# Maximum principles at infinity on Riemannian manifolds and the Ahlfors-Khas'minskii duality Joint works with 

M. Rigoli, B. Bianchini, A.G. Setti, P. Pucci, M. Magliaro, D. Valtorta and L.F. Pessoa

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Banff, April 2017

- $(X,\langle\rangle$,$) Riemannian, u \in C^{2}(X)$.

Finite maximum principle: if $x_{0} \in X$ max point of $u$,

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u\left(x_{0}\right)=\sup _{x} u, \quad\left|\nabla u\left(x_{0}\right)\right|=0, \quad \nabla^{2} u\left(x_{0}\right) \leq 0
$$

$X$ noncompact, $u$ bounded above. Look for $\left\{x_{k}\right\} \subset X$ satisfying:
(Ekeland') $\left.\quad u^{\prime}\left(x_{k}\right) \rightarrow \sup _{x} u, \quad \mid \nabla u^{( } x_{k}\right) \mid \rightarrow 0$
(Omori)
$u\left(x_{k}\right) \rightarrow \sup _{x} u$,
$\left|\nabla u\left(x_{k}\right)\right| \rightarrow 0$,
$\nabla^{2} u\left(x_{k}\right) \leq \frac{1}{k}\langle$,
(You) $u\left(x_{k}\right) \rightarrow \sup _{x} u$,



- $(X,\langle\rangle$,$) Riemannian, u \in C^{2}(X)$.
- Finite maximum principle: if $x_{0} \in X$ max point of $u$,

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$(Y a u) \quad u\left(x_{k}\right) \rightarrow \sup _{X} u, \quad\left|\nabla u\left(x_{k}\right)\right| \rightarrow 0, \quad \Delta u\left(x_{k}\right) \leq \frac{1}{k}$
- $(X, \mathrm{~d})$ metric. Then [Ekeland '74, Weston '77, Sullivan '81]
$X$ complete $\quad \Longleftrightarrow \quad X$ has the Ekeland principle


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An Example [H. Omori '67, M.-Rigoli '10]
$\varphi: X^{m} \rightarrow \mathbb{R}^{2 m-1}$ isometric immersion, $X$ complete.
$v \in \mathbb{S}^{2 m}$. Non-degenerate cone:


Suppose $\varphi(X) \subset C_{V, \varepsilon}$. Fix $T=\left\langle\varphi\left(x_{0}\right), v\right\rangle, a \in(0, \varepsilon)$. Define


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## Theorem

If $-B^{2} r^{2} \leq$ Sect $\leq 0$, for a constant $B>0$, then $X$ cannot be contained into a non-degenerate cone of $\mathbb{R}^{2 m-1}$.

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\begin{aligned}
& u(x)=\sqrt{T^{2}+a^{2}|\varphi(x)|^{2}}-\langle\varphi(x), v\rangle \\
& \Longrightarrow u<T \quad \text { on } x, \quad \varphi(\{u>0\}) \text { is bounded. }
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& x \in\{u>0\}, W \in T_{x} X,|W|=1 \\
& \nabla^{2} u(W, W)=\frac{a^{2}(1+\langle\|(W, W), \varphi\rangle}{\sqrt{T^{2}+a^{2}|\varphi|^{2}}-\langle\|(W, W), v\rangle-\frac{a^{4}\langle W, \varphi)^{2}}{\left(T^{2}+a^{2}|\varphi|^{2}\right.}} \\
& \geq \frac{a^{2} T^{2}}{\left(T^{2}+a^{2}|\varphi|^{2}\right)^{3 / 2}}+|\|(W, W)|\{\cdots\} . \\
& {[\text { Otsuki]: } \exists W: \|(W, W)=0}
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\nabla^{2} u(W, W) \geq \frac{a^{2} T^{2}}{\left(T^{2}+a^{2}|\varphi|^{2}\right)^{3 / 2}} \geq c>0 \quad \text { on }\{u>0\}
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- X Riemannian. Omori (Yau) principle holds if

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\begin{aligned}
& \exists w \in C^{2}(X \backslash K) \quad(K \text { cpt. }) \text { with } \\
& w \leq 0 \text { on } X \backslash K, \quad w(x) \rightarrow-\infty \text { as } x \text { diverges, } \\
& |\nabla w| \leq 1, \quad \nabla^{2} w \geq-\langle,\rangle \quad(\Delta w \geq-1), \quad \lambda>0 .
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## ADVANTAGES:

- Weak formulation: Weak Laplacian principle holds iff $\forall u \in C^{2}(X)$ bounded above and solving

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for some $f \in C(\mathbb{R})$, then $f\left(\sup _{X} u\right) \leq 0$.


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$\liminf _{r \rightarrow+\infty} \frac{\log \operatorname{vol}\left(B_{r}\right)}{r^{2}}<+\infty \quad \Longrightarrow \quad \Delta$ has weak Laplacian principle.



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$0<\mathcal{A} \in C\left(\mathbb{R}^{+}\right), \quad 0<b \in C(X)$,
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Def: $(b /)^{-1} \Delta_{\mathcal{A}}$ has weak max. principle if

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\operatorname{div}(\mathcal{A}(|\nabla u|) \nabla u) \geq b(x) f(u) l(|\nabla u|) \quad \text { on } \Omega_{\gamma} \quad \Rightarrow f\left(\sup _{x} u\right) \leq 0
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[Mitidieri-Pohozaev, Mitidieri-D'Ambrosio '12, Farina-Serrin '11]

RELATION WITH STOCHASTIC ANALYSIS $X$ Riemannian, $p(x, y, t)$ heat Kernel of $X$.

## Brownian motion

 Alexandrov compactification.If it is 1 for some (any) $(x, t)$, we say that $X$ is stochastically complete.
$Y$ Martingale on $X$. If

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\mathcal{P}\left(Y_{t} \in X: Y_{0}=x\right)=1 \quad \text { for each }(x, t)
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we say that $X$ is martingale complete [M. Emery]

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 $X^{m}$ Riemannian, $F$ subequation, $F_{0} \doteq F \cup\{r \leq 0\}$.Definition
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## [Ahlfors, Alias-Miranda-Rigoli-Albanese]

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[Berestycki - Hamel - Rossi '07]

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F has the Khas'minskii property iff for each ( $K, h$ ) pair, there exists $w \in F(X \backslash K), \quad h \leq w \leq 0, \quad w(x) \rightarrow-\infty$ as $x \rightarrow \infty$.
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Immersions and submersions
(1) $\sigma: X^{m} \rightarrow Y^{n}$ isometric immersion, proper, $\|$ II $\|_{\infty}<+\infty$.

For $k \leq m$, consider $F_{k}=\left\{\lambda_{k}(A) \geq f(r)\right\}$
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\begin{aligned}
& \left\{\sum_{j=n-k+1}^{n} \lambda_{j}(A) \geq f(r)\right\} \cup \widetilde{E} \text { has Ahlfors on } Y \\
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