Online Covering with Sum of ℓ_q-Norm Objectives

Viswanath Nagarajan

Industrial and Operations Engineering
University of Michigan

Joint work with Xiangkun Shen

Outline

Motivation

- Traditional design and analysis of algorithms assumes complete knowledge of the entire input.
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- This assumption may not be realistic in practice.
- Online optimization: deals with input uncertainty.
  Makes decisions without *any* information about the future.

Online Algorithms Setting

1. Inputs are revealed incrementally over time.
2. Each input needs to be satisfied as soon as revealed.
3. Any decision made earlier cannot be revised.

Many computational problem are intrinsically online as immediate decisions are required. Examples: scheduling, paging, routing...

Competitive ratio is worst-case ratio between

- online objective, and
- optimal offline objective.

Online Primal-Dual Approach

A powerful algorithmic technique applied for a wide variety of problems.
[Alon, Awerbuch, Azar, Buchbinder, Naor 03]...

- Formulate a linear programming (LP) relaxation.
- Solving the LP online
- Obtain an online rounding algorithm for the fractional solution.

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- Formulate a linear programming (LP) relaxation.
- Solving the LP online – highly nontrivial (unlike offline setting).
- Obtain an online rounding algorithm for the fractional solution.

Online Packing-Covering LPs

- Special class of LPs with wide applications.
- All entries a_{ji}, c_i are non-negative.

Primal problem: covering

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_{ji} x_i \geq 1, \quad \forall j \in [m], \\ & x \geq \mathbf{0}. \end{aligned}$$

Dual problem: packing

$$\begin{aligned} \max \quad & \sum_{j=1}^{m} y_j \\ \text{s.t.} \quad & \sum_{j=1}^{m} a_{ji} y_j \leq c_i, \quad \forall i \in [n], \\ & y \geq \mathbf{0}. \end{aligned}$$

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- Covering LPs: $\Theta(\log d)$-competitive [Buchbinder, Naor 09] [Gupta, N. 14].
- Packing LPs: $\Theta(\log d\rho)$-competitive [Buchbinder, Naor 09].

  $d =$ row sparsity, $\rho = a_{max}/a_{min}$.

Online Mixed Packing-Covering LPs

- Mixed LPs: $O(\log t \log d\rho\kappa)$ and $\Omega(\log t \log d)$ [Azar, Bhaskar, Fleischer, Panigrahi 13].

Mixed packing/covering

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \sum_{i=1}^{n} a_{ji} x_i \geq 1, \quad \forall j \in [m], \\ & \sum_{i=1}^{n} p_{kj} x_j \leq \lambda, \quad \forall k \in [t], \\ & x \geq \mathbf{0}. \end{aligned}$$

Beyond Online LPs

- Many online applications with convex/concave objectives:
  energy-efficient scheduling [Bansal, Pruhs, Stein 09], matching [Devanur, Jain 12]
  paging [Menache, Singh 15] , network routing [Gupta, Krishnaswamy, Pruhs 12],
  combinatorial auctions [Blum, Gupta, Mansour, Sharma 11]...
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  combinatorial auctions [Blum, Gupta, Mansour, Sharma 11]...
- Can we extend the online primal-dual approach by designing online algorithms for *general classes* of convex programs?
- Recent results by [Azar Cohen Panigrahi 14] [Buchbinder Chan Gupta N. Naor 14] [Chen Huang Kang 15] and [Eghbali Fazel Mesbahi 16]

Online Convex Covering

- Minimize convex objective s.t. linear covering constraints (online).
- Many applications: mixed packing-covering, capacitated facility location, welfare maximization with production costs.

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  Competitive ratio $O(p \log d)^p$ where $p = \sup_{x \geq 0} \frac{x^T \nabla f(x)}{f(x)}$.
  [Buchbinder Chan Gupta N. Naor 14]

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- When constraint k arrives i.e., $\sum_{i=1}^{n} a_{ki} x_i \geq 1$ update:
  Primal: increase each x_i at rate $\frac{\partial x_i}{\partial \tau} = \frac{a_{ki} x_i + \frac{1}{d}}{\nabla_i f(x)}$.
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- This leads to $O(p \log \rho d)^p$ ratio where $\rho = a_{max}/a_{min}$.

