Propagators and distinguished states
on curved spacetimes

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QFT on curved spacetimes

Quantum fields propagating on fixed (M, g):

- Interesting **quantum effects** even without interaction
- Starting point to understand ***geometry ↔ quantum* coupling**
- **Quantum** $:\phi^n:$ and $:T_{\mu\nu}:$ dramatically different from classical

  e.g. [Dappiaggi, Fredenhagen, Pinamonti '08], [Hack '13]

Curved background helpful even for flat setting!

- Better understanding of (local, covariant) **renormalization**
- $\Lambda > 0$ acts as **infrared regularizer** (see e.g. $P(\varphi)_2$)
- **AdS/CFT** or **dS/CFT**

Hadamard condition and microlocal analysis

70's: key conceptual ideas

mid 90's-00's: connection with microlocal analysis

[Radzikowski '96], [Brunetti, Fredenhagen '00], [Hollands & Wald '01]

recent progress: global/asymptotic aspects

Essential ingredient: **Hadamard condition** on two-point functions of states $\omega(\phi(t,\mathrm{x})\phi(t',\mathrm{x}'))$ (rigorously defined by [Kay, Wald '91])

Formalizes condition of "same short-distance behaviour as Minkowski vacuum". Intuition:

$$\left(\mathrm{i}^{-1}\partial_t - \sqrt{-\Delta_{\mathrm{x}} + m^2}\right) u(t,\mathrm{x}) \in C^\infty(M).$$

[Radzikowski '96] Hadamard condition ⇒

2-pt function = '**geometrical, singular part**'

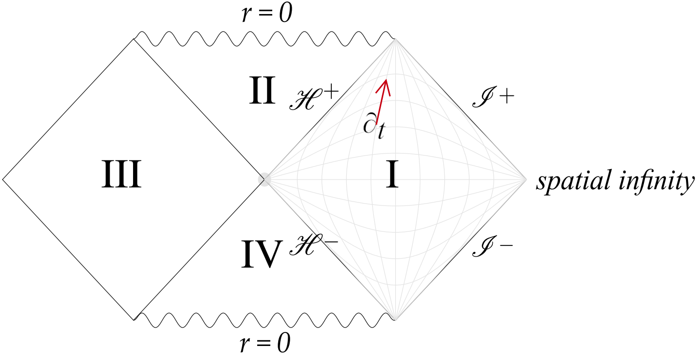
+ '**state-dependent, smooth part**'

Consequences of Hadamard condition

- Finite $:\phi^n:$ and $:T_{\mu\nu}:$
  - ⇒ **Perturbative interacting theory** [Brunetti, Fredenhagen '00], [Hollands, Wald '01-'08], cf. [Fredenhagen, Rejzner '13]
  - ⇒ **Quantum energy inequalities** [Fewster '00]
- Hadamard condition delicate at horizons
  - ⇒ With symmetries, it enforces β-KMS condition at $\beta = \beta_{\rm H}$ at $\mathscr{H}$ [Kay, Wald '91]
  - ⇒ Trouble at Cauchy horizons, hence **chronology protection mechanism** [Kay, Radzikowski, Wald '97]
- Asymptotic aspects
  - ⇒ For black holes, implies $\beta_{\rm H}$-thermal behaviour at infinity [Fredenhagen, Haag '90]
  - ⇒ Universal **asymptotics at conformal infinity** of asym. de Sitter spacetimes [Hollands '13], [Vasy, W. '18]
- *Analytic version* implies **Reeh-Schlieder property** [Strohmaier, Verch, Wollenberg '02]
- *Holographic version* implies well-defined **theory on boundary of AdS** spacetimes [W. '17]

Examples *(stationary spacetimes)*

- **Ground** and β-**KMS states** associated to time-like Killing vector field.



- Analytic version satisfied on analytic spacetimes [Strohmaier, Verch, Wollenberg '02]

Non-examples

- If we propagate Cauchy data of ground or β-KMS states, generically not Hadamard state
- Ground state in exterior Schwarzschild does not extend to a Hadamard state

Examples *(asymptotic constructions)*

If spacetime has good asymptotic structure, consider **asymptotically ground** or β-**KMS** state.

- *Conformal scattering*
  - conformal m, asymptotically flat [Moretti '08]
  - general m, cosmological spacetimes [Dappiaggi, Moretti, Pinamonti '09]
- *Standard scattering*
  - $m > 0$, asymptotically static spacetimes [Gérard, W. '17]
- *Geometric scattering*
  - $m = 0$, asymptotically Minkowski [Vasy, W. '18]
  - $m > 0$ asymptotically de Sitter (global chart) [Vasy, W. '18]

Examples *(black hole spacetimes)*

✓ **Hartle-Hawking-Israel state** on spacetimes with bifurcate Killing horizon

= invariant under t-translations, Hadamard in regions I, II, III, IV, and:

- β_{H}-KMS in exterior region

Conjectured in '76. Uniqueness [Kay, Wald '91]. Rigorous construction and Hadamard property established by [Sanders '15] (static case), reworked and generalized by [Gérard '18].

Examples *(black hole spacetimes)*

✓ **Unruh state** on Schwarzschild

= final state resulting from collapse into (idealized) black hole

- (asymptotically) β_{H}-KMS at $\mathscr{H}^-$, (asymptotically) ground state at $\mathscr{I}^-$
- Hadamard in regions I, II

Conjectured in '76. Rigorous construction and Hadamard property established by [Dappiaggi, Moretti, Pinamonti '11].

