Topological models of emergent dynamics

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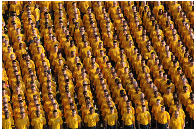
Examples of collective behavior

- Biology – swarming of insects, bird flocking, fish schools;

- Social science – opinion dynamics, social networks, economics
- Traffic dynamics, crowds, swarming of robots, material production, cosmology

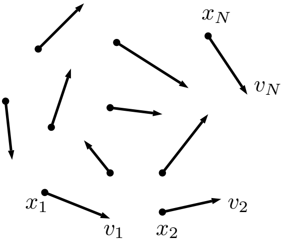


- Gossiping, phases of Tour de France...

Agent-based models of collective behavior describe dynamics of a number of objects:

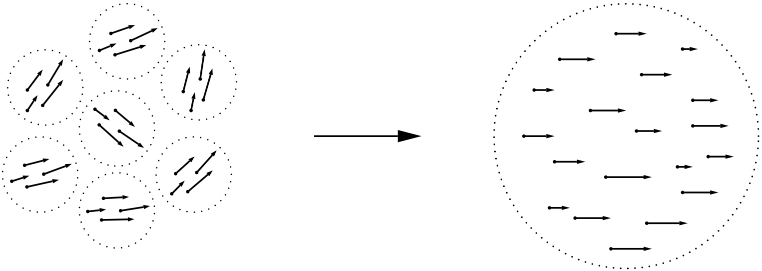
$$\mathbf{x}_i \in \Omega \subset \mathbb{R}^n, \quad i = 1, \dots, N$$
$$\mathbf{v}_i = \dot{\mathbf{x}}_i$$

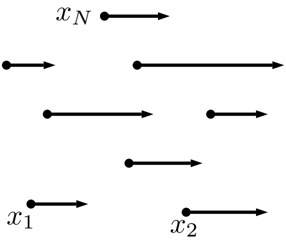
governed by mutual communication - adjustment of velocity or position to that of nearby neighbors.



Emergent dynamics

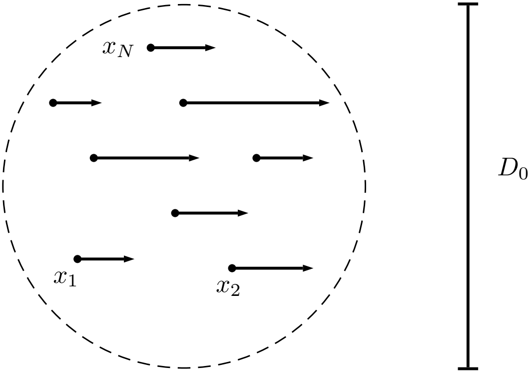
From local self-organization to global emergence:





The long time dynamics is characterized by three phenomena:

- *alignment*: $\lim_{t\to\infty}\max_i|\mathbf{v}_i-\bar{\mathbf{v}}|=0$,



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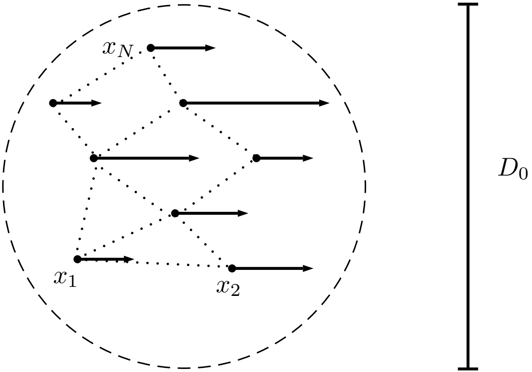

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- *flocking*: $\sup_{i,j} |\mathbf{x}_i - \mathbf{x}_j| \leq D_0 < \infty$,

- *strong flocking*: $\mathbf{x}_i - \mathbf{x}_j \to \mathbf{x}_{ij}$, as $t \to \infty$,