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- This leads to $O(p \log \rho d)^p$ ratio where $\rho = a_{max}/a_{min}$.
- Better dual update (needs dual decrease) gives $\Theta(p \log d)^p$ ratio.

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Analysis idea [Buchbinder Chan Gupta N. Naor 14].

- Let $\bar{x}$ be final primal solution.
- Prove *pointwise* bound $A^T y \leq \alpha \cdot \nabla f(\bar{x})$. *Uses gradient monotonicity.*
- This allows bounding the dual objective by roughly $\mathbf{1}^T y$.
- Rest of analysis similar to *linear* case with cost vector $\nabla f(\bar{x})$.

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  Q: Good competitive ratios for other convex objectives?

Outline

Our Results

- Natural class with non-monotone gradients: sums of ℓ_q-norms.

  Eg. $f(x) = \|x\|_2 \Rightarrow \nabla f(x) = \frac{x}{\|x\|_2}$

Theorem

There is an $O(\log d\rho)$-competitive online algorithm for minimizing sum of ℓ_q-norm objectives subject to linear covering constraints.

$d \approx$ row-sparsity of constraints. $\rho = a_{max}/a_{min}$.

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Applications:

- Covering: non-uniform multicommodity buy-at-bulk network design. Improves [Ene, Chakrabarty, Krishnaswamy, Panigrahi 15].
- Packing: throughput maximization with ℓ_p-norm edge capacities. Generalizes [Awerbuch, Azar, Plotkin 93] for ℓ_∞-norm.

Sum of ℓ_q-Norm Objectives

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$$\begin{aligned} \min \quad & \sum_{e=1}^{r} c_e \|x(S_e)\|_{q_e} \\ \text{s.t.} \quad & Ax \geq \mathbf{1}, \\ & x \geq \mathbf{0}. \end{aligned}$$

- Each $S_e \subseteq [n]$ is any subset of variables.

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- Each $S_e \subseteq [n]$ is any subset of variables.
- The dual can be derived using Lagrangian duality.
- If all $|S_e| = 1$ then reduces to packing/covering LPs.

Simplification: disjoint S_e

Lemma

If there is a poly-time α-competitive algorithm for instances with disjoint S_e*, then there is a poly-time $O(\alpha)$-competitive algorithm for all instances.*

Disjoint S_e allows for cleaner algorithm/analysis.

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For any pair of feasible primal-dual solutions x and (y, μ), we have

$$\sum_{e=1}^{r} c_e \|x(S_e)\|_{q_e} \geq \sum_{k=1}^{m} y_k.$$

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- This follows from the following inequalities:

$$y^T \mathbf{1} \leq y^T Ax = \mu^T x \leq \sum_{e=1}^{r} \sum_{i \in S_e} \mu_i \cdot x_i \leq \sum_{e=1}^{r} \|\mu(S_e)\|_{p_e} \cdot \|x(S_e)\|_{q_e} \leq \sum_{e=1}^{r} c_e \cdot \|x(S_e)\|_{q_e}$$

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- Strong duality also holds since Slater's condition is satisfied.

Algorithm

Algorithm for ℓ_q-norm packing/covering

When the k^{th} request $\sum_{i=1}^{n} a_{ki} x_i \geq 1$ arrives

Let τ be a continuous variable denoting the current time.;

while *the constraint is unsatisfied, i.e.,* $\sum_{i=1}^{n} a_{ki} x_i < 1$ **do**

For each i with $a_{ki} > 0$, increase x_i at rate $\frac{\partial x_i}{\partial \tau} = \frac{a_{ki} x_i + \frac{1}{d}}{\nabla_i f(x)} = \frac{a_{ki} x_i + \frac{1}{d}}{c_e x_i^{q_e - 1}} \|x(S_e)\|_{q_e}^{q_e - 1}$;

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- The algorithm is identical to the one in [Azar, Buchbinder, Chan, Chen, Cohen, Gupta, Huang, Kang, Nagarajan, Naor, Panigrahi 16] for convex functions with monotone gradients.
- New ideas needed in analysis.

