Scalings and saturation in infinite-dimensional control problems with applications to SPDEs

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Regularity and Blow-up of Navier-Stokes Type PDEs using Harmonic and Stochastic Analysis
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Outline of the Talk

I Control and Stochastic PDEs.
II Scaling Arguments.
III Subsumption and Saturation.
VI Applications.

Collaborators

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Part I:

Control and Stochastic PDEs.

Degenerate Stochastic PDEs

$$d\boldsymbol{u} + (L\boldsymbol{u} + N(\boldsymbol{u}))dt = \sum_{k=1}^{d} \sigma_k dW^k, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in H \tag{1}$$

- L linear, unbounded. N multi-linear.
- dW^k i.i.d. gaussian white noise. $\sigma_k \in H$. $d << \infty$ ('degenerate noise').

Basic Questions

(i) Robust observability of statistics (Unique ergodcity of invariant measures).
(ii) Realizable outcomes (Support Properties).

Model equations: Navier-Stokes, Boussinesq, Korteweg-de Vries (KdV).

Physical motivations: Stochastic forcing in (1) models large scale stirring driving turbulent flow.

The Low Mode Control Problem

$$\frac{d\boldsymbol{u}}{dt} + L\boldsymbol{u} + N(\boldsymbol{u}) = \sum_{k=1}^{d} \alpha_k(t)\sigma_k \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in H \tag{2}$$

- L linear, unbounded. N multi-linear. $\sigma_k \in H$. $d << \infty$.
- $\boldsymbol{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_d(t))$ actuators (replace white noise).

Goal: Characterize the accessibility sets

$$\mathcal{A}(\boldsymbol{u}_0, T) := \{\boldsymbol{u}(T, \boldsymbol{u}_0, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \text{ piecewise continuous}\}. \tag{3}$$

- $\mathcal{A}(\boldsymbol{u}_0, T)$ has basic implication for the SPDE associated to (2).
- Interactions between forcing and non-linearity N.

The Markovian Framework

$$du + F(u)dt = \sigma dW = \sum_{k=1}^{d} \sigma_k dW^k, \quad u(0) = u_0 \in H. \tag{4}$$

Markov transition functions: $P_t(u_0, A) = \mathbb{P}(u(t, u_0) \in A)$, $u_0 \in H$, $A \in \mathcal{B}(H)$

$$P_t\phi(u_0) := \int_H \phi(u) P_t(u_0, du) = \mathbb{E}\phi(u(t, u_0)); \quad (\phi : H \to \mathbb{R}),$$

$$\mu P_t(A) := \int_H P_t(u, A) d\mu(u); \quad (\mu \text{ probability measure on H}),$$

evolving observables and probability laws.

$\mu \in \Pr(H)$ is an Invariant Measure (IM) if

$$\mu P_t = \mu \quad \text{for all } t \geq 0.$$

Unique Ergodicity

$$d\boldsymbol{u} + F(\boldsymbol{u})dt = \sigma dW, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in H, \quad P_t(\boldsymbol{u}_0, A) = \mathbb{P}(\boldsymbol{u}(t, \boldsymbol{u}_0) \in A)$$

$$P_t\phi(\boldsymbol{u}_0) := \mathbb{E}\phi(\boldsymbol{u}(t, \boldsymbol{u}_0)), \quad \mu P_t(A) := \int_H \mathbb{P}(\boldsymbol{u}(t, \boldsymbol{u}_0) \in A) d\mu(\boldsymbol{u}_0).$$

Existence/Uniqueness/Attractivity of Invariant Measures (IM)

(i) **Smoothing properties of P_t**: (Hypo)ellipticity of the Kolmogorov Equation? Recall that $V(t, u) := P_t\phi(u)$ solves

$$\partial_t V = \frac{1}{2} Tr[(\sigma\sigma^*) D^2 V] - \langle F(u), DV \rangle, \quad V(0) = \phi.$$

(ii) **Irreducibility**: Common states $\boldsymbol{v}^*$ can be reached by the dynamics

$$\inf_{\boldsymbol{u} \in B(M, \boldsymbol{v}^*)} P_t(\boldsymbol{u}, B(\epsilon, \boldsymbol{v}^*)) > 0 \quad \text{for all } M > 0, \epsilon > 0.$$

(iii) **Lyupunov Structure**: There is a $\mathfrak{L} : H \to \mathbb{R}^+$ w/ $\mathfrak{L}(\boldsymbol{u}) \to \infty$ as $\boldsymbol{u} \to \infty$ s.t.

$$P_t\mathfrak{L} \leq f(t)\mathfrak{L} + C \quad \text{with } f(t) \to 0 \text{ as } t \to \infty.$$

- (i), (ii) are reducible geometric to control problems.
- References: Doeblin (30's), Doob-Khasminskii (40, 60's), Harris (50's), Hairer-Mattingly (00's).