Environmental averaging models

Vicsek discrete model, 1995:

$$\begin{cases} \mathbf{v}_i(n+1) = v_0 \dfrac{\sum_{j:|x_j - x_i| < r_0} \mathbf{v}_j}{\left|\sum_{j:|x_j - x_i| < r_0} \mathbf{v}_j\right|} + \text{perturbation} \\ \mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{v}_i(n+1). \end{cases}$$

Kuramoto synchronization model, $\theta_i \in \mathbb{T}^1$:

$$\dot{\theta}_i = \frac{\lambda}{N} \sum_j \sin(\theta_j - \theta_i) + \omega_i.$$

Dynamic alignment:

$$\dot{\mathbf{p}}_i = \lambda \sum_{j \in N_i} a_{ij}(t)(\mathbf{p}_j - \mathbf{p}_i), \qquad \sum_j a_{ij}(t) = 1.$$

where N_i is a set of 'active' agents in local proximity, $\mathbf{p}_i \in \mathbb{R}^n$, e.g. $\mathbf{p} = \mathbf{x}$ or $\mathbf{p} = \mathbf{v}$.

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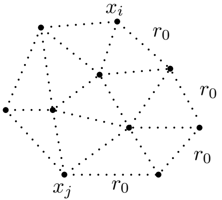

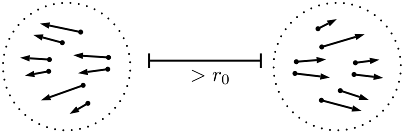

Alignment relies on propagation of connectivity:

- Analysis: Jadbabaie, Lin, Morse (2003)
- Numerics: by Gomez-Serrano, Graham, Le Boudec (2010),
- Analysis and Numerics: Motsch, Tadmor (2014).

Cucker-Smale model (2007)

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\lambda}{N} \displaystyle\sum_{j=1}^{N} \phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i), \end{cases} \qquad (\mathbf{x}_i, \mathbf{v}_i) \in \Omega \times \mathbb{R}^n \qquad (1)$$

Here, ϕ is a positive, non-increasing influence function which regulates communication between agents in Ω.

Theorem (Cucker, Smale)

Let $\phi(r) = \frac{H}{(1+r^2)^{\beta/2}}$. Then every solution aligns exponentially and flocks strongly for $\beta \leq 1$, and conditionally if $\beta > 1$.

S.-Y. Ha, J.-G. Liu (2009); E. Tadmor, C. Tan (2014): generally any kernel with "fat tail" $\int_0^\infty \phi(r) dr = \infty$ implies exponential alignment and strong flocking.

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ESA: Darwin mission

Status

Study ended in 2007

Objective

Darwin was a phase-A study performed by ESA. It studied a constellation of spacecraft to find Earth-like planets - the most likely places where life as we know it could develop. Darwin proposed to survey 1000 of the closest stars looking for small, rocky planets.

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Quest for locality: part I

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Alignment under the same conditions. No symmetry.

– S.-Y. Ha, J.-G. Liu (2009); Peszek (2014-15); Carrillo, Choi, Mucha, Peszek, (2017): singular communication kernel

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From microscopic to kinetic to macroscopic

S.-Y. Ha, E. Tadmor (2008); S.-Y. Ha , J.-G. Liu (2009); T. Karper, A. Mellet and K. Trivisa (2015): kinetic version of CS-MT model

$$\partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f,f) + \frac{1}{\varepsilon} \nabla_v \cdot ((u-v)f) = 0,$$

where

$$Q(f,f)(x,v,t) = \int_{\mathbb{R}^{2n}} \phi(|x-y|)(v_* - v) f(x,v,t) f(y,v_*,t) dv_* dy.$$

Evolution of macroscopic density and momentum

$$\rho(x,t) = \int_{\mathbb{R}^n} f(x,v,t) dv, \quad \rho u(x,t) = \int_{\mathbb{R}^n} v f(x,v,t) dv$$

can be derived from kinetic formulation via hydrodynamic limit $\varepsilon \to 0$ (Kang, Vasseur (2014); Figalli, Kang, (2018)):

$$f(x,v,t) \to \rho(x,t) \delta(v - u(x,t))$$

We obtain the following coupled system

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \displaystyle\int_{\mathbb{R}^n} \phi(|x-y|)(u(y,t) - u(x,t))\rho(y,t)\,\mathrm{d}y \end{cases}$$

