Communication Complexity with Small Advantage

Thomas Watson

University of Memphis

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Small advantage:
$\mathsf{R}_{1/2+\epsilon}$ = randomized c.c., success probability $\geq 1/2 + \epsilon$

Classic results — revisited

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$\vdots$

other functions?

Climbing the polynomial hierarchy

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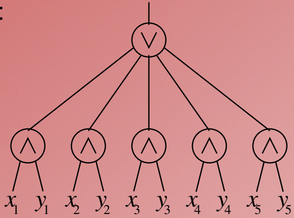
Small advantage: open

Function definitions

SET-INTERSECTION:



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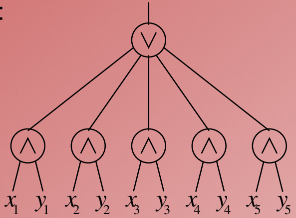
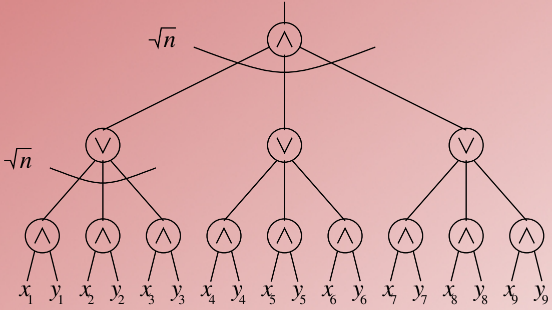


Tribes:



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- Fails in general ($\textsc{Gap-Hamming}$)

Our approach for TRIBES

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4-step approach:

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4-step approach:

1. Conditioning and direct sum
2. Uniformly covering a pair of gadgets
3. Relating information and probabilities for inputs
4. Relating information and probabilities for transcripts

Preliminaries

Idea from [BM'13]:

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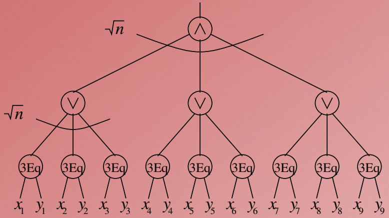
$3\textsc{Eq}$ has nice symmetry properties

- Exploit to get linear terms to perfectly cancel

1. Conditioning and direct sum

info cost $o(\epsilon \cdot n)$ $\rightsquigarrow$ info cost $o(\epsilon)$



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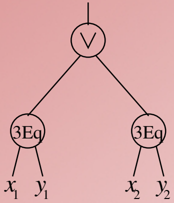

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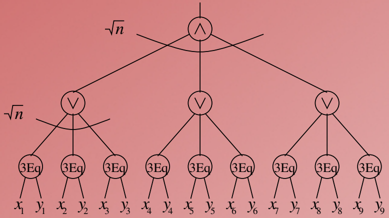

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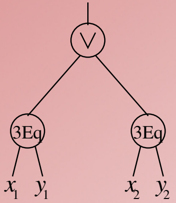

Want to show:

advantage $\leq O$(info cost)

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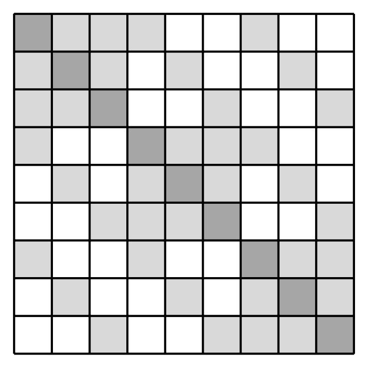
Uniformly cover



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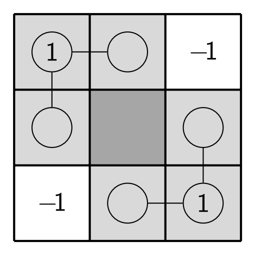

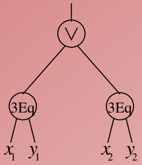

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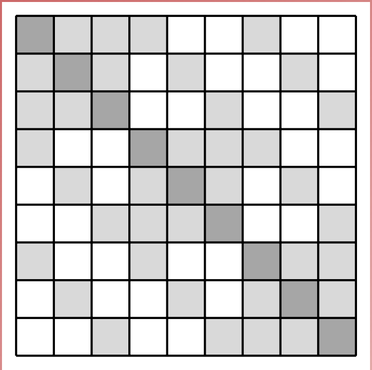