Support of a Borel measure

Given $\mu \in \mathrm{Pr}(H)$, $\mathrm{supp}(\mu) = \{\boldsymbol{u} \in H : \mu(B(\boldsymbol{u}, \epsilon)) > 0, \text{ for every } \epsilon > 0\}$.

$$d\boldsymbol{u} + F(\boldsymbol{u})dt = \sigma dW, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in H, \quad P_t(\boldsymbol{u}_0, A) = \mathbb{P}(\boldsymbol{u}(t, \boldsymbol{u}_0) \in A) \tag{5}$$

$$\frac{d\boldsymbol{v}}{dt} + F(\boldsymbol{v}) = \boldsymbol{\alpha} \cdot \sigma, \quad \boldsymbol{v}(T, \boldsymbol{u}_0, \alpha) = \boldsymbol{u}(T, \boldsymbol{u}_0, \int_0^{\cdot} \alpha).$$

Theorem: Controllability ⇒ Support (Stroock-Varadhan)

(i) $\mathrm{supp}(P_T(\boldsymbol{u}_0, \cdot)) = \overline{\mathcal{A}(\boldsymbol{u}_0, T)} := \overline{\{\boldsymbol{v}(T, \boldsymbol{u}_0, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \text{ piecewise continuous}\}}$.

(ii) Suppose $\overline{\mathcal{A}(\boldsymbol{u}_0, T)} = H$ then $\mathrm{supp}(\mu) = H$ for every IM of (5).

Support of a Borel measure

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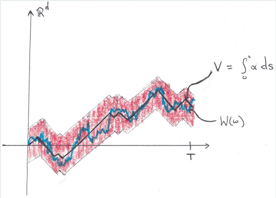
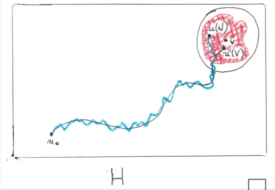
(ii) Suppose $\overline{\mathcal{A}(\boldsymbol{u}_0, T)} = H$ then $\text{supp}(\mu) = H$ for every IM of (5).

Proof (additive noise ⇒ $W \mapsto \boldsymbol{u}(T, \boldsymbol{u}_0, W)$ continuous.).

$\Leftarrow$ Fix $\boldsymbol{v} \in \overline{\mathcal{A}(\boldsymbol{u}_0, T)}$, $\epsilon > 0$.
Find $V = \int_0^{\cdot} \alpha ds$ & $\delta > 0$...
$\mathbb{P}(\sup_{t \in [0,T]} |V(t) - W(t)| < \delta) > 0$

$\Rightarrow$ Fix $\boldsymbol{v} \in \text{supp}(P_T(\boldsymbol{u}_0, \cdot))$, $\epsilon > 0$. Find $W(\omega)$ s.t.
$\|\boldsymbol{u}(T, \boldsymbol{u}_0, W(\omega)) - \boldsymbol{v}\| < \epsilon/2 \ldots$





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Overview and Previous Work

Geometric Control in Infinite Dimensions

Scaling: Short powerful burst controls yield dynamics following rays.

Saturation: Accounting framework which sidesteps multiple time scales and other nightmares.

Algebraic Conditions: Hormander type algebraic conditions.

Applications: Ergodicity and support properties for degenerate stochastic KdV & Boussinesq, Prevention of blowup for 3D Euler.

Previous Work

- Jurdevic-Kupka– Geometric control in finite dimensions.
- Agrachev-Sarychev's Approach. See also Shirikyan, Nersisyan, Nersesyan.
- Bracket Analysis: E-Mattingly, Mattingly-Hairer, Romito, GH-Foldes-Richards-Thomann.
- Non-degenerate noise: DaPrato-Zabczyk, Flandoli-Maslowski.
- ∞-Dim Malliavin Calculus: Hairer-Mattingly, Mattingly-Pardoux.

Part II:

Scaling Arguments

Directly Controlled Modes, Ray Semigroups

$$\frac{d}{dt}\boldsymbol{u} + L\boldsymbol{u} + N(\boldsymbol{u}) = \boldsymbol{h}, \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{defining } \Phi_t^{\boldsymbol{h}}\boldsymbol{u}_0 := \boldsymbol{u}(t, \boldsymbol{u}_0, \boldsymbol{h})$$

$$\frac{d}{dt}\boldsymbol{v} = g, \boldsymbol{v}(0) = \boldsymbol{v}_0 \qquad \text{defining } \rho_t^{\boldsymbol{g}}\boldsymbol{u}_0 := \boldsymbol{v}(t, \boldsymbol{u}_0, \boldsymbol{g})$$

Scaling to a Ray: Introduce a parameter $\lambda >> 0$

$$\boldsymbol{v}_\lambda(t) := \Phi_{t/\lambda}^{\lambda \boldsymbol{h}}\boldsymbol{u}_0 \quad \text{for } \boldsymbol{h} \in H,$$

solves

$$\frac{d}{dt}\boldsymbol{v}_\lambda + \frac{1}{\lambda}(L\boldsymbol{v}_\lambda + N(\boldsymbol{v}_\lambda)) = h, \;\; \boldsymbol{v}_\lambda(0) = \boldsymbol{u}_0$$

