Mahler measure and the Vol-Det Conjecture

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The Vol-Det Conjecture

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$$\mathrm{vol}(K) < 2\pi \log \det(K).$$

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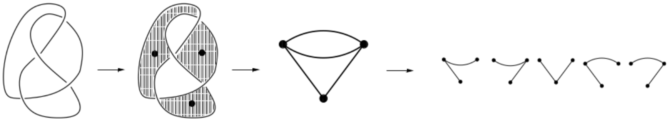
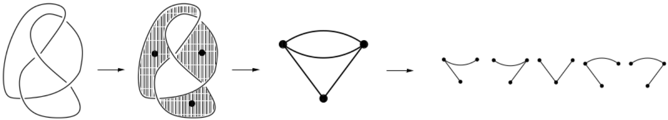
Alternating:	Link has an alternating diagram
Hyperbolic:	$S^3 - K = \mathbb{H}^3/\Gamma$, finite-volume
$\mathrm{vol}(K) =$	Hyperbolic volume of $S^3 - K$

$$
\begin{aligned}
\det(K) = &\quad |\det(M + M^T)|, && M = \text{Seifert matrix} \\
= &\quad |V_K(-1)| = |\Delta_K(-1)|, && V_K,\, \Delta_K = \text{Jones, Alexander poly} \\
= &\quad \#\,\text{spanning trees } \tau(G_K), && G_K = \text{Tait (checkerboard) graph}
\end{aligned}
$$



$$\begin{aligned} \det(K) = & \quad |\det(M + M^T)|, & M = \text{Seifert matrix} \\ = & \quad |V_K(-1)| = |\Delta_K(-1)|, & V_K,\ \Delta_K = \text{Jones, Alexander poly} \\ = & \quad \#\,\text{spanning trees } \tau(G_K), & G_K = \text{Tait (checkerboard) graph} \end{aligned}$$



- Verified for all alternating knots ≤ 16 crossings (≈ 1.7 million knots).
- (S. Burton '17) Proved for all 2-bridge knots and alternating 3-braids.
- (C-Kofman-Purcell '16) The constant 2π is sharp.

Mahler measure and hyperbolic volume

Mahler measure of polynomial $p(z)$ is defined as

$$\mathrm{m}(p(z)) = \frac{1}{2\pi i}\int_{S^1} \log|p(z)|\,\frac{dz}{z} \overset{\text{Jensen}}{=} \sum_{\alpha_i \text{ roots of p}} \max\{\log|\alpha_i|, 0\}.$$

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2-variable Mahler measure:

$$\mathrm{m}(p(z,w)) = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log|p(z,w)| \frac{dz}{z}\frac{dw}{w}.$$

2-variable Mahler measures are related to hyperbolic volume due to their relationship with dilogarithms.

Examples

$\mathrm{vol}($  $) = 2v_{tet} = 2.0298\ldots$

(Smyth '81) $\mathrm{vol}($  $) = 2\pi\,\mathrm{m}(1 + x + y) = \frac{2\sqrt{3}}{2} L(\chi_{-3}, 2)$

(Boyd '00) $\mathrm{vol}($  $)$

$$= \pi\,\mathrm{m}(A(L, M))$$

$$= \pi\,\mathrm{m}(M^4 + L(1 - M^2 - 2M^4 - M^6 + M^8) - L^2 M^4)$$

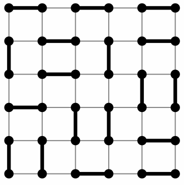
(Kenyon '00) $\mathrm{vol}($  $)$

$$= \frac{2\pi}{5}\,\mathrm{m}(p(z, w))$$

$$= \frac{2\pi}{5}\mathrm{m}\left(6 - w - \frac{1}{w} - z - \frac{1}{z} - \frac{z}{w} - \frac{w}{z}\right)$$

Dimers

A dimer covering of a graph G is a set of edges that covers every vertex exactly once, i.e. a perfect matching.



The dimer model is the study of the set of dimer coverings of G.
Let $Z(G) = \#$ dimer coverings of G.

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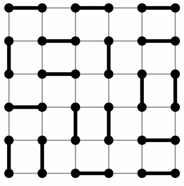


The dimer model is the study of the set of dimer coverings of G.
Let $Z(G) = \#$ dimer coverings of G.

