From infrared problems to non-commutative recurrence

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joint work with Wojciech Dybalski [2]

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Non-commutative recurrence

1. Let $\hat{a}$, $\hat{b}$ be (possibly non-commuting) operators on X.
2. Let $x_n \in X$ be a sequence s.t.

$$x_n = \hat{a}x_{n-1} + \hat{b}x_{n-2}$$

with initial conditions x_0 and $x_1 = \hat{a}x_0$.
3. Problem: Determine x_n.
4. Solution 1.

$$x_n = \sum_{j_1+2j_2=n} \{\hat{a}^{(j_1)}, \hat{b}^{(j_2)}\} x_0,$$

where $\{\hat{a}^{(j_1)}, \hat{b}^{(j_2)}\}$ is the sum over all possible distinct permutations of factors $\hat{a}$, $\hat{b}$ each one appearing $j_1, j_2 \geq 0$ times, respectively. [Jivulescu, Messina, Napoli, Petruccione 07/08, Puhlfürst 15]

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$$x_n = \mathrm{Q}_{\hat{a},\hat{b}}[\exp(\sum_{i=1}^{n-1} b_{i+1,i}\partial_{a_{i+1}}\partial_{a_i})a_n \dots a_1]x_0,$$

   where $\mathrm{Q}_{\hat{a},\hat{b}}[a_n \dots b_{j+1,j} \dots b_{j'+1,j'} \dots a_1] = \hat{a} \dots \hat{b} \dots \hat{b} \dots \hat{a}$.

Non-commutative recurrence

Lemma

The relation $x_n = \hat{a}x_{n-1} + \hat{b}x_{n-2}$ *with* $x_1 = \hat{a}x_0$ *is solved by*

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Proof. Write $\delta_i := b_{i+1,i}\partial_{a_{i+1}}\partial_{a_i}$ and $d_{n-1} := \sum_{i=1}^{n-1} \delta_i$ and compute

$$\begin{aligned}
&\exp(d_{n-1})a_n \dots a_1 \\
&\quad = \{\exp(d_{n-2})\exp(\delta_{n-1})a_n a_{n-1} \dots a_1\} \\
&\quad = \{\exp(d_{n-2})a_n a_{n-1} \dots a_1\} + \{\exp(d_{n-2})(\delta_{n-1})a_n a_{n-1} \dots a_1\} \\
&\quad = a_n\{\exp(d_{n-2})a_{n-1} \dots a_1\} + b_{n,n-1}\{\exp(d_{n-3})a_{n-2} \dots a_1\}. \ \square
\end{aligned}$$

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- Proving infrared regularity of physical quantities which suffer from superficial infrared divergencies even after the implementation of multi-scale techniques

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Outline

Nelson model with many electrons/atoms

Definition

The Nelson model with many atoms/electrons is given by:

(1) Hilbert space $\mathcal{H} = \Gamma(L^2(\mathbb{R}^3)_{\mathrm{at/el}}) \otimes \Gamma(L^2(\mathbb{R}^3)_{\mathrm{ph}})$.

(2) Hamiltonian $H = H_{\mathrm{at/el}} + H_{\mathrm{ph}} + H_I$, where

(a) $H_{\mathrm{at/el}} = \int d^3p \, \frac{p^2}{2m} \, c^*(p)c(p)$,

(b) $H_{\mathrm{ph}} = \int d^3k \, |k| a^*(k)a(k)$,

(c) $H_I = \int d^3p \, d^3k \, \lambda \frac{\tilde{\rho}(k)}{\sqrt{2|k|}}(c^*(p+k)a(k)c(p) + \mathrm{h.c.})$.

(3) Momentum operator: $\hat{P} = \int d^3p \, p \, c^*(p)c(p) + \int \, d^3k \, k \, a^*(k)a(k)$.

Nelson model with one electron/atom

Definition

The Nelson model with one electron is given by:

(1) Hilbert space $\mathcal{H}^{(1)} = L^2(\mathbb{R}^3)_{\mathrm{at/el}} \otimes \Gamma(L^2(\mathbb{R}^3)_{\mathrm{ph}})$.

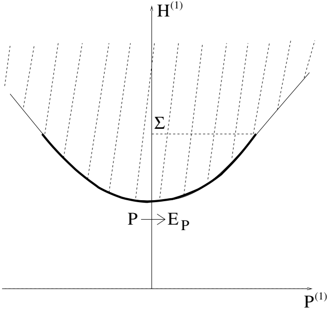
(2) Hamiltonian $H^{(1)} = \frac{p^2}{2m} + H_{\mathrm{ph}} + \phi(G_x)$, where

(a) $H_{\mathrm{ph}} = \int d^3k \, |k| a^*(k) a(k)$,

(b) $\phi(G_x) = \int d^3k \, \lambda \frac{\tilde{\rho}(k)}{\sqrt{2|k|}} (e^{-ikx} a^*(k) + e^{ikx} a(k))$.

(3) Momentum operator: $\hat{P}^{(1)} = p + \int d^3k \, k \, a^*(k) a(k)$.