Subject to suitable estimates we expect

$$\lim_{\lambda \to \infty} \|\Phi_{t/\lambda}^{\lambda \boldsymbol{h}}\boldsymbol{u}_0 - \rho_t^{\boldsymbol{h}}\boldsymbol{u}_0\| = 0$$

The 'Nonlinear-Twist'

$$\frac{d}{dt}\boldsymbol{u} + L\boldsymbol{u} + N(\boldsymbol{u}) = \boldsymbol{h}, \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{defining } \Phi_t^h \boldsymbol{u}_0 := \boldsymbol{u}(t, \boldsymbol{u}_0, \boldsymbol{h})$$

$$\frac{d}{dt}\boldsymbol{v} = \boldsymbol{g}, \boldsymbol{v}(0) = \boldsymbol{v}_0 \qquad \text{defining } \rho_t^{\boldsymbol{g}} \boldsymbol{u}_0 := \boldsymbol{v}(t, \boldsymbol{u}_0, \boldsymbol{g})$$

Accentuating 'Resonant' Terms in N:

$$\boldsymbol{w}_\lambda(t) = \rho_{1/\lambda}^{-\lambda^m \boldsymbol{g}} \Phi^0_{t/\lambda^m} \rho_{1/\lambda}^{\lambda^m \boldsymbol{g}} \boldsymbol{u}_0$$

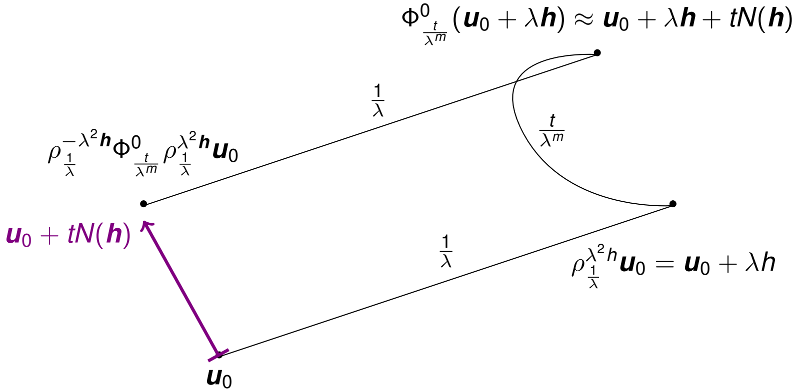
which solves

$$\frac{d}{dt}\boldsymbol{w}_\lambda + \frac{1}{\lambda^m}(L(\boldsymbol{w}_\lambda + \lambda \boldsymbol{g}) + N(\boldsymbol{w}_\lambda + \lambda g)) = 0, \;\; \boldsymbol{w}_\lambda(0) = \boldsymbol{u}_0$$

Recalling that N is m-multi-linear $N(\lambda \boldsymbol{g}) = \lambda^m N(\boldsymbol{g})$ we expect

$$\lim_{\lambda \to \infty} \|\rho_{1/\lambda}^{-\lambda^2 \boldsymbol{g}} \Phi^0_{t/\lambda^m} \rho_{1/\lambda}^{\lambda^2 \boldsymbol{g}} \boldsymbol{u}_0 - \rho_t^{N(\boldsymbol{g})} \boldsymbol{u}_0\| = 0$$

The ‘Nonlinear Twist’



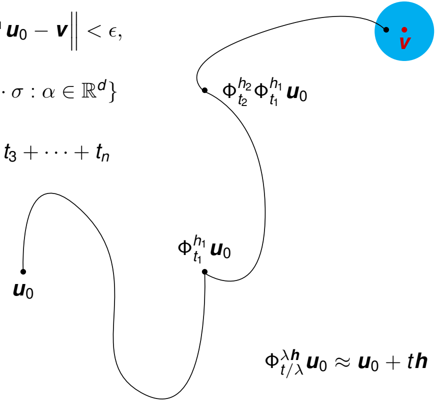
The Goal: In Pictures.

$$\frac{d}{dt}\boldsymbol{u} + L\boldsymbol{u} + N(\boldsymbol{u}) = \boldsymbol{h}, \quad \Phi_t^{\boldsymbol{h}}\boldsymbol{u}_0 := \boldsymbol{u}(t, \boldsymbol{u}_0, \boldsymbol{h}).$$

$$\left\| \Phi_{t_n}^{\boldsymbol{h}_n} \cdots \Phi_{t_2}^{\boldsymbol{h}_2} \Phi_{t_1}^{\boldsymbol{h}_1} \boldsymbol{u}_0 - \boldsymbol{v} \right\| < \epsilon,$$

$$\boldsymbol{h}_j \in \mathcal{F}_0 := \{\alpha \cdot \sigma : \alpha \in \mathbb{R}^d\}$$