Analysis Outline

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  Partition time into phases where Φ_e increases by factor θ

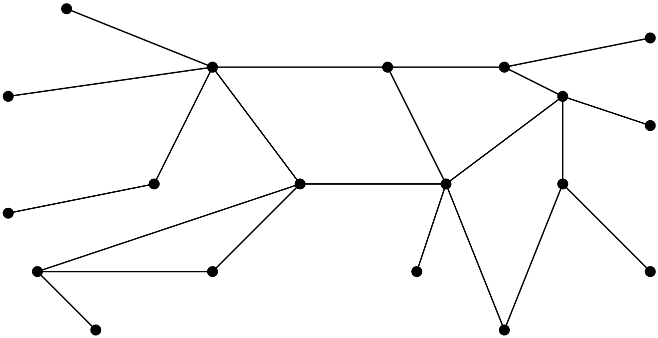
  Bound increase in $\|\mu(S_e)\|_p$ separately for each phase

  Choose θ (depends on q_e) so that overall increase $O(\log \rho d)$

Covering Application: Non-Uniform Buy-at-Bulk

- An undirected graph $G = (V, E)$,
- Monotone subadditive cost function g_e on each edge $e \in E$,
- Source/destination (s_i, t_i) pairs arrive online.

Find an $s_i - t_i$ path P_i for each $i \in [m]$ minimizing $\sum_{e \in E} g_e(load_e)$.
$load_e$ = number of paths using e.



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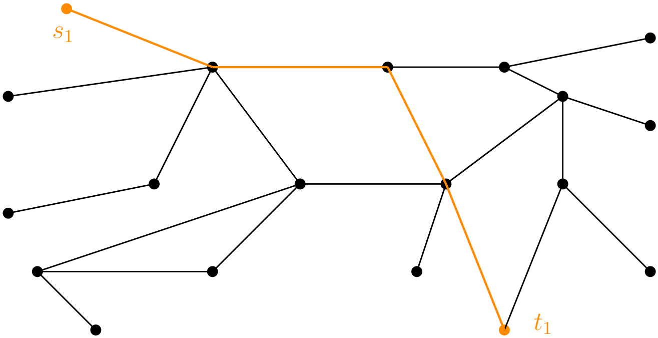

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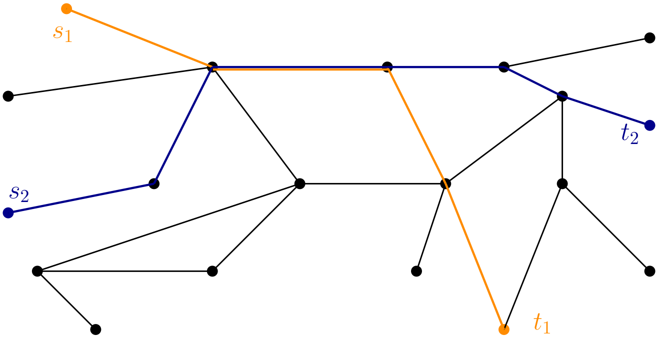

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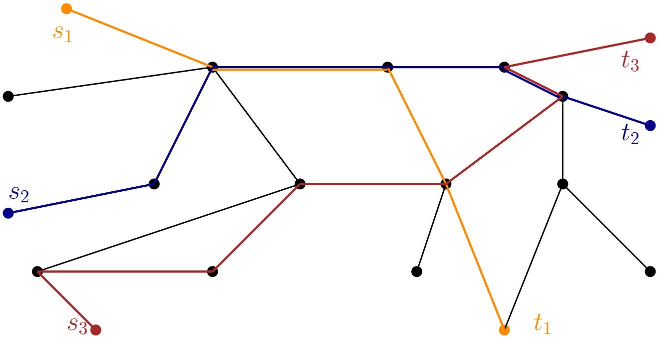
Find an $s_i - t_i$ path P_i for each $i \in [m]$ minimizing $\sum_{e \in E} g_e(load_e)$.
$load_e$ = number of paths using e.