$(x,t) : \mathbb{R}^n \times [0,\infty)$. The velocity equation is Burgers with commutator forcing:

$$u_t + u \cdot \nabla u = \mathcal{L}_\phi(\rho u) - \mathcal{L}_\phi(\rho)u$$

where

$$\mathcal{L}_\phi f = \int_{\mathbb{R}^n} \phi(|x-y|)(f(y) - f(x))dy, \quad \text{or} \quad \mathcal{L}_\phi f = \phi \star f.$$

All g.w.p. results are in 1D: the system in 1D has a special conserved quantity

$$e = u_x + \mathcal{L}_\phi \rho, \quad e_t + (ue)_x = 0.$$

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Logistic: $\frac{d}{dt}e = (\phi \star \rho - e)e.$

Theorem

Case of smooth $\phi > 0$ on $\mathbb{R}$ (Carrillo, Choi, Tadmor, Tan, 2014): If $e_0(x_0) < 0$, then blow-up. If $e_0 \geq 0$, then global solution $(u, \rho) \in W^{1,\infty} \times L^\infty$, and provided $\int_0^\infty \phi(r)dr = \infty$, then

$$|u(t) - \bar{u}|_\infty \leq Ce^{-\delta t}, \quad \operatorname{diam}_x \operatorname{supp} \rho(\cdot, t) \leq D_\infty < \infty,$$

Case of smooth $\phi > 0$ on $\mathbb{T}$ (Tadmor, S, 2017): if $e_0 > 0$ and $(u, \rho) \in W^{2,\infty} \times W^{1,\infty}$, then

$$|u(t) - \bar{u}|_\infty + |u_x|_\infty \leq Ce^{-\delta t}, \quad \rho(\cdot, t) \to \rho_\infty(\cdot - \bar{u}t).$$

In multi-D (Ha, Kang, Kwon, 2014; He, Tadmor, 2017): smooth solutions flock and align, but s.i.d. for global existence.

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Case of singular ϕ on $\mathbb{T}$: Let $0 < \alpha < 2$ on the periodic torus $\mathbb{T}$. For any non-vacuous initial condition $(u_0, \rho_0) \in H^4 \times H^{3+\alpha}$, $\exists!$ global solution:

$$|u(t) - \bar{u}|_\infty + |u_{xxxx}(t)|_2 \le Ce^{-\delta t}, \tag{2}$$

and there is exponential strong flocking towards $(\bar{u}, \bar{\rho})$, where and $\bar{\rho} = \rho_\infty(x - t\bar{u}) \in H^{3+\alpha}$,

$$|\rho(t) - \bar{\rho}(t)|_{H^{3+\alpha-\varepsilon}} \le Ce^{-\delta t}, \qquad t > 0. \tag{3}$$

In multi-D smooth solutions align as well, but s.i.d. for global existence (RS, 2018 (Hölder); Danchin,Mucha,Peszek,Wroblewski, 2018 (Besov)).

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The nature of the system is fractional parabolic:

$$\rho_t + u\rho_x + e\rho = \rho\Lambda_\alpha\rho$$
$$m_t + um_x + em = \rho\Lambda_\alpha m, \quad m = u\rho.$$