Neutral particle ('atom')



Suppose that the 'charge' of the massive particle is zero, i.e. $\tilde{\rho}(0) = 0$.
Then (generically):

$$\mathcal{H}_{\rm sp} := \{\text{Spectral subspace of the lower boundary}\} \neq \{0\}$$

Renormalized creation operators of 'atoms'

1. For $h \in C_0^\infty(\mathbb{R}^3)$ consider $\Psi_h \in \mathcal{H}_{\rm sp}$ given by

$$\Psi_h := \Pi^* \int^{\oplus} d^3P\, h(P)\psi_P.$$

2. Let us define the renormalized creation operator of Ψ_h:

$$\hat{c}^*(h) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d^3p d^{3n}k\, h(p) f_p^n(k_1, \dots, k_n) a^*(k_1) \dots a^*(k_n) c^*(p - \underline{k}),$$

where $\{f_P^n\}_{n\in\mathbb{N}}$ are the wave-functons of ψ_P.

3. With this definition

$$\hat{c}^*(h)\Omega = \Psi_h.$$

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Asymptotic creation operators of ‘atoms’

Definition

For $h \in C_0^\infty(\mathbb{R}^3)$ let us define

$$\hat{c}_t^*(h) := \mathrm{e}^{iHt}\hat{c}^*(e^{-iEt}h)\mathrm{e}^{-iHt}.$$

$\hat{c}_{\mathrm{out}}^*(h) := \lim_{t\to\infty} \hat{c}_t^*(h)$ (if it exists) is called the asymptotic creation operator of the 'atom' smeared with h.

Scattering states of two atoms

Theorem

For $h_1, h_2 \in C_0^\infty(\mathbb{R}^3)$ with disjoint supports the limits

$$\Psi^{\mathrm{out}}_{h_1,h_2} := \lim_{t\to\infty} \hat{c}_t^*(h_1)\hat{c}_t^*(h_2)\Omega$$

exist and span a subspace naturally isomorphic to $\mathcal{H}_{\mathrm{sp}} \otimes_{\mathrm{a}} \mathcal{H}_{\mathrm{sp}}$.

1. This theorem was proven before at fixed infrared cut-off (Fröhlich 73) and assuming non-zero photon mass (Albeverio 73).
2. We treat the case of massless photons and $\tilde{\rho}(k) = \chi(k)|k|^\alpha$, $\alpha > 0$, $\chi(k) > 0$ near zero (no infrared cut-off in H).
3. However, we have to replace f_P^n with f_{P,σ_t}^n in $\hat{c}_t^*(h_i)$ for $\lim_{t\to\infty} \sigma_t = 0$.

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Idea of the proof

1. Let $\Psi_t := \mathrm{e}^{itH}\hat{c}^*(h_{1,t})\hat{c}^*(h_{2,t})\Omega$, with $h_{i,t}(P) = e^{-itE_P}h_i(P)$.

$$\partial_t \Psi_t = \mathrm{e}^{itH} i[[H_I, \hat{c}^*(h_{1,t})], \hat{c}^*(h_{2,t})]\Omega.$$

2. This can be expressed by integrals of the form

$$\int d^3\tilde{r}\,\lambda \frac{\tilde{\rho}(\tilde{r})}{\sqrt{2|\tilde{r}|}} e^{-i(E_{p-\tilde{r}}+E_{q+\tilde{r}})t} h_1(p-\tilde{r}) h_2(q+\tilde{r}) f^{n+1}_{q+\tilde{r}}(r,\tilde{r}) f^m_{p-\tilde{r}}(k).$$

3. (Non-) stationary phase gives integrable decay of $\partial_t\Psi_t$, provided we can control derivatives of $P \mapsto f^n_P(k)$ up to second order.
4. We replace $P \mapsto f^n_P(k)$ with $P \mapsto f^n_{P,\sigma_t}(k)$ with $\lim_{t\to\infty}\sigma_t = 0$. Using non-commutative recurrence relations we will show

$$|\partial^\beta_P f^n_{P,\sigma_t}(k_1,\dots,k_n)| \leq \frac{1}{\sqrt{n!}} \frac{(c\lambda)^n}{\sigma_t^{\delta_\lambda}} \prod_{i=1}^n \frac{\chi_{[\sigma,\kappa)}(k_i)}{|k_i|^{3/2}}.$$

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Ground-state wave-functions

1. Let $\psi_{P,\sigma} \in \Gamma(L^2(\mathbb{R}^3))$ be ground-states of $H^{(1)}_{P,\sigma}$ i.e.

$$H^{(1)}_{P,\sigma}\psi_{P,\sigma} = E_{P,\sigma}\psi_{P,\sigma},$$

   where $\sigma > 0$ is the infrared cut-off in the interaction

2. Let $\{f^n_{P,\sigma}(k_1,\ldots,k_n)\}_{n\in\mathbb{N}_0}$ be the wave functions of $\psi_{P,\sigma}$:

$$f^n_{P,\sigma}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n!}}\langle\Omega, b(k_1)\ldots b(k_n)\psi_{P,\sigma}\rangle.$$

