Semiclassical Robin Laplacians: Miscellaneous of linear and nonlinear results

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IRMAR

This talk is devoted to recent results about the semiclassical Robin Laplacian under linear and nonlinear aspects:

i. Weyl formulae,
ii. tunneling effect,
iii. semiclassical concentration-compactness.

I will give an overview on collaborations with **A. Kachmar**, **S. Fournais**, **B. Helffer**, **P. Keraval**, **L. Le Treust**, and **J. Van Schaftingen**.

This first part is devoted to the semiclassical analysis of the operator

$$\mathcal{L}_h = -h^2\Delta\,,$$

with domain

$$\mathrm{Dom}(\mathcal{L}_h) = \{u \in H^2(\Omega)\ :\ \mathbf{n}\cdot h^{\frac{1}{2}}\nabla u = u \text{ on } \Gamma\}\,,$$

where $\mathbf{n}$ is the *outward* pointing normal and $h > 0$ is the semiclassical parameter. In this talk, $\Gamma = \partial\Omega$ will be smooth.
The associated quadratic form is given by

$$\forall u \in H^1(\Omega)\,,\quad \mathcal{Q}_h(u) = \int_\Omega |h\nabla u|^2\mathrm{d}\mathbf{x} - h^{\frac{3}{2}}\int_\Gamma |u|^2\mathrm{d}s\,,$$

where $\mathrm{d}s$ is the standard surface measure on the boundary.
We denote by $\mu_n(h)$ the eigenvalues of $\mathcal{L}_h$.

Selected authors

Many people have worked on this subject and on different aspects: **Exner**, **Frank**, **Freitas**, **Geisinger**, **Helffer**, **Kachmar**, **Keraval**, **Krejčiřík**, **Levitin**, **Minakov**, **Pankrashkin**, **Parnovski**, **Popoff**, etc.

An effective Hamiltonian for low lying eigenvalues

Among this wide literature, let us point out one result. Let us introduce

$$\mathcal{M}_h^{\text{eff}} = -h + h^2\mathcal{L}^\Gamma - h^{\frac{3}{2}}\kappa(s)\,,$$

For all fixed $n \in \mathbb{N} \setminus \{0\}$, there holds[1]

$$\mu_n(h) = \mu_n^{\text{eff}}(h) + \mathcal{O}(h^2)\,.$$

[1] cf. Pankrashkin-Popoff

Questions

The known results about the asymptotic expansions of the Robin eigenvalues only concern individual eigenvalues.

Are the effective Hamiltonians effective enough to describe more eigenvalues and prove for instance Weyl asymptotics?

Questions

When there are symmetries, are the effective Hamiltonians effective enough to describe the tunneling effect[2], say in two dimensions?

This question might be difficult since the "effective Hamiltonians" are only effective modulo $\mathcal{O}(h^2)$ in smooth cases!

[2] *i.e.* exponentially small gap between eigenvalues

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Weyl formulae for the Robin Laplacian in the semiclassical limit

with **A. Kachmar** and **P. Keraval**

Confluentes Mathematici (2017)

Theorem

For $\varepsilon_0 \in (0,1)$, $h > 0$, *we let*

$$\mathcal{N}_{\epsilon_0, h} = \{n \in \mathbb{N}^* : \mu_n(h) \leq -\varepsilon_0 h\}\,.$$

There exist positive constants h_0, C_+, C_- *such that, for all* $h \in (0, h_0)$ *and* $n \in \mathcal{N}_{\varepsilon_0, h}$,

$$\mu_n^-(h) \leq \mu_n(h) \leq \mu_n^+(h)\,,$$

where $\mu_n^\pm(h)$ *is the n-th eigenvalue of* $\mathcal{L}_h^{\mathrm{eff},\pm}$ *defined by*

$$\mathcal{L}_h^{\mathrm{eff},+} = -h + (1 + C_+ h^{\frac{1}{2}}) h^2 \mathcal{L}^\Gamma - \kappa h^{\frac{3}{2}} + C_+ h^2\,,$$

and

$$\mathcal{L}_h^{\mathrm{eff},-} = -h + (1 - C_- h^{\frac{1}{2}}) h^2 \mathcal{L}^\Gamma - \kappa h^{\frac{3}{2}} - C_- h^2\,.$$