$$T = t_1 + t_2 + t_3 + \cdots + t_n$$



$$\Phi_{t/\lambda}^{\lambda \boldsymbol{h}} \boldsymbol{u}_0 \approx \boldsymbol{u}_0 + t\boldsymbol{h} \quad \text{when } \lambda \gg 0$$

$$\Phi_{\frac{1}{\lambda\mu}}^{-\mu\lambda^2 \boldsymbol{h}} \Phi_{\frac{1}{\lambda^m}}^{0} \Phi_{\frac{1}{\lambda\mu}}^{\mu\lambda^2 \boldsymbol{h}} \boldsymbol{u}_0 \approx \boldsymbol{u}_0 + tN(\boldsymbol{h}) \quad \text{when } \mu \gg \lambda \gg 0$$

Outlook

$$\mathcal{B}_0 = \text{span}\Big\{\sigma_1, \dots, \sigma_d\Big\},\ \mathcal{B}_n = \text{span}\Big\{\mathcal{B}_{n-1} \cup \{N(\boldsymbol{h}) : \boldsymbol{h} \in \mathcal{B}_{n-1}\}\Big\},\ \mathcal{B}_\infty = \overline{\bigcup_{n\geq 0} \mathcal{B}_n}.$$

We might expect

$$\boldsymbol{u}_0 + \mathcal{B}_\infty \subseteq \overline{\mathcal{A}(\boldsymbol{u}_0, T)} = \overline{\{\Pi_{j=1}^n \Phi_{t_j}^{\alpha_j \cdot \sigma} \boldsymbol{u}_0 : \alpha_j \in \mathbb{R}^d, t_1 + \cdots + t_n = T\}}$$

Complications

- Multi-scale nightmare: (μ, λ) for $\mathcal{B}_1$ becomes $(\mu_1, \dots, \mu_{k(n)}, \lambda)$ for $\mathcal{B}_n$.
- Arguments for relaxed time. Small time to fixed time $T > 0$?
- What's in $\mathcal{B}_\infty$?
- We need to be able to flow forwards & backwards along $N(\boldsymbol{h})$ for $\boldsymbol{h} \in \mathcal{B}_n$.
- Rigorous bounds to justify approximations.

Part III:

Subsumption, Saturation.

Saturation Formalism

Let $\mathcal{S}$ be the continuous (local) semi-groups on a phase space H. $\mathcal{F}, \mathcal{G} \subseteq \mathcal{S}$.

Definition: Accessible Sets, The Saturate

$$\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, \leq T) = \bigcup_{t \in (0,T]} \{\Phi^n_{t_n} \cdots \Phi^1_{t_1} \boldsymbol{u} : \Phi^j \in \mathcal{F}, \sum_j t_j = t\} \quad \text{(Accessible Sets)}$$

Subsumption: $\mathcal{G} \preccurlyeq \mathcal{F}$ if $\overline{\mathcal{A}_{\mathcal{G}}(\boldsymbol{u}, \leq T)} \subseteq \overline{\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, \leq T)}$, for all $\boldsymbol{u} \in H$, $T > 0$.

Equivelence: $\mathcal{G} \sim \mathcal{F}$ if $\mathcal{G} \preccurlyeq \mathcal{F}$ and $\mathcal{F} \preccurlyeq \mathcal{G}$

Saturate: $\text{Sat}(\mathcal{F}) := \bigcup_{\mathcal{G} \preccurlyeq \mathcal{F}} \mathcal{G}$

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The Saturation Theorem (Sidesteps the Multi-scale Nightmare!)

(i) $\mathcal{G} \preccurlyeq \mathcal{F} \iff$ For $\Psi \in \mathcal{G}$, ϵ, $T > 0$,$\boldsymbol{u}_0 \in H$ there exists $\Phi^j \in \mathcal{F}$, $t_j > 0$, s.t.

$$\|\Psi \boldsymbol{u}_0 - \Phi^n \cdots \Phi^1 \boldsymbol{u}_0\| < \epsilon, \quad t_1 + \cdots + t_n \leq T.$$

(ii) $\mathcal{F} \sim \text{Sat}(\mathcal{F})$

Relaxed Accessibility to Exact Time

$$\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, \leq T) = \bigcup_{t \in (0,T]} \mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, t) = \bigcup_{t \in (0,T]} \{\Phi^n_{t_n} \cdots \Phi^1_{t_1} \boldsymbol{u} : \Phi^j \in \mathcal{F}, \sum_j t_j = t\}$$

Scaling arguments only identify $\boldsymbol{u} \in \overline{\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, \leq T)}$.

