t, Q_t, \pi_t)$ TFAE:

- Bochner inequality:

$$\Gamma_{2,t}(\mu,\psi) \geq \frac{1}{2}\partial_t\Gamma_t(\mu,\psi) \quad \text{a.e. } t, \forall \mu \in \mathcal{P}(\mathcal{X}_t), \psi \in \mathbb{R}^{\mathcal{X}_t} \ ,$$

- gradient estimate:

$$\Gamma_t(\mu, P_{t,s}\psi) \leq \Gamma_s(\hat{P}_{t,s}\mu, \psi) \quad \forall s \leq t, \mu \in \mathcal{P}(\mathcal{X}_t), \psi \in \mathbb{R}^{\mathcal{X}_s} \ ,$$

- transport estimate:

$$\mathcal{W}_t(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq \mathcal{W}_s(\mu,\nu) \quad \forall s \leq t, \mu, \nu \in \mathcal{P}(\mathcal{X}_t) \ ,$$

- dynamic convexity of entropy: for a.e. t and all $\mathcal{W}_t$-geodesics $(\mu^a)_{a\in[0,1]}$

$$\partial_a\Big|_{a=1-}\mathrm{Ent}_t(\mu^a) - \partial_a\Big|_{a=0+}\mathrm{Ent}_t(\mu^a) \geq -\frac{1}{2}\partial_t\mathcal{W}^2_{t-}(\mu^0,\mu^1) \ .$$

If these properties hold $(\mathcal{X}_t, Q_t, \pi_t)_t$ is called a super Ricci flow.

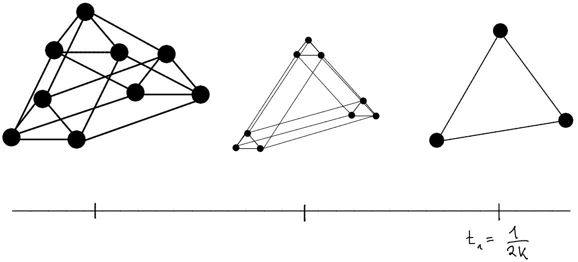
Examples

Collapse:

Let $(\mathcal{Y}, Q^Y, \pi^Y), (\mathcal{Z}, Q^Z, \pi^Z)$ be Markov triples with $\text{Ric}(\mathcal{Y}) \geq 0$, $\text{Ric}(\mathcal{Z}) \geq \kappa > 0$. Then the time-dependent triple

$$(\mathcal{X}_t, Q_t, \pi_t) := \begin{cases} (\mathcal{Y}, Q^Y, \pi^Y) \otimes (\mathcal{Z}, L_t Q^Z, \pi^Z) \,, & 0 \leq t < t_1 := 1/2\kappa \,, \\ (\mathcal{Y}, Q^Y, \pi^Y) \,, & t \geq t_1; \,, \end{cases}$$

with $L_t = 1/(1 - 2\kappa t)$ is a super Ricci flow.



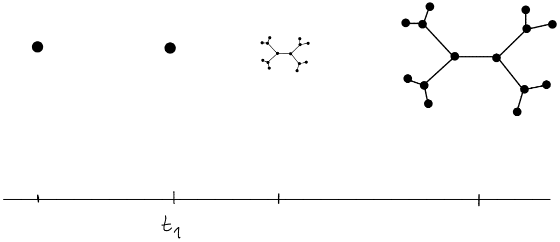
Examples

Explosion:

Let $(\mathcal{Y}, Q^Y, \pi^Y), (\mathcal{Z}, Q^Z, \pi^Z)$ be Markov triples with $\mathrm{Ric}(\mathcal{Y}) \geq 0$, $\mathrm{Ric}(\mathcal{Z}) \geq \kappa < 0$.
Then the time-dependent triple

$$(\mathcal{X}_t, Q_t, \pi_t) := \begin{cases} (\mathcal{Y}, Q^Y, \pi^Y) \ , & 0 \leq t \leq t_1 \ , \\ (\mathcal{Y}, Q^Y, \pi^Y) \otimes (\mathcal{Z}, L_t Q^Z, \pi^Z) \ , & t \geq t_1 \ , \end{cases}$$

with $L_t = -1/2\kappa(t - t_1)$ is a super Ricci flow.



Ideas for the proof

- Bochner inequality and gradient estimate: consider interpolation

$$\phi(r) = \mathcal{A}_r(\hat{P}_{t,r}\mu, P_{r,s}\psi)$$

Bochner $\Leftrightarrow$ $\phi'(r) \leq 0$ on (t_i, t_{i+1}); ϕ continuous across singular times

- gradient estimate and transport estimate: uses dual discrete transport problem

$$\mathcal{W}_t(\mu^0, \mu^1)^2 = \sup\left\{\langle \phi^1, \mu^1\rangle - \langle \phi^0, \mu^0\rangle \ : \ \langle \partial_a \phi^a, \nu\rangle + \frac{1}{2}\Gamma_t(\nu, \phi^a) \leq 0 \ \forall \nu\right\}$$

by gradient estimate, if ϕ^a is a HJ-subsolution at s, $P_{t,s}\phi^a$ is a HJ-subsol. at t

- from gradient estimates to dynamic convexity: core argument passing through dynamic evolution variational inequality

- from dynamic convexity to Bochner: one has that

$$\text{Hess}(\text{Ent}_t)(\mu^a)[\cdot] = \Gamma_{2,t}(\mu^a, \cdot)\ , \quad -\partial_t|\dot{\mu}^a|_t^2 = \partial_t\Gamma_t(\mu^a, \psi^a)$$

Stability of super Ricci flows

Let $(\mathcal{X}^n, Q_t^n, \pi_t^n)_t$ discrete time-dep. Markov triples that converge to time-dep. continuous mm-space $(X, d_t, m_t)_t$. More precisely:

there exist maps $i_n : \mathcal{P}(\mathcal{X}_n) \to \mathcal{P}(X)$ s.t.

- whenever $i_n(\mu^n) \to \mu$, $i_n(\nu^n) \to \nu$ we have for all t:

$$\mathrm{Ent}(\mu|\pi_t) \leq \liminf_n \mathrm{Ent}(\mu^n|\pi_t^n) , \qquad W_2(\mu,\nu) \leq \liminf_n \mathcal{W}^n(\mu^n,\nu^n) .$$

- for each μ, ν ex. $i_n(\mu^n) \to \mu$, $i_n(\nu^n) \to \nu$ s.t. for all t:

$$\mathrm{Ent}(\mu|\pi_t) = \lim_n \mathrm{Ent}(\mu^n|\pi_t^n) , \quad W_2(\mu,\nu) = \lim_n \mathcal{W}^n(\mu^n,\nu^n) .$$

Theorem:

If $(\mathcal{X}^n, Q_t^n, \pi_t^n)_t$ are discrete super Ricci flows, then (X, d_t, m_t) is a super Ricci flow in the sense of [STURM '16].

Conclusion

- Existence and uniqueness of solutions to the heat equation on time-dependent weighted graphs with time-dependent base set
- 4 equivalent characterizations of discrete super Ricci flows using heat flow and discrete optimal transport
- consistency with synthetic notion of super Ricci flow for mm-spaces

Further questions:

- How to characterize minimal discrete super Ricci flows?
- Can one construct such flows starting from a given Markov triple $(\mathcal{X}_0, Q_0, \pi_0)$?
- Existence/uniqueness of heat flow on singular continuous space-times?

Thank you for your attention!

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