Non-examples *(black hole spacetimes)*

On **Kerr** spacetime, ∂_t not everywhere time-like. Superradiance for bosons: no positive conserved energy.

- Strong evidence for non-existence of state at the same time maximally symmetric and Hadamard [Kay, Wald '91]

  (Kay-Wald no-go theorem is based on superradiance)
- In exterior, β-KMS states are not Hadamard, even for $\beta = \beta_{\mathrm{H}}$. [Pinamonti, Sanders, Verch '18]

Conjectures on black holes

One can conjecture:

- No global **Hartle-Hawking-Israel state on Kerr**, even for fermions?
- **Unruh state on Kerr** might exist, but "asymptotically ground/KMS state" should refer to *different* Killing vector fields
  - Evidence from [Ottewill, Winstanley '00]
- More realistic star models?
  - Scattering description of Hawking effect [Bachelot '97], [Häfner '09], [Bouvier, Gérard '13], [Drouot '17], though case of bosons on Kerr still open.

General existence

To have large domain of definition of $:\phi^n:$, and set up semi-classical Einstein equations one wants **large classes of Hadamard states**.

- ✓ Existence by deformation [Fulling, Narcowich, Wald '81]
  - ✓ Same technique applies to AdS spacetimes [W. '17]
- ✓ More direct construction by pseudodifferential operators [Junker '96], [Gérard, W. '14]
- ✓ Existence and construction of **analytic Hadamard states** by Wick rotation [Gérard, W. '17]

Linearized gauge theories more difficult if non-zero background [Gérard, W. '15]

Zoom on Wick rotation

Real analytic metric:

$$\mathbf{g} = -N^2(t)dt^2 + \mathbf{h}_{jk}(t)(dy^j + w^j(t)dt)(dy^k + w^k(t)dt),$$

and Wick-rotated *complex metric*:

$$\mathbf{k} = N^2(\mathrm{i}s)ds^2 + \mathbf{h}_{jk}(\mathrm{i}s)(dy^j + \mathrm{i}w^j(\mathrm{i}s)ds)(dy^k + \mathrm{i}w^k(\mathrm{i}s)ds).$$

Klein-Gordon and 'complex Laplace-Beltrami' operators:

$$P = -|\mathbf{g}|^{-\frac{1}{2}}\partial_a|\mathbf{g}|^{\frac{1}{2}}\mathbf{g}^{ab}\partial_b + m^2, \quad K = -|\mathbf{k}|^{-\frac{1}{2}}\partial_a\mathbf{k}^{ab}|\mathbf{k}|^{\frac{1}{2}}\partial_b + m^2.$$

Theorem [W. '18]

$\exists$ a two-point function Λ^+, and a $\mathscr{D}'(\Sigma^2)$-valued holomorphic function F s.t.

$$\begin{aligned} K^{-1}(s,s') &= F(\mathrm{i}s,\mathrm{i}s'), \quad s>0,\ s'<0,\\ \Lambda^+(t,t') &= F\big((t,t')+\mathrm{i}\Gamma 0\big), \quad t,t' \in]-\delta,\delta[, \end{aligned}$$

i.e., $(s,s') \to 0$ from $\Gamma = \{s>0,\ s'<0\}$.

🛠 generalized **Calderón projectors**, cf. [Gérard '17-18], [Schapira '17].

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$\mathrm{e}^{\mathrm{i}t\sqrt{-\Delta+m^2}}f$ vs. $\mathrm{e}^{-\mathrm{i}t\sqrt{-\Delta+m^2}}f$ becomes $\mathrm{e}^{-s\sqrt{-\Delta+m^2}}f$ vs. $\mathrm{e}^{s\sqrt{-\Delta+m^2}}f$

Global propagators

Even more surprisingly (to mathematicians), Lorentzian propagators can be treated by global Hilbert space analysis.

- global *advanced/retarded* propagators & *in-out* Feynman propagators
  - extremely effective in non-linear problems, e.g. resolution of Kerr-de Sitter stability conjecture [Hintz, Vasy '16]
  - for fermions, index formula for chiral anomalies [Bär, Strohmaier '16]
  - correct 'Hadamard' microlocal behaviour [Gell-Redman, Haber, Vasy '16], [Gérard, W. '18], [Vasy, W. '18]
- self-adjointness of $-\Box_g + m^2$ [Dereziński, Siemssen '17], [Vasy '17]

? Towards rigorous S_{eff}, rather than $\delta S_{\text{eff}}/\delta g^{\mu\nu}$

Of course, if exists, $(-\Box_g + m^2)^z$ not local. But locality conjectured after removing poles, cf. Riemannian case [Dang, Zhang '18]

Summary & outlook

- **Hadamard condition** is a fundamental ingredient of QFT on curved spacetimes.
- Now, better access to global aspects tied to microlocal ones.
  - Perturbative interacting QFT on AdS spacetimes?
  - Preferred state on Kerr?
  - Well-posedness theorems for semi-classical Einstein equations?

Yet more questions...

- Use global Feynman propagators and their relationship with geometry?
- Singular potentials, bound state QED?

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Thank you for your attention!