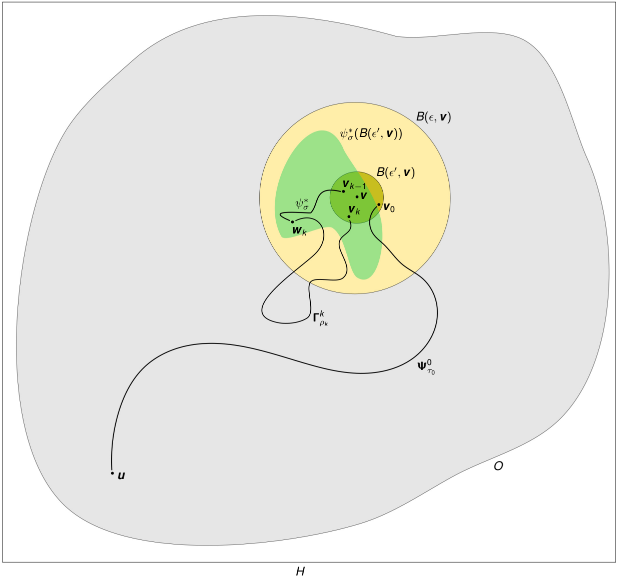
Conversion Lemma ($\boldsymbol{u} \in \overline{\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, \leq T)} \Rightarrow^{???} \boldsymbol{u} \in \overline{\mathcal{A}_{\mathcal{F}}(\boldsymbol{u}, T)}$)

Let $\mathcal{F}$ be a collection of continuous semigroups and $O \subseteq H$, open. Then

$$O \subseteq \overline{A_{\mathbf{F}}(\boldsymbol{u}, \leq T)} \quad \Rightarrow \quad O \subseteq \overline{A_{\mathbf{F}}(\boldsymbol{u}, T)}.$$

Corollary

If $\overline{\mathcal{A}_{\text{Sat}(\mathcal{F})}(\boldsymbol{u}, \leq T)} = H$ then $\mathcal{F}$ is approximately controllable ($A_{\mathbf{F}}(\boldsymbol{u}, T) = H$).



- Fix $\boldsymbol{u} \in O$, $T > 0$, $\epsilon > 0$.
- Pick any $\psi^* \in \mathcal{F}$ and then $\epsilon' > 0$ such that
  $$\sigma := \inf_{\substack{\tilde{v} \in B(\boldsymbol{v}, \epsilon') \\ s > 0}} \left\{ d(\psi_s^* \tilde{\boldsymbol{v}}, \boldsymbol{v}) > \epsilon \right\}$$
  is strictly positive.
- Find $\tau_0 \leq T$ and
  $$\boldsymbol{\Phi}^0 := \Phi^{1,0} \cdots \Phi^{n,0} \in \mathcal{F}$$
  such that $d(\boldsymbol{\Phi}^0_\tau \boldsymbol{u}, v) < \epsilon'$.
- Pick n such that
  $$\tau_0 + n\sigma \leq T < \tau_0 + (n+1)\sigma.$$
- Inductively, let $\rho_k \leq \frac{T - \tau_0 + n\sigma}{n}$
  $$\boldsymbol{\Gamma}^k = \Gamma^{k,1} \cdots \Gamma^{k,n_k} \in \mathcal{F}$$
  s.t.
  $$d(\boldsymbol{\Gamma}^k_{\rho_k} \phi_\sigma^* \boldsymbol{v}_{k-1}, \boldsymbol{v}) < \epsilon'.$$

More Refined Controls.

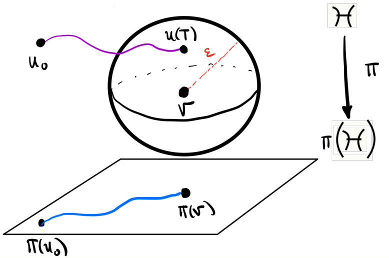
Exact Control on Finite Dimensional Projections π?

For $\boldsymbol{u}, \boldsymbol{v} \in H$, $T, \epsilon > 0$ does there exist $\Phi^1, \ldots, \Phi^n \in \mathcal{F}$ such that

$$\|\Phi^1 \cdots \Phi^n \boldsymbol{u} - \boldsymbol{v}\| < \epsilon$$

and

$$\pi(\Phi^1 \cdots \Phi^n \boldsymbol{u}) = \pi(\boldsymbol{v})?$$



Uniform Saturation

$$\mathfrak{G} = \{\Phi : [0,\infty) \times H \times Z \to H | \Phi \text{ continuous}, \Phi_0^{\boldsymbol{g}} \boldsymbol{u} = \boldsymbol{u}, \Phi_{t+s}^{\boldsymbol{g}} \boldsymbol{u} = \Phi_t^{\boldsymbol{g}} \Phi_s^{\boldsymbol{g}} \boldsymbol{u}\}$$

Definitions: $\mathfrak{G}, \mathfrak{F} \subset \mathfrak{S}$

(Subsumption) $\mathfrak{G} \preccurlyeq_u \mathfrak{F}$, if $\forall \Psi \in \mathfrak{G}$, compact $K_I \subseteq H$, $K_p \subseteq \mathcal{P}(\Psi)$, ϵ, $T > 0$, there exist $\Phi^1, \dots, \Phi^n \in \mathfrak{F}$, $t_1, \dots, t_n > 0$, continuous $f_i : K_p \to \mathcal{P}(\Phi^i)$ s.t.