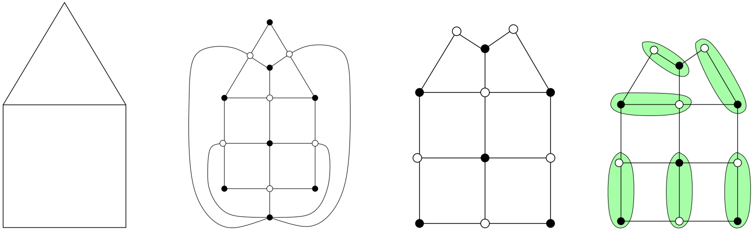
Theorem (Kasteleyn '63) If G is a balanced bipartite graph,

$$Z(G) = \det(\kappa),$$

where κ is a Kasteleyn matrix.

Dimers and Spanning trees

For any finite plane graph G, overlay G and its dual G^*, delete a vertex of G and G^* (in the unbounded face) and delete all incident edges to get balanced bipartite graph G^b.



G $\quad$ $G \cup G^*$ $\quad$ G^b $\quad$ A dimer on G^b

Theorem (Burton-Pemantle '93, Propp '02) $\tau(G) = Z(G^b)$.

Toroidal dimer model

Let G^b be a finite balanced bipartite toroidal graph.

Kasteleyn matrix $\kappa(z, w)$ for toroidal dimer model on G^b is defined by:

1. Choose Kasteleyn weighting (signs on edges), such that each face with 0 mod 4 edges has odd # of signs.
2. Choose oriented scc's $\gamma_z,\ \gamma_w$ on T^2 that are basis of $H_1(T^2)$.
3. Orient each edge e from black to white. Let

$$\mu_e = z^{\gamma_z \cdot e} w^{\gamma_w \cdot e}.$$

4. Order the black and white vertices.

Then $\kappa(z, w)$ is the $|B| \times |W|$ adjacency matrix with entries $\pm\mu_e$.

Toroidal dimer model

Let $\mathcal{G}^b$ be a biperiodic balanced bipartite planar graph, i.e. invariant under translations by 2-dim lattice Λ.

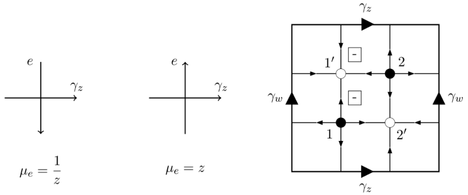
The characteristic polynomial of the toroidal dimer model on $\mathcal{G}^b$ is

$$p(z, w) = \det \kappa(z, w).$$

Theorem (Kenyon-Okounkov-Sheffield '06) Let $\mathcal{G}^b$ be biperiodic balanced bipartite graph, and $G_n = \mathcal{G}^b / n\Lambda$ be a toroidal exhaustion of $\mathcal{G}^b$. Then the partition function satisfies:

$$\log Z(\mathcal{G}^b) := \lim_{n\to\infty} \frac{1}{n^2} \log Z(G_n) = \mathrm{m}(p(z, w)).$$

Example



$$\kappa(z,w) = \begin{bmatrix} -1-1/z & 1+w \\ 1+1/w & 1+z \end{bmatrix}, p(z,w) = -\left(4 + \frac{1}{w} + w + \frac{1}{z} + z\right).$$

(Boyd '98) $\pi\,\mathrm{m}(p(z,w)) = \mathrm{vol}($  $)v_{oct} = 4C = 3.6638\ldots$

where v_{oct} = hyperbolic volume of regular ideal octahedron and C =Catalan's constant.

Biperiodic links

A biperiodic link $\mathcal{L}$ is a link projection which can be isotoped to be invariant under translations by 2-dim lattice Λ.

$\mathcal{L}/\Lambda = L \subset \mathbb{R}^3/\Lambda = T^2 \times I = S^3 - \text{Hopf link}$
$\implies L \cup \text{Hopf link} \subset \mathrm{S}^3$

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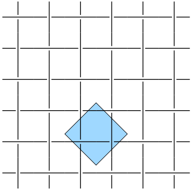
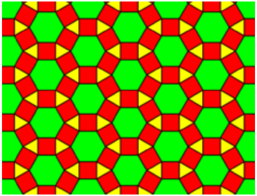
Theorem (C-Kofman-Purcell '18) Let L be a link in $T^2 \times I$ with a weakly prime alternating diagram on $T^2 \times \{0\}$ with no bigons. If L has no cycle of tangles, then $(T^2 \times I) - L$ is hyperbolic.