Theorem

We have the following Weyl estimates

i. *for the low lying eigenvalues:*

$$\forall E \in \mathbb{R}\,, \quad \mathsf{N}\left(\mathcal{L}_h, -h + E h^{\frac{3}{2}}\right) \underset{h\to 0}{\sim} \mathsf{N}\left(h^{\frac{1}{2}}\mathcal{L}^{\Gamma} - \kappa, E\right)\,.$$

ii. *for the non positive eigenvalues:*

$$\mathsf{N}\left(\mathcal{L}_h, 0\right) \underset{h\to 0}{\sim} \mathsf{N}\left(h\mathcal{L}^{\Gamma}, 1\right)\,.$$

Strategy

The proof follows the following steps:

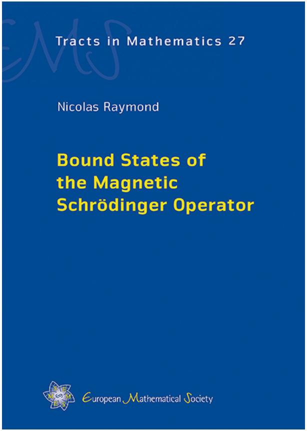
i. the low lying eigenfunctions are localized, in the Agmon sense, near the boundary at a scale of order $h^{\frac{1}{2}}$,

ii. we introduce the tubular "coordinates" $(s,t) \in \Gamma \times (0, Dh^{\frac{1}{4}})$ near the boundary and rescale the normal variable $(\sigma, \tau) = (s, h^{-\frac{1}{2}}t)$,

iii. we implement a standard Born-Oppenheimer reduction,

iv. we estimate the "Born-Oppenheimer correction".

Propaganda



The last two points that are consequences of ideas and strategies developed in many different contexts in the last 30 years.

See for instance **Helffer-Sjöstrand**, **Martinez-Sordoni**, **Jecko**, **Panati-Spohn-Teufel**, or the book (Chapter 11)

Bound States of the Magnetic Schrödinger Operator

EMS Tracts (27) (2017)

Flavor of the proof

We divide the operator by h. We let $\hbar = h^{\frac{1}{4}}$ and we are reduced to study the operator acting on $L^2(\widehat{a}\mathrm{d}\Gamma\mathrm{d}\tau)$ and expressed in the coordinates (σ, τ).

$$\widehat{\mathcal{V}}_T = \Gamma \times (0, T)\,,$$

$$\widehat{V}_T = \{u \in H^1(\widehat{\mathcal{V}}_T) \ : \ u(\sigma, T) = 0\}\,,$$

$$\widehat{\mathcal{D}}_T = \{u \in H^2(\widehat{\mathcal{V}}_T) \cap \widehat{V}_T \ : \ \partial_\tau u(\sigma, 0) = -u(\sigma, 0)\}\,,$$

$$\widehat{\mathcal{Q}}_\hbar^T(u) = \int_{\widehat{\mathcal{V}}_T} \left(\hbar^4 \langle \nabla_\sigma u, \widehat{g}^{-1} \nabla_\sigma u\rangle + |\partial_\tau u|^2\right) \widehat{a}\mathrm{d}\Gamma\mathrm{d}\tau - \int_\Gamma |u(\sigma, 0)|^2 \mathrm{d}\Gamma\,,$$

$$\widehat{\mathcal{L}}_\hbar^T = -\hbar^4 \widehat{a}^{-1} \nabla_\sigma (\widehat{a}\widehat{g}^{-1} \nabla_\sigma) - \widehat{a}^{-1} \partial_\tau \widehat{a} \partial_\tau\,.$$

Here $T = D\hbar^{-1}$.