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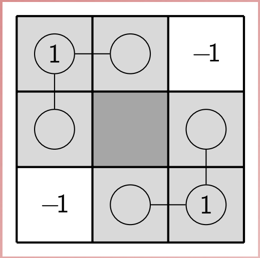

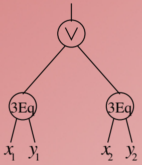

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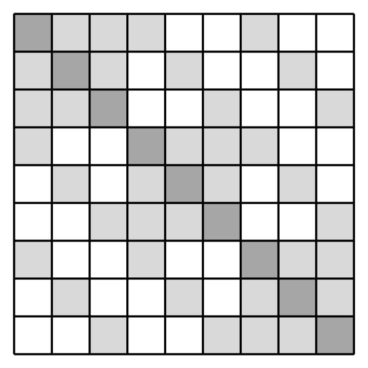

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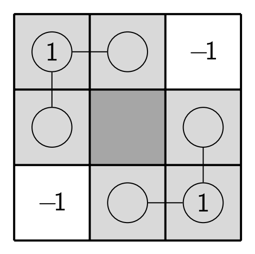

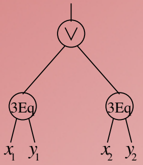

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$\Downarrow$

adv $\leq$ $\left(\sum_{\text{light gray}} \text{acc prob}\right) - \left(\sum_{\text{white}} \text{acc prob}\right)$
$\leq O(\text{info cost})$

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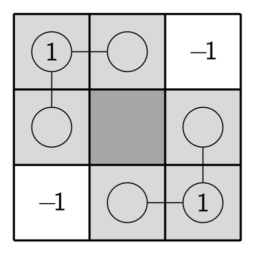

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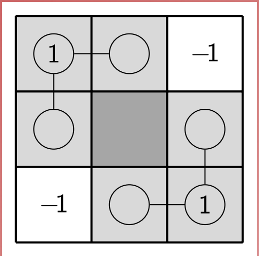

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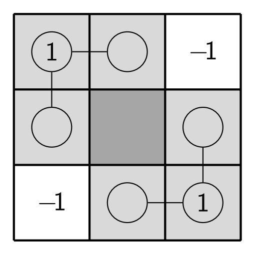

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[BM'13] setting: yes

Our setting: ....

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Our transcript lemma:

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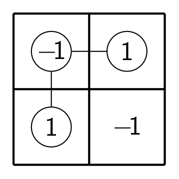

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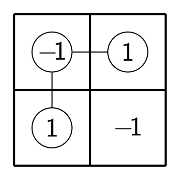

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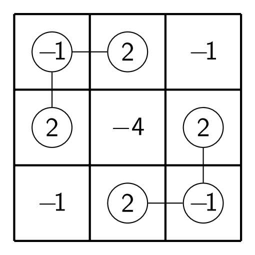

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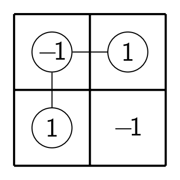

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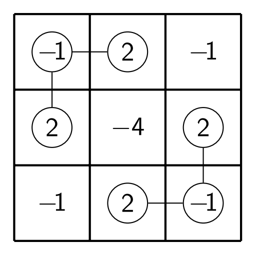

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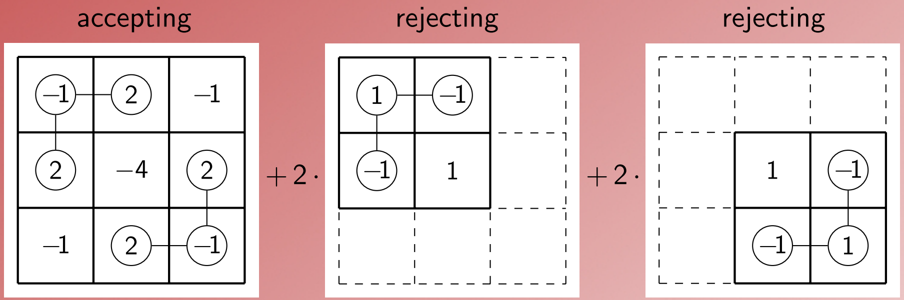