- u satisfies maximum principle.
- The e-quantity

$$e = u_x + \Lambda_\alpha\rho,$$

relates higher order terms while itself being of lower order. Indeed,

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{e}{\rho} = 0.$$

So, $|e| \leq C\rho$. One can lift this to higher order $|e^{(k)}| \leq C|\rho^{(k)}|$.
- If $0 < \alpha < 2$, the density remains bounded above and below uniformly in time:

$$c \leq \rho(x,t) \leq C$$

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Equations fall into a general class of forced fractional parabolic equations with L^∞ drift

$$v_t + b \cdot \nabla_x v = \int K(x,h,t)(v(x+h) - v(x))dh + f$$

where

$$K(x,h,t) = \rho(x) \frac{1}{|h|^{1+\alpha}}.$$

The kernel falls under Silverstre (2012): there exists a $\gamma > 0$ such that

$$|\rho, u|_{C^\gamma(\mathbb{T}\times[t+1,t+2))} \leq C(|\rho, u|_{L^\infty(t,t+2)} + |e|_{L^\infty(t,t+2)}).$$

Bootstrapping to higher classes via enhancement of dissipation (inhomogeneous Constantin-Vicol nonlinear maximum principle).

Theorem (T. Leslie, 2018)

Existence and uniqueness of solutions to forced system in classes

$$u, \rho, e \in L^\infty \text{ (weak) }; \qquad u, \rho, e \in W^{1,\infty} \text{ (strong)}$$

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Quest for Locality: part II

Suppose the agents can interact only locally (Vicsek protocol):

$$\phi(r) = \frac{1}{r^{n+\alpha}} \chi_{\{r<R_0\}}.$$

Static kernels may not define flocking dynamics:

- The bound of the density in 1D is only polynomial:

$$\rho(x,t) \gtrsim \frac{1}{1+t}.$$

- The best assumption under which alignment can be established is

$$\rho(x,t) \gtrsim \frac{1}{\sqrt{1+t}}.$$

Topological v.s. Metric Model

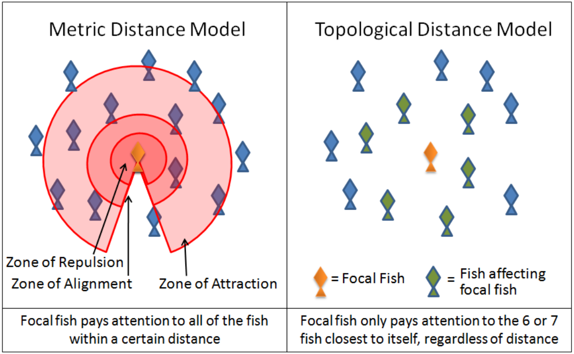

Figure: By Murphd84 - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=6524538

New communication protocol

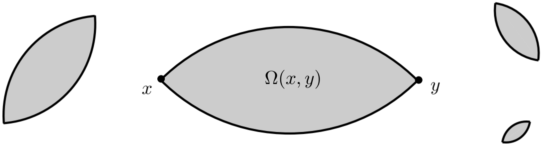

Figure: Communication domains between agents

1. Every agent x_i has a finite influence range, $B(x_i, r_0)$.
2. Agent x_i influences agent x_j via intermediaries in domain Ω_{x_i,x_j}
3. The quantity $m_{ij} = \sum_{k:\, x_k \in \Omega_{ij}} m_k$ measures collective power of the intermediaries.

Based on the outlined principles, we make the following choice:

$$\phi_{ij}(x) = \frac{1}{m_{ij}^{\tau}} \psi(|x_i - x_j|), \tag{4}$$

$$\begin{cases} \dot{x}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \lambda \sum_{j=1}^{N} m_j \phi_{ij}(x)(\mathbf{v}_j - \mathbf{v}_i). \end{cases} \tag{5}$$

In the macroscopic approximation, m_{ij} is replaced by "mass"

$$d(x,y,t) = \left[\int_{\Omega(x,y)} \rho(z,t)\, \mathrm{d}z \right]^{\frac{1}{n}}.$$

The kernel adopts a hybrid metric and topological distance:

$$\phi(x,y) = \frac{1}{d(x,y,t)^{\tau}|x-y|^{n+\alpha-\tau}} \chi_{|x-y|<R_0}, \text{ where } \tau > 0.$$

The hydrodynamic topological model reads

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \int_{\mathbb{R}} \phi(x,y)(u(y,t) - u(x,t))\rho(y,t)\, \mathrm{d}y \end{cases}$$