$$\sup_{u \in K_I, p \in K_p} \left\| \prod_{i=1}^{n} \Phi_{t_i}^{i, f_i(p)} u - \Psi_T^p u \right\| < \epsilon, \quad \sum t_i \leq T. \tag{6}$$

(Saturation) $\mathrm{Sat}_u(\mathfrak{F}) = \bigcup_{\mathcal{G} \preccurlyeq_u \mathcal{F}} \mathfrak{G}$

- As above $\mathrm{Sat}_u(\mathfrak{F}) \sim_u \mathfrak{F}$.
- To obtain (6) we need estimates like

$$\lim_{\lambda \to \infty} \sup_{\substack{\boldsymbol{h} \in K, \\ \boldsymbol{u}_0 \in \tilde{K}}} \|\Phi_{t/\lambda}^{\lambda \boldsymbol{h}} \boldsymbol{u}_0 - \rho_t^{\boldsymbol{h}} \boldsymbol{u}_0\| = 0 \quad \lim_{\lambda \to \infty} \sup_{\substack{\boldsymbol{g} \in K, \\ \boldsymbol{u}_0 \in \tilde{K}}} \|\rho_{1/\lambda}^{-\lambda^2 \boldsymbol{g}} \Phi_{t/\lambda^m}^0 \rho_{1/\lambda}^{\lambda^2 \boldsymbol{g}} \boldsymbol{u}_0 - \rho_t^{N(\boldsymbol{g})} \boldsymbol{u}_0\| = 0$$

- Morally speaking $\mathrm{Sat}_u(\mathfrak{F}) = H$ yields results for exact control on finite dimensional projections with Brouwer fixed point arguments.

Positivity of the Density on Finite-Dim Projections

Let $\Omega = C_0((-\infty, \infty); \mathbb{R}^d)$ and $\Phi : [0, \infty) \times H \times \Omega \to H$ be a (continuous, markov) cocycle:

$$\phi_{t+s}(\boldsymbol{u}, V) = \phi_t(\phi_s(\boldsymbol{u}, V), \theta_s V), \quad \phi_0(\boldsymbol{u}, V) = \boldsymbol{u} \tag{7}$$

where $\theta_s V(t) := V(t+s) - V(s)$. The associated Malliavin Matrix is given as

$$M_t(\boldsymbol{u}, V) := D_V \phi_t(\boldsymbol{u}, V)(D_V \phi_t(\boldsymbol{u}, V))^* \tag{8}$$

Theorem (GH-Herzog-Mattingly '17, Mattingly-Pardoux '06)

Let $\pi : H \to \mathbb{R}^m$ be a projection and assume

(i) For all $\boldsymbol{u}, \boldsymbol{v} \in H$, $t > 0$, there exists $V \in \Omega$ s.t. $\pi(\boldsymbol{v}) = \pi(\phi_t(\boldsymbol{u}, V))$.

(ii) For $t > 0$, $\mathbb{P}(\langle M_t(u, W)\xi, \xi\rangle$ for all $\xi \in H \setminus \{0\}) = 1$

Then $\pi(\phi_t(\boldsymbol{u}, W))$ is continuously distributed on $\mathbb{R}^m$ w/ an a.e. positive density.

Remarks on the Malliavin Matrix

Given a cocycle $\phi_t(\boldsymbol{u}_0, W)$ we defined $M_t(\boldsymbol{u}, W) := D_W\phi_t(\boldsymbol{u}, W)(D_W\phi_t(\boldsymbol{u}, W))^*$.

Full Rank Tangent Spaces

- Exact controllability on $\pi(H) \Rightarrow$ for every $x \in \mathbb{R}^m$ there exists $V_x \in \Omega$ such that $x = \pi(\phi_t(\boldsymbol{u}, V_x))$.
- For any Cameron-Martin perturbation H
$$\pi(\phi_t(u, V_x + \epsilon H)) \approx x + \epsilon\pi(D_w\phi_t(u, V_x)H)$$
- Invertibility of $\pi M_t(u, V_x)\pi$ implies tangent space around x is of full rank. (Take $H_\xi = (D_w\phi_t(u, V_x))^*\pi(\pi M_t(t, V_x)\pi)^{-1}\xi$ for $\xi \in \mathbb{R}^m$.)

Part IV:

Applications.

Damped and Stochastically Forced KdV

$$du + (u_{xxx} + \gamma u + \frac{1}{2}(u^2)_x)dt = f + \sum_{k \in \mathcal{Z}} \sigma_k dW^k \tag{9}$$

Theorem (GH-Martinez-Richards '18)

(i) For any invariant measures μ of (9)

$$\int \|u\|^R_{H^m} \mu(du) < \infty \quad \text{for all } m \geq 0, R \geq 1, \quad \mu(C^\infty) = 1 \tag{10}$$

(ii) Suppose $\sigma_k(x) = \sin(kx)$, $k > 0$, $\sigma_k(x) = \cos(kx)$, $k < 0$, then there is an N s.t. if $\mathcal{Z} \supset [-N, \dots, N]$ has a unique invariant measure μ.

