Optimal principal eigenfunction for elliptic operators with large drift

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Joint work with F. Hamel, E. Russ

Plan of the talk

- Isoperimetric problem
 - Faber-Krahn inequality
 - Adding a drift

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 - Principal eigenvalue for nonlinear operators
- Asymptotics of the optimal principal eigenfunction
 - The conjectures
 - The (partial) answers

Isoperimetric problem for the principal eigenvalue

Let λ_{Ω} denote the principal eigenvalue of $-\Delta$ in a bounded domain Ω , i.e.,

$$\begin{cases} -\Delta \varphi = \lambda_{\Omega} \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial \Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

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Answer (Faber, Krahn 1920s): $\Omega =$ the ball.

Tool: Schwarz symmetrization.

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Adding a drift $v \in L^{\infty}(\Omega)$. Let $\lambda_{\Omega,v}$ denote the principal eigenvalue:

$$\begin{cases} -\Delta \varphi - \mathbf{v} \cdot \nabla \varphi = \lambda_{\Omega, \mathbf{v}} \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial \Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

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Question

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Answer (Hamel-Nadirashvili-Russ, Ann. of Math. 2011):

$$\Omega=$$
 the ball, $v(x)=- aurac{x}{|x|}.$

Tool: new type of symmetrization.

Optimization in a fixed domain

Let Ω be a given bounded smooth domain. For $v \in L^{\infty}(\Omega)$, let λ_v be the principal eigenvalue:

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$$\forall \tau \geq 0, \qquad \lambda(\tau) := \inf\{\lambda_{\mathbf{v}} : \|\mathbf{v}\|_{L^{\infty}(\Omega)} \leq \tau\}.$$

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Theorem (Hamel-Nadirashvili-Russ)

The infimum is achieved by a unique \underline{v} . Furthermore

$$\underline{v} = au rac{
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eq 0,$$

where φ_{τ} is the associated principal eigenfunction.

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Optimal principal eigenfunction

Principal eigenvalue for nonlinear operators

 $arphi_ au \in \mathcal{C}^2(\Omega)$ satisfies the nonlinear eigenvalue problem

$$\begin{cases} -\Delta \varphi_{\tau} - \tau |\nabla \varphi_{\tau}| = \lambda(\tau) \varphi_{\tau} & \text{in } \Omega \\ \varphi_{\tau} = 0 & \text{on } \partial \Omega \\ \varphi_{\tau} > 0 & \text{in } \Omega. \end{cases}$$

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Can we say that $\lambda(\tau)$ is THE principal eigenvalue?

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1-homogeneous \implies Berestycki-Nirenberg-Varadhan approach:

 $\lambda_{gen} = \sup\{\lambda \ : \ (-\Delta - \tau |\nabla| - \lambda I) \text{ admits a positive supersolution}\}.$

- Pucci (Felmer-Quaas)
- 1-homogeneous (Quaas-Sirakov)
- p and ∞ Laplacian (Kawohl-Lindqvist, Birindelli-Demengel, Juutinen)
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 $=\min\{\lambda : \exists \varphi > 0, \ -\Delta \varphi - \tau |\nabla \varphi| \le \lambda \varphi \text{ in } \Omega, \ \varphi = 0 \text{ on } \partial \Omega\}.$

Furthermore, the above extrema are attained only by (a multiple of) φ_{τ} .

Theorem (Hamel-Nadirashvili-Russ)

$$e^{-R\tau} \leq \lambda(\tau) \leq e^{-r\tau}, \qquad B_r(x_1) \subset \Omega \subset B_R(x_2).$$

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Conjecture 1(-)

Let $(x_{\tau})_{\tau}$ be maximal points for $(\varphi_{\tau})_{\tau}$. Then

$$d(x_{\tau}) o \max_{\overline{\Omega}} d \quad as \ \tau \to +\infty.$$

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Theorem (-)

$$\lim_{\tau \to +\infty} \frac{\varphi_{\tau}(x)}{\|\varphi_{\tau}\|_{\infty}(1 - e^{-\tau d(x)})} = 1, \qquad \text{uniformly w.r.t. } x \in \Omega.$$

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 $-\Psi'' - (au + arepsilon) \Psi' \ge \lambda(au) \Psi \quad ext{ in } (0,r)$

with $\varepsilon > 0$ independent of τ (using $\lambda(\tau) \leq e^{-r\tau}$ because $B_r(x_1) \subset \Omega$).

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 $\psi := \Psi \circ d$

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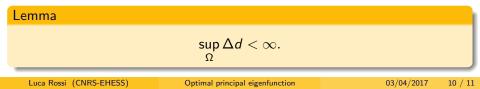
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$$\psi_{ au}(\mathbf{x}) := arphi_{ au}\left(\mathbf{x}_{ au} + rac{\mathbf{x}}{ au}
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Lemma

For $B_R(x_0) \subset \Omega$, $\varepsilon > 0$ and τ large enough,

$$\forall \ 2\frac{N-1}{\tau} \leq r \leq r' \leq R, \qquad \min_{\partial B_{r'}(x_0)} \varphi_{\tau} \geq \frac{1-\varepsilon r'}{1-\varepsilon r} \min_{\partial B_{r}(x_0)} \varphi_{\tau}.$$

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Covering argument.

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