3. We need, for $|\beta| = 0,1,2$ and $\delta_\lambda \to 0$ for $\lambda \to 0$

$$|\partial^\beta_P f^n_{P,\sigma}(k_1,\ldots,k_n)| \leq \frac{1}{\sqrt{n!}}\frac{(c\lambda)^n}{\sigma^{\delta_\lambda}}\prod_{i=1}^n \frac{\chi_{[\sigma,\kappa)}(k_i)}{|k_i|^{3/2}}.$$

Fröhlich formula for ground-state wave-functions

1. Define $\boldsymbol{f}^n(k_1, \ldots, k_n) := b(k_1)\ldots b(k_n)\psi \qquad b(k) = a(k)e^{ikx}$
2. Using $H\psi = E\psi$, we obtain

$$\boldsymbol{f}^n(k_1, \ldots, k_n) = (-)R_n \sum_{i=1}^{n} v^\sigma(k_i)\boldsymbol{f}^{n-1}(k_1, \ldots \check{i} \ldots, k_n),$$

where $R_n := (H_{P-\underline{k}_n,\sigma} - E_{P,\sigma} + |\underline{k}|_n)^{-1}, \quad v^\sigma(k_i) := \lambda \frac{\chi_{[\sigma,\kappa)}(k_i)}{\sqrt{2|k_i|}}$.

3. This has the form $x_n = \hat{a}x_{n-1} + \hat{b}x_{n-2}$ with $\hat{b} = 0$.
4. Solution:

$$\boldsymbol{f}^n = (-1)^n n! P_{\mathrm{sym}}(v_n^\sigma \ldots v_1^\sigma)(R_n \ldots R_1)\psi.$$

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Derivatives of Fröhlich formula

1. Fröhlich formula:

$$\boldsymbol{f}^n = (-1)^n n! P_{\mathrm{sym}}(v_n^\sigma \dots v_1^\sigma)(R_n \dots R_1)\psi.$$

2. To show: $\|\partial_P^\beta \boldsymbol{f}^n(k_1, \dots, k_n)\| \le \frac{c^n}{\sigma^{\delta_\lambda}} \prod_{i=1}^n \frac{v^\sigma(k_i)}{|k_i|}$ for $|\beta| \le 2$.
3. Rules of the game:

$$\|R_i\| \le c|\underline{k}|_i^{-1}$$

$$\partial_P \psi = R\Lambda\psi, \quad \partial_P R_i = R_i \Lambda^i R_i, \ \text{where } \Lambda := \nabla_P(E - H)$$

$$\|R\Lambda\psi\| \le \frac{c}{\sigma^{\delta_\lambda}}, \quad \|\bar{Q}^\perp R\Lambda R\Lambda\psi\| \le \frac{c}{\sigma^{\delta_\lambda}} \text{ where } \bar{Q} = |\psi\rangle\langle\psi|.$$

4. This gives the first derivative. But not the second one:

$$\boldsymbol{f}^1(k_1) = -v_1^\sigma R_1\psi \quad \Rightarrow \quad \partial_P^2 \boldsymbol{f}^1(k_1) \ni -v_1^\sigma (R_1\Lambda^1 R_1) R\Lambda\psi$$

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1. $\boldsymbol{f}^{W,n}(k_1,\ldots,k_n) := b_W(k_1)\ldots b_W(k_n)\psi,$

$$b_W(k) := W^* b(k) W = b(k) + |k|^{-1} v_*^\sigma(k),$$
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2. To control $\partial_P^\beta \boldsymbol{f}^n$ it suffices to control $\partial_P^\beta \boldsymbol{f}^{W,n}$.
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$$\times (R_{i_1} \hat{\partial}_{i_1} \hat{\partial}_{i_1 - 1}) \dots (R_{i_\ell} \hat{\partial}_{i_\ell} \hat{\partial}_{i_\ell - 1}) \Big[(R_n \Lambda_n) \dots (R_2 \Lambda_2)(R_1 \Lambda_1) \Big] \psi,$$

2. To show: $\|\partial_P^\beta \boldsymbol{f}^{W,n}(k_1, \dots, k_n)\| \le \frac{c^n}{\sigma^{\delta_\lambda}} \prod_{i=1}^n \frac{v^\sigma(k_i)}{|k_i|}$ for $|\beta| = 2$.
3. Rules of the game:

$$\|R_i\| \le c |\underline{k}|_i^{-1}$$

$$\partial_P \psi = R\Lambda\psi, \quad \partial_P R_i = R_i \Lambda^i R_i, \ \text{where } \Lambda := \nabla_P (E - H)$$

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Derivatives of the novel formula

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Example

1. Consider a contribution to the two-point function

$$\begin{aligned}
\partial_P^2 \boldsymbol{f}^{W,2} &\ni (-1) P_{\mathrm{sym}} v^{\sigma}_{*;1} v^{\sigma}_{*;2} (\partial_P R_2 \hat{\partial}_2 \hat{\partial}_1) \left[(R_2 \Lambda_2)(R_1 \Lambda_1) \right] \partial_P \psi \\
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