Prior Work

Theorem (Ene, Chakrabarty, Krishnaswamy, Panigrahi 15)

There is an $O(\alpha\beta\gamma \log^5 n)$-competitive randomized online algorithm.

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Key component was an $O(\log^3 n)$-competitive algorithm for the LP:

$$
\begin{aligned}
\min \quad & \sum_{r\in V}\sum_{e\in E} c_e \cdot x_{e,r} + \sum_{r\in V}\sum_{e\in E} \ell_e \cdot \sum_{u\in\mathcal{T}} f_{r,u,e} \\
\text{s.t.} \quad & \sum_{r\in V} z_{ir} \geq 1, \qquad \forall i \in [m] \\
& \{f_{r,s_i,e} : e \in E\} \text{ is a flow from } s_i \text{ to } r \text{ of } z_{ir} \text{ units}, \quad \forall r \in V, i \in [m] \\
& \{f_{r,t_i,e} : e \in E\} \text{ is a flow from } r \text{ to } t_i \text{ of } z_{ir} \text{ units}, \quad \forall r \in V, i \in [m] \\
& f_{r,u,e} \leq x_{e,r}, \qquad \forall u \in \mathcal{T}, e \in E \\
& x, f, z \geq 0
\end{aligned}
$$

Our Result for Buy-at-Bulk

- Using our fractional algorithm, we obtain a tight $O(\log n)$-competitive ratio for a convex *reformulation* of the same LP.

$$\min \quad \sum_{r\in V}\sum_{e\in E} c_e \cdot \left(\max_{u\in\mathcal{T}} f_{r,u,e}\right) + \sum_{r\in V}\sum_{e\in E} \ell_e \cdot \sum_{u\in\mathcal{T}} f_{r,u,e}$$

$$\text{s.t.} \quad \sum_{r\in R_s} f_{r,s_i}(S_r) + \sum_{r\in R_t} f_{r,t_i}(T_r) \geq 1,\ \forall i \in [m],\ \forall (R_s, R_t) \text{ partition of } V,$$

$$\forall S_r : s_i - r \text{ cut}, \forall r \in R_s,\ \forall T_r : r - t_i \text{ cut}, \forall r \in R_t,$$

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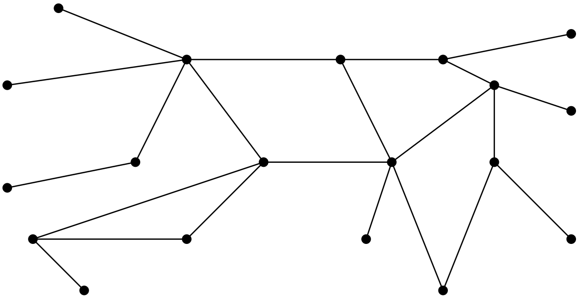
Theorem (This paper + [Ene, Chakrabarty, Krishnaswamy, Panigrahi 15])

$O(\alpha\beta\gamma\log^3 n)$-competitive ratio for non-uniform buy-at-bulk.

Packing Application: Throughput Maximization

- A directed graph $G = (V, E)$ with edge capacities,
- Unit demand requests (s_i, t_i) arrive online.

Find paths for max number of $s_i - t_i$ requests s.t. $load_e \leq c_e, \forall e$.



Theorem ([Awerbuch, Azar, Plotkin 93] [Buchbinder, Naor 09])

There is an $O(\log m)$-competitive online algorithm where $m = \#$ edges. Assumes that each capacity is $\Omega(\log m)$.