Theorem (Strong solutions align. Tadmor, RS (2018))

Let (u,ρ) *be a global smooth solution to the topological model on* $\mathbb{T}^n$ *with* $\tau \geq n$ *and*

$$\rho(x,t) \gtrsim \frac{1}{1+t}. \tag{6}$$

Then

$$|u(t)-\bar{u}|_\infty \to 0.$$

- Why $\tau \geq n$?

$$\lim_{\alpha\to 2}(2-\alpha)[\mathcal{L}_\phi(\rho u) - u\mathcal{L}_\phi(\rho)] = \frac{1}{\rho^{\gamma-1}}\Delta u + \frac{2-\gamma}{\gamma}\nabla u \nabla \rho, \qquad \gamma = \frac{\tau}{n}.$$

- In 1D, the density bound holds *automatically* due to a remarkable "survival" of the e-quantity:

$$e = u_x + \mathcal{L}_\phi \rho; \quad e_t + (ue)_x = 0.$$

$$\rho_t \geq -C\rho^2.$$

Hence, (6).

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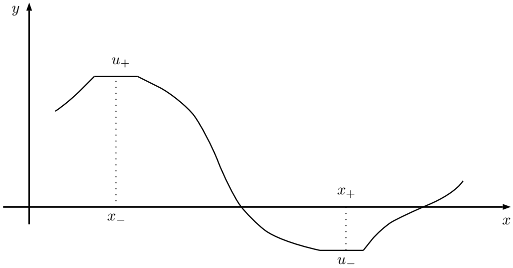
Elements of the proof



Let us fix a $\delta > 0$ and consider the sets

$$G_\delta^+(t) = \{y : u(y,t) < u_+(1-\delta)\}$$
$$G_\delta^-(t) = \{y : u(y,t) > u_-(1+\delta)\}$$

These sets are small that w.r.t to density measure $\mathrm{d}m_t = \rho\,\mathrm{d}x$

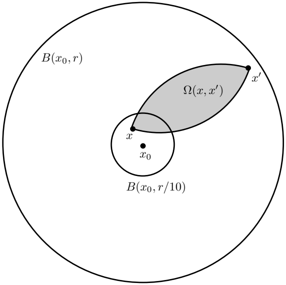
$$\int_0^\infty \mathbb{E}_t(G_\delta^\pm(t)|B(x_\pm, r))\,\mathrm{d}t < \infty. \tag{7}$$

We consider the averages with respect to m_t-measure:

$$u_{x,r} = \frac{1}{m_t(B(x,r))} \int_{B(x,r)} u(z,t)\, \mathrm{d}m_t(z).$$

We use the weighted Campanato semi-norm:

$$[u]_\rho^2 = \sup_{x_0, r<r_0} \int_{|x-x_0|<r/10} |u(x) - u_{x_0,r}|^2 \rho(x)\, \mathrm{d}x.$$

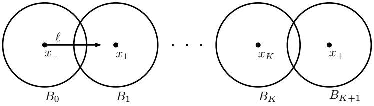


Crucial estimate:

$$[u]_\rho^2 \le \int_{\substack{|x-x_0|<\frac{1}{10}r \\ |x'-x_0|<r}} \frac{1}{m_t(B(x_0,r))} |u(x)-u(x')|^2 \rho(x)\rho(x')\, \mathrm{d}x'\, \mathrm{d}x$$

$$\le \int_{|x-x'|<\frac{11}{10}r} \frac{1}{d^n(x,x')} |u(x)-u(x')|^2 \rho(x)\rho(x')\, \mathrm{d}x'\, \mathrm{d}x$$

$$\le \int_{|x-x'|<\frac{11}{10}r} \phi(d(x,x'),x,x') |u(x)-u(x')|^2 \rho(x)\rho(x')\, \mathrm{d}x'\, \mathrm{d}x$$



