Distances between classes of sphere-valued maps

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Based on a joint work with **H. Brezis** and **P. Mironescu**

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  (in both cases "energy scales like length")

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(3) What is the maximal distance of a map $u \in \mathcal{E}_{\mathbf{a},\mathbf{d}}$ to $\mathcal{E}_{\mathbf{b},\mathbf{e}}$, i.e.,

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Upper bound: via the "dipole construction".

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Can be used to construct pairs of dipoles with an arbitrary small energy-difference!

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- Set $\Sigma_p(u)=\inf_{v\sim u}\int_\Omega|\nabla v|^p=\inf_{\varphi\in W^{1,p}(\Omega)}\int_\Omega|\nabla(ue^{-i\varphi})|^p$.

Summary of results [BMS]:

(i) For every $u_0,v_0\in W^{1,p}(\Omega;\mathbb{S}^1)$:

$$\text{dist}^p_{W^{1,p}}(\mathcal{E}(u_0),\mathcal{E}(v_0)):=\inf_{u\sim u_0,v\sim v_0}\int_\Omega|\nabla(u-v)|^p\geq\left(\frac{2}{\pi}\right)^p\Sigma_p(u_0\overline{v}_0).$$

(ii) $\text{dist}^p_{W^{1,p}}(\mathcal{E}(u_0),\mathcal{E}(1))=\left(\frac{2}{\pi}\right)^p\Sigma_p(u_0)$,
but in general strict inequality "$>$" may occur in (i).

(iii) For every $u_0,v_0\in W^{1,p}(\Omega;\mathbb{S}^1)$ with $v_0\not\sim u_0$ we have:

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Thank you for your attention!