(iii) If $\{-1, 1\} \subset \mathcal{Z}$ (9) is approximately controllable in H^m for $m \geq 2$.

(Conjectured: Unique Ergodcity in the 'Hypo-elliptic Case', (iii)).

- Invariants at all orders (complete integrability) for the free equation (Miura-Gardner-Kruskal, Lax, Zakharov-Faddeev).
- Finite dimensional attractors for the deterministic-damped driven system (Ghidaglia, Goubet, Debussche-Odasso, Jolly-Sadigov-Titi).

Nontrivial task find functionals to exploit these structures for sKdV.

Braketology for Even Degree Nonlinearities

Bidirectionality?

Recall that $\rho_{1/\lambda}^{-\lambda^2\alpha \boldsymbol{g}}\Phi^0_{t/\lambda^m}\rho_{1/\lambda}^{\lambda^2\alpha \boldsymbol{g}}\boldsymbol{u}_0 \approx \rho_t^{\alpha^m N(\boldsymbol{g})}\boldsymbol{u}_0$ but we need obtain $\rho_t^{\gamma N(\boldsymbol{g})} \in \mathrm{Sat}(\mathcal{F}_0)$, $\gamma \in \mathbb{R}$ to iterate!

KdV-Burgers $N(u,v) := \partial_x(uv)$, $\sigma_k(x) = \sin(kx)$, $\tilde{\sigma}_k = \cos(kx)$

$$N(\alpha\sigma_k + \beta\sigma_\ell) = -\alpha^2 k\tilde{\sigma}_{2k} - \alpha\beta\left((k+\ell)\tilde{\sigma}_{k+\ell} + (k-\ell)\tilde{\sigma}_{k-\ell}\right) + \beta^2\ell\tilde{\sigma}_{2\ell}$$
$$N(\alpha\sigma_k + \beta\tilde{\sigma}_\ell) = -\alpha^2 k\tilde{\sigma}_{2k} + \alpha\beta\left((k+\ell)\sigma_{k+\ell} - (k-\ell)\sigma_{k-\ell}\right) + \beta^2\ell\tilde{\sigma}_{2\ell}$$
$$N(\alpha\tilde{\sigma}_k + \beta\tilde{\sigma}_\ell) = \alpha^2 k\tilde{\sigma}_{2k} + \alpha\beta\left((k+\ell)\tilde{\sigma}_{k+\ell} - (k-\ell)\tilde{\sigma}_{k-\ell}\right) + \beta^2\ell\tilde{\sigma}_{2\ell}.$$

Even Modes

$$N(\alpha\sigma_k \pm \alpha\tilde{\sigma}_k) = \pm 2\alpha^2 k\sigma_{2k}, \quad N(\alpha\sigma_k) = -2\alpha^2\tilde{\sigma}_{2k}, \quad N(\beta\tilde{\sigma}_k) = 2\beta^2\tilde{\sigma}_{2k}. \quad (11)$$

Odd Modes

$$N(\alpha\sigma_{2m+2}+\beta\tilde{\sigma}_1) = -\alpha^2(2m+2)\tilde{\sigma}_{4m+4} + \alpha\beta((2m+3)\sigma_{2m+3} - (2m+1)\sigma_{2m+1}) + \beta^2\tilde{\sigma}_2,$$
$$N(\alpha\sigma_{2m+2}+\beta\sigma_1) = -\alpha^2(2m+2)\tilde{\sigma}_{4m+4} - \alpha\beta((2m+3)\tilde{\sigma}_{2m+3} + (2m+1)\tilde{\sigma}_{2m+1}) + \beta^2\tilde{\sigma}_2.$$

The Boussinesq Equation with degenerate forcing

$$d\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi - \nu_1 \Delta \boldsymbol{u})dt = \boldsymbol{g}\theta dt, \quad \nabla \cdot \boldsymbol{u} = 0,$$
$$d\theta + (\boldsymbol{u} \cdot \nabla \theta - \kappa \Delta \theta)dt = hdt + \sigma_\theta dW.$$

Theorem (Földes-GH-Richards-Thomann '13, GH-Herzog-Mattingly '17)

Suppose that

$$\sigma_\theta dW = \cos(\mathbf{k} \cdot \mathbf{x})dW^1 + \sin(\mathbf{k} \cdot \mathbf{x})dW^2 + \cos(\tilde{\mathbf{k}} \cdot \mathbf{x})dW^3 + \sin(\tilde{\mathbf{k}} \cdot \mathbf{x})dW^4,$$

for some $\mathbf{k}, \tilde{\mathbf{k}} \in \mathbb{Z}^2$ *with* $\mathbf{k} \cdot \tilde{\mathbf{k}}^\perp \neq 0$. *Then*