Combined flattening-enstrophy bound:

$$\int_0^\infty (\mathbb{E}_t(G_\delta^\pm(t)|B(x_\pm,r)) + [u(t)]_\rho^2)\,\mathrm{d}t < \infty.$$

Thus, for any $\varepsilon > 0$ there exists a time $t > 0$ as large as we like so that

$$\mathbb{E}_t(G_\delta^\pm(t)|B(x_\pm,r)) + [u(t)]_\rho^2 < \frac{\varepsilon}{t}. \qquad (8)$$

Hence

$$\sup_{x_0, r<r_0} \int_{|x-x_0|<r/10} |u(x) - u_{x_0,r}|^2\,\mathrm{d}x < \varepsilon$$

Slide x_0 from $x_+(t)$ to $x_-(t)$ in finite number of steps, hence

$$|u_+ - u_-| < \varepsilon + \delta$$

Theorem (Global existence on $\mathbb{T}^1$. Tadmor, RS (2018))

Let $0 < \alpha < 2$, and suppose a given initial condition $u_0 \in H^4$, $\rho_0 \in H^{3+\alpha/2}$ satisfies the following assumptions:

(i) *no vacuum $c_0 < \rho_0(x) < C_0$, and*

(ii) *if $\tau > \alpha$ then, in addition, $\mathcal{M}^\tau \left|\frac{e_0}{\rho_0}\right|_\infty < \frac{R_0^{\tau-\alpha}}{\tau-\alpha}$.*

Then there exists a global in time solution to the (τ, α)-model in the same class.

- Reduces to propagation of a modulus continuity of ρ:

$$\rho_t + u\rho_x + e\rho = \mathcal{L}_K[\rho].$$

where

$$K(x,y,t) = \frac{\rho(x)}{d(x,y,t)^\tau |x-y|^{n+\alpha-\tau}} \chi_{|x-y|<R_0}$$

$$\frac{\chi_{|x-y|<R_0}}{\Lambda |x-y|^{n+\alpha}} \leq K(x,y,t) \leq \frac{\Lambda}{|x-y|^{n+\alpha}}.$$

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- For $1 < \alpha < 2$, we apply fractional Schauder estimates:

$$e = u_x + \mathcal{L}_\phi \rho, \quad \partial_x^{-1} \mathcal{L}_\phi \rho = -u + \partial_x^{-1} e \in L^\infty.$$

$$\int [f(x+z) + f(x-z) - 2f(x)] H(x,z,t) \, \mathrm{d}z = g(x) \in L^\infty,$$

where $f = \partial_x^{-1}(\rho - \mathcal{M})$. Schauder (Tianling Jin and Jingang Xiong, 2015) implies $f \in C^{1+\gamma}$.

- For $0 < \alpha < 1$, we apply Silvestre regularization theorem.

$$u = -\partial_x^{-1} \mathcal{L}_\phi \rho + \partial_x^{-1} e \in C^{1-\alpha}.$$

- For $\alpha = 1$:

$$\rho_t + u\rho_x + e\rho = \rho \mathcal{L}_{\mathrm{sym}}[\rho],$$
$$\omega_t + u\omega_x + e = \mathcal{L}_{\mathrm{sym}}[e^\omega], \quad \omega = \log \rho.$$

DeGiorgi method in nonlinear forced settings (Caffarelli, Vasseur (2008) SQG; Caffarelli, Chen, Vasseur (2011) with symmetry; Vazquez, de Pablo, Rodriguez, Quiros (2017) nonlinear but no drift; Golse, Imbert, Mouhot, Vasseur (2018), kinetic Fokker-Plank).

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Theorem (Unconditional flocking for 1D local topological kernels)

Consider the one-dimensional system on $\mathbb{T}$ *with local* (τ, α)*-kernel with topological singularity of order* $1 \leq \tau \leq \alpha < 2$*. Then any non-vacuous smooth initial data* $\rho_0 > 0, u_0$ *gives rise to a unique global solution which aligns*

$$|u(t, \cdot) - u_\infty|_\infty \to 0.$$

Thank you!