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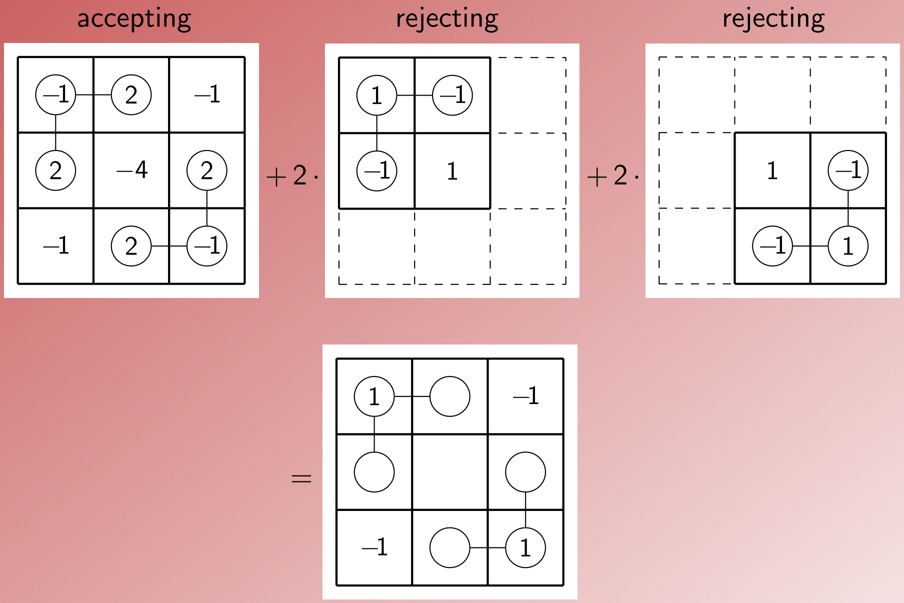

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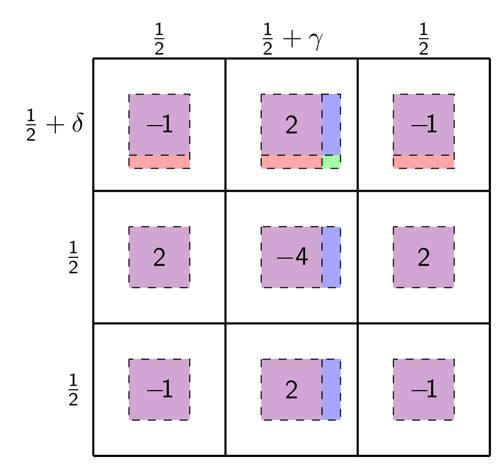

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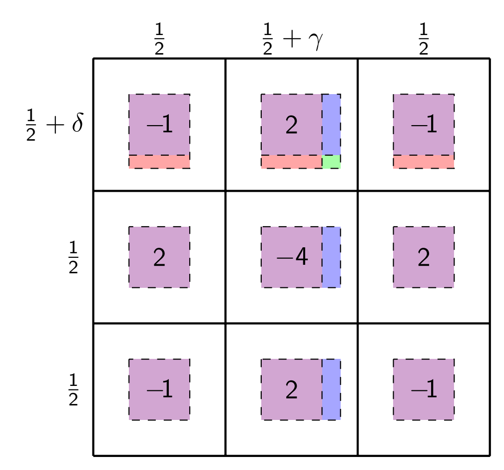

lin comb of probabilities
$= 2 \cdot$ green area
$= \Theta(\delta\gamma)$

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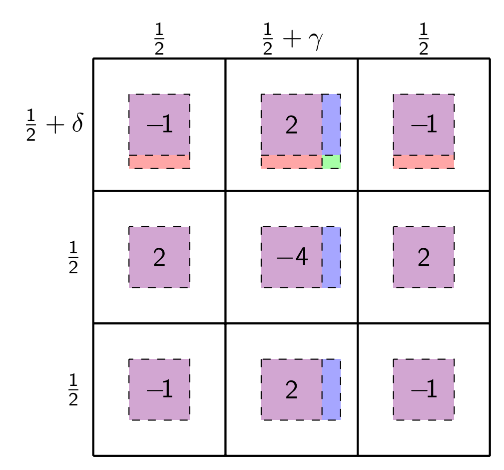

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$\leq$

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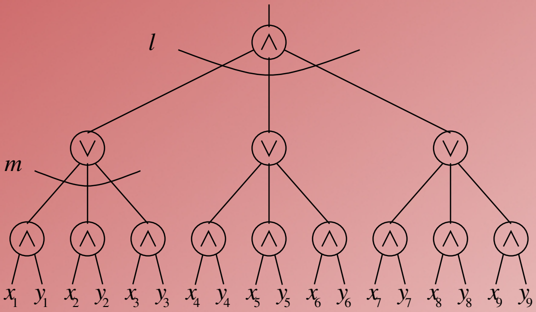
Generalized TRIBES