Flavor of the proof

We must focus on the (one dimensional) transverse operator whose quadratic form is

$$q_B^{\{T\}}(u) = \int_0^T |\partial_\tau u|^2 \widehat{a} \mathrm{d}\tau - |u(\sigma, 0)|^2 ,$$

where $\widehat{a}(\sigma, \tau) = 1 - B\tau$, with $B = \hbar^2 \kappa(\sigma)$ and $T = \hbar^{-1}$. Many results have been obtained on the first eigenvalues of this family of operators with two parameters B and T such as the estimates:

$$\lambda_1(q_B^{\{T\}}) = -1 - \kappa(\sigma)\hbar^2 + \mathcal{O}(\hbar^4) , \qquad \lambda_2(q_B^{\{T\}}) \geq -C\hbar \geq -\frac{\varepsilon_0}{2} .$$

Flavor of the proof

We introduce the normalized groundstate $u_B^{\{T\}}$ of $\mathcal{H}_B^{\{T\}}$ and use it to perform an orthogonal decomposition of $\widehat{\mathcal{L}}_\hbar^T$.
The only thing that we must care about is the commutation defect between ∇_σ and $u_B^{\{T\}}$. This commutator is called "Born-Oppenheimer correction" and we may prove that it is of order $\hbar^4$.

Comments

a. This method has recently been used to establish abstract dimensional reduction results via norm-resolvent convergence, see **Reduction of dimension as a consequence of norm-resolvent convergence and applications** with **D. Krejčiřík**, **J. Royer** and **P. Siegl**

b. This kind of dimensional reductions is considered in the thesis of **P. Keraval** in a pseudo-differential context.

c. Our method can be applied to the **Dirac operator**, acting on $L^2(\Omega, \mathbb{C}^4)$,

$$\alpha \cdot D + m\beta \,,$$

with the **MIT bag condition** $i\beta\alpha \cdot \mathbf{n}\psi = \psi$ on $\partial\Omega$, see **On the MIT bag model in the non-relativistic limit** with **N. Arrizabalaga** and **L. Le Treust**. CMP (2017)

Tunneling for the Robin Laplacian in smooth planar domains.
with **B. Helffer** and **A. Kachmar**.
Communications in Contemporary Mathematics (2017)

Now $d = 2$.

Assumption

The boundary of Ω is a smooth, simple and closed curve.

Asymptotics for non degenerate wells

Among other things, **Helffer** and **Kachmar** have proved the following. If the curvature admits a unique maximum (at a_1) that is non degenerate, then, we have

$$\mu_n(h) = -h - \kappa_{\max} h^{\frac{3}{2}} + (2n-1)\gamma h^{\frac{7}{4}} + o(h^{\frac{7}{4}}) \,,$$

where

$$\gamma = \sqrt{\frac{-\max \kappa''(a_1)}{2}} \,.$$

Assumption

We assume that

i. *Ω is symmetric with respect to the y-axis.*

ii. *The curvature κ on the boundary Γ attains its maximum at exactly two points a_1 and a_2 which are not on the symmetry axis. We write*

$$a_1 = (a_{1,1}, a_{1,2}) \in \Gamma \quad \text{and} \quad a_2 = (a_{2,1}, a_{2,2}) \in \Gamma\,,$$

such that $a_{1,1} > 0$ and $a_{2,1} < 0$.

iii. *The second derivative of the curvature (w.r.t. arc-length) at a_1 and a_2 is negative.*

Heuristics: tunneling formula

As noticed before, it seems to be sufficient to consider an operator in the form

$$\mathcal{M}_{\hbar}^{\mathrm{circ}} = \hbar^2 D_s^2 + \mathfrak{v}(s)$$

on the circle of length $2L$ and where $\mathfrak{v}$ has two non degenerate minima s_{le} and s_{ri}.
In our case, we have

$$\mathfrak{v}(\sigma) = \kappa_{\mathsf{max}} - \kappa