(i) *There is a unique invariant measure* μ *on* $H = (L^2(\mathbb{T}^2))^2$. μ *is geometrically ergodic and for any regular observable* Φ, $U_0 \in H$

$$\frac{1}{T}\int_0^T \Phi(U(t, U_0)) \to \int_{(L^2(\mathbb{T}^2))^2} \Phi(U) d\mu(U) \quad \textit{almost surely.}$$

(ii) *The IM* μ *has full support:*

$$\mu(B(U_0, \epsilon)) > 0 \quad \textit{for every } U_0 \in H, \epsilon > 0.$$

Moreover, for any finite dimensional projection $\Pi : H \to \mathbb{R}^N$

$$\Pi U(t, U_0) \textit{ is continuously distributed with support } \mathbb{R}^N.$$

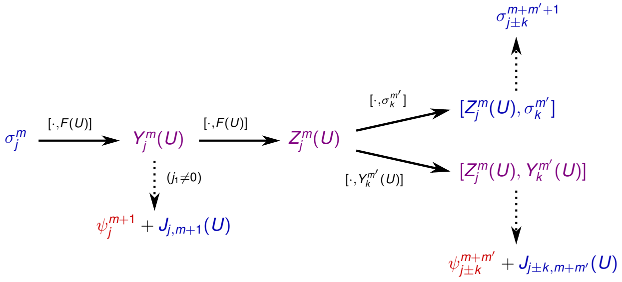
The Lie Bracket Structure

$$d\omega + (\boldsymbol{u}\cdot\nabla\omega - \nu_1\Delta\omega)dt = \alpha g \partial_x\theta dt, \quad \boldsymbol{u} = K * \omega$$

$$d\theta + (\boldsymbol{u}\cdot\nabla\theta - \nu_2\Delta\theta - h)dt = \sigma_\theta dW = \sum_{k=1,2,l=0,1} \sigma_k^l dW^{k,l}.$$

$$\sigma_k^0(x) := (0, \cos(k\cdot x))^T, \quad \sigma_k^1(x) := (0, \sin(k\cdot x))^T$$

$$\psi_k^0(x) := (\cos(k\cdot x), 0)^T, \quad \psi_k^1(x) := (\sin(k\cdot x), 0)^T.$$



Low Mode Control Prevents Blow up for 3D Euler

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{g} + \sum_{\mathbf{k}\in\mathcal{Z}, l,m\in 0,1} \alpha_{\mathbf{k},l,m}(t)\mathbf{e}_{\mathbf{k},l,m} \tag{12}$$

$$\mathbf{e}_{\mathbf{k},l,m}(\mathbf{x}) = 2\mathbf{a}_{\mathbf{k}}^l \mathrm{Re}(i^m e^{-i\mathbf{k}\cdot\mathbf{x}}) \quad \mathbf{a}_{\mathbf{k}}^0\cdot\mathbf{k} = \mathbf{a}_{\mathbf{k}}^1\cdot\mathbf{k} = \mathbf{a}_{\mathbf{k}}^1\cdot\mathbf{a}_{\mathbf{k}}^0$$

$$H = C^\infty, \quad d(\boldsymbol{u},\boldsymbol{v}) = \sum_{m\geq 0} 1 \wedge \|\boldsymbol{u}-\boldsymbol{v}\|_{H^m}$$

Theorem (GH-Herzog-Mattingly '17)

Suppose that $\{(1,0,0),(0,1,0),(0,0,1)\} \subseteq \mathcal{Z}$*. Then*

- (12) *is approximately controllable on H and exactly controllable on finite dimensional projections.*
- *In particular* $\exists$ *a smooth* $\alpha : [0,\infty) \to \mathbb{R}^d$ *s.t.* $\boldsymbol{u}(\cdot,\boldsymbol{u}_0,\alpha)$ *exists globally.*

Commentary:

- No 'energy budget' imposed for this result.
- 'Braketology' due to Romito.
- Previous similar no-blowup results: Shirikyan, Nersisyan using the Agrachev-Sarychev approach.

Summary

- Statistical properties of stochastic evolution equations lead to geometric control problems of independent interest.
- We developed a scaling and saturation framework which provides a powerful, flexible framework to tackle 'low mode' control problems.
- Each model still requires a separate analysis:

  Algebraic: Interactions between stochastic and nonlinear terms. Hormander type conditions.

  Analytic: Rigorous PDE type estimates.

References

- “Scaling and Saturation in Infinite-Dimensional Control Problems with Applications to Stochastic Partial Differential Equations” (w/ D. Herzog, J. Mattingly).
- “The Damped-Driven Korteweg-de Vries Equation with Degenerate Random Forcing” (w/ V. Martinez, G. Richards).
- “Ergodic and Mixing Properties of The Boussinesq Equations with a Degenerate Random Forcing” (w/ J. Földes, G. Richards, E Thomann) Journal of Functional Analysis, Volume 269, Issue 8, 15 October 2015, Pages 2427-2504.