We will assume L is a toroidal link as above.

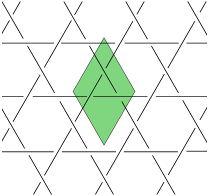
Examples



Square Weave



RhombiTriHexagonal Link



Triaxial link

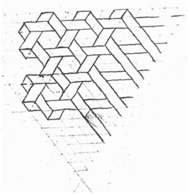


Figure from Gauss's 1794 notebook

Diagrammatic convergence

We say that a sequence of alternating links K_n Følner converges almost everywhere to the biperiodic alternating link $\mathcal{L}$, denoted by $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$, if the respective projection graphs $\{G(K_n)\}$ and $G(\mathcal{L})$ satisfy the following: There are subgraphs $G_n \subset G(K_n)$ such that

1. $G_n \subset G_{n+1}$, and $\bigcup G_n = G(\mathcal{L})$,
2. $\lim\limits_{n\to\infty} |\partial G_n|/|G_n| = 0$, where $|\cdot|$ denotes number of vertices, and $\partial G_n \subset G(\mathcal{L})$ consists of the vertices of G_n that share an edge in $G(\mathcal{L})$ with a vertex not in G_n,
3. $G_n \subset G(\mathcal{L}) \cap (n\Lambda)$, where $n\Lambda$ represents n^2 copies of the Λ-fundamental domain for the lattice Λ such that $L = \mathcal{L}/\Lambda$,
4. $\lim\limits_{n\to\infty} |G_n|/c(K_n) = 1$.

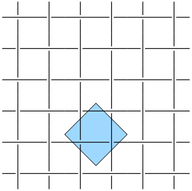
We denote this as $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$.

Dimers and Determinant Density Convergence

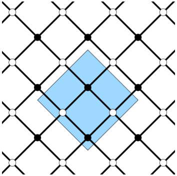
Theorem (C-Kofman '16) Let $\mathcal{L}$ be a biperiodic alternating link, with toroidally alternating quotient link L. Let $p(z,w)$ be the characteristic polynomial of the associated toroidal dimer model on $\mathcal{G}^b_{\mathcal{L}}$. Let $\{K_n\}$ be a sequence of alternating links, then

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{\log \det(K_n)}{c(K_n)} = \frac{\mathrm{m}(p(z,w))}{c(L)}.$$

Example: Infinite square weave $\mathcal{W}$



$\mathcal{W}$ & L



$\mathcal{G}^b_{\mathcal{W}}$ & G^b_L

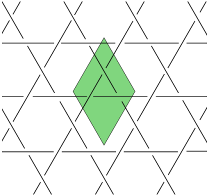
$\mathrm{vol}(T^2 \times I - L) = 2\, v_{oct} = 7.32772\ldots.$

$p(z,w) = -(4 + 1/w + w + 1/z + z).$

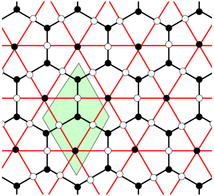
(Boyd '98) $2\pi\, \mathrm{m}(p(z,w)) = 2\, v_{oct}$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi \mathrm{m}(p(z,w))}{2} = v_{oct} = \frac{\mathrm{vol}(T^2 \times I - L)}{c(L)}$$

Example: Triaxial link $\mathcal{L}$



$\mathcal{L}$ & L



$\mathcal{G}^b_{\mathcal{L}}$ & G^b_L

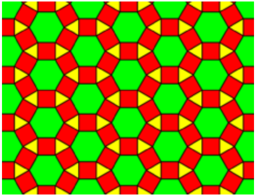
$$\mathrm{vol}(T^2 \times I - L) = 10\, v_{tet} = 10.14941\ldots.$$

$$p(z,w) = 6 - 1/w - w - 1/z - z - w/z - z/w.$$

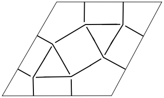
(Boyd '98) $2\pi\, \mathrm{m}(p(z,w)) = 10\, v_{tet}$