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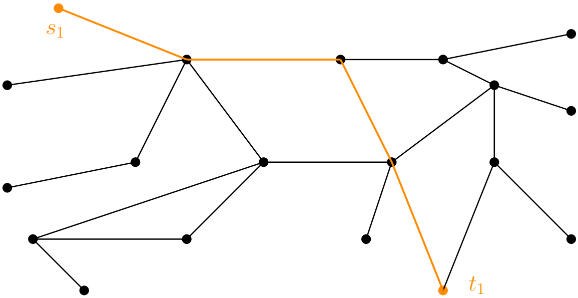

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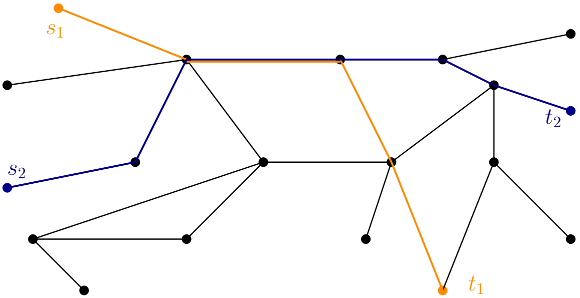

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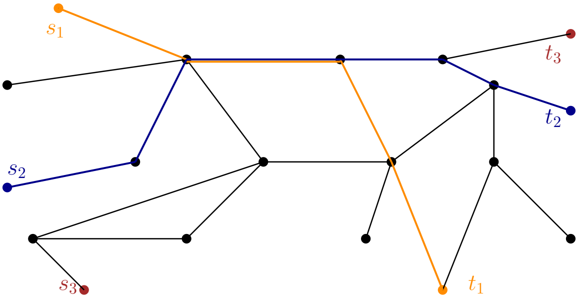

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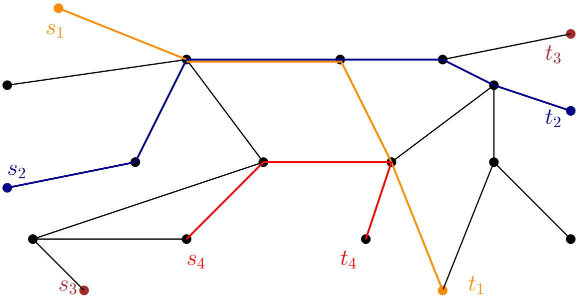

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- A directed graph $G = (V, E)$,
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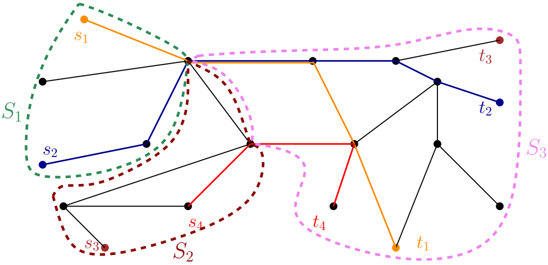

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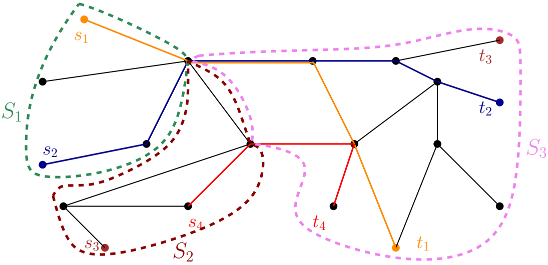

Theorem (This paper)

There is an $O(\log m)$-competitive online algorithm where $m = \#$ edges. Assumes that each $c_j = \Omega(\log m) \cdot |S_j|^{1/p}$.

Outline

Conclusion

General framework for online packing-covering with convex objectives.

- Gave nearly tight $O(\log d\rho)$ ratio for sum of ℓ_q-norms. Analysis goes beyond "monotone gradient" assumption in prior works.
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Thank you!