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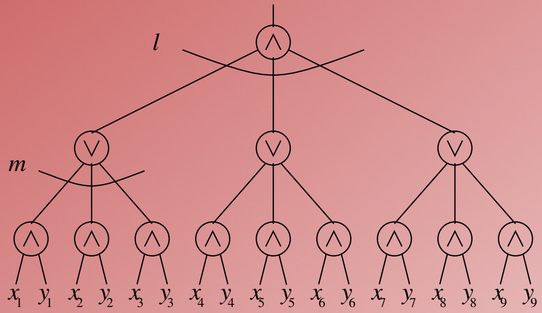

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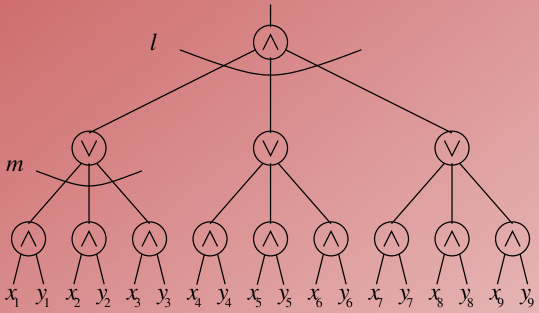

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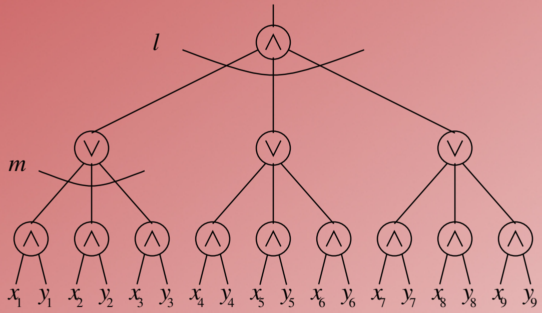

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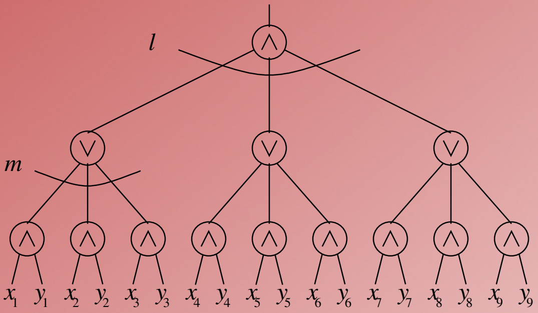

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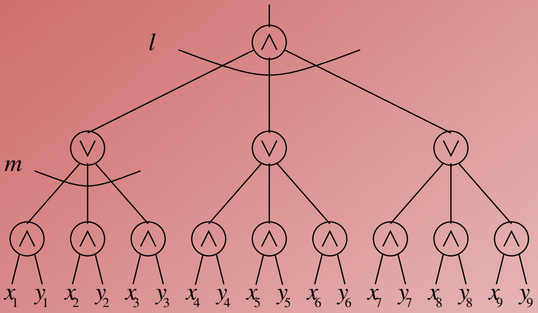

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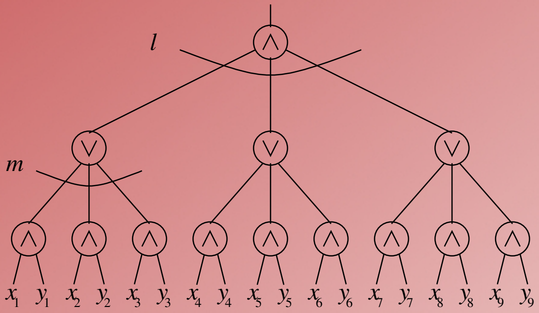
for decision trees

Generalized Tribes



$\Omega(\epsilon \cdot \ell m)$: still holds

$O(\epsilon \cdot \ell m)$: if $\epsilon \geq \Omega(1/\ell)$

What if $\epsilon \leq o(1/\ell)$?

$\ell = 2$: $O(\sqrt{\epsilon} \cdot m)$

$\Omega(\sqrt{\epsilon} \cdot m)$
for decision trees

open:
$\Omega(\sqrt{\epsilon} \cdot m)$
for communication?

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Also: Simplified proof of $\mathsf{UP} \cap \mathsf{coUP} \not\subseteq \mathsf{BPP}$ [Klauck, CCC'03]

The end