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, \mathrm{m}(p(z,w))}{3} = \frac{10 v_{tet}}{3} = \frac{\mathrm{vol}(T^2 \times I - L)}{c(L)}$$

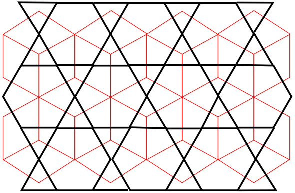
Example: Rhombitrihexagonal link $\mathcal{L}$



$G(\mathcal{L})$



L



$G_{\mathcal{L}}^{b}$

$$\mathrm{vol}(T^2 \times I - L) = 10\, v_{tet} + 3\, v_{oct} = 21.14100\ldots.$$

$$p(z,w) = 6\,(6 - 1/w - w - 1/z - z - w/z - z/w)$$

$$2\pi \mathrm{m}(p(z,w)) = 2\pi \log 6 + 10 v_{tet} = 21.40737\ldots > \mathrm{vol}(T^2 \times I - L)$$

Mahler measure and the Vol-Det Conjecture

Vol-Det Conjecture: For any alternating hyperbolic link K,

$$\mathrm{vol}(K) < 2\pi \log \det(K).$$

Idea: Use biperiodic alternating links to obtain infinite families of links satisfying the Vol-Det Conjecture.

1. Prove using explicit Mahler measure computation that $\mathrm{vol}(T^2 \times I - L) < \mathrm{m}(p(z,w))$.
2. Use Determinant Density Convergence and geometry of links in $T^2 \times I$ to prove that if $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$, then K_n satisfies the Vol-Det Conjecture for almost all n.

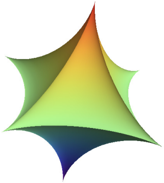
e.g. Rhombitrihexagonal link $\mathcal{L}$.

Bipyramid Volume

Let B_n denote the hyperbolic regular ideal bipyramid whose link polygons at the two coning vertices are regular n–gons. The hyperbolic volume of B_n is given by

$$\mathrm{vol}(B_n) = n\left(\int_0^{2\pi/n} -\log|2\sin(\theta)|\,d\theta + 2\int_0^{\pi(n-2)/2n} -\log|2\sin(\theta)|\,d\theta\right)$$

E.g. B_4 = regular ideal octahedron



Let L be an alternating link in $T^2 \times I$. For a face f of L, let $|f|$ denote the degree of the face. Define the bipyramid volume of L as

$$\mathrm{vol}^{\lozenge}(L) = \sum_{f \in \{\text{faces of } L\}} \mathrm{vol}(B_{|f|}).$$

Theorem (C-Kofman-Purcell '18) Let L be an alternating link in $T^2 \times I$. Then

$$\mathrm{vol}(T^2 \times I - L) \leq \mathrm{vol}^{\lozenge}(L)$$

Note: This is a sharp upper bound for volume of links in the thickened torus e.g. Square weave and the Triaxial link attain this upper bound !

New Conjecture

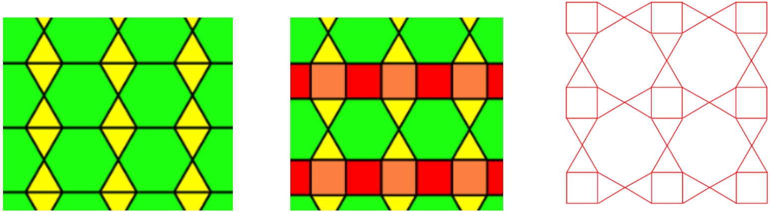
Conjecture 1 (C-Kofman-Lalín '18) Let $\mathcal{L}$ be a biperiodic alternating link, $L = \mathcal{L}/\Lambda$ and let $p(z,w)$ be the characteristic polynomial for the toridal dimer model on $\mathcal{G}^b_{\mathcal{L}}$. Then

$$\mathrm{vol}^{\lozenge}(T^2 \times I - L) \leq 2\pi\, \mathrm{m}(p(z,w)).$$

Theorem (C-Kofman-Lalín '18) Let $\mathcal{L}$ satisfy Conjecture 1 and let $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$. Then K_n satisfies the Vol-Det Conjecture for almost all n.

New Examples

We have explicit Mahler measure computations which verify Conjecture 1 for more examples:



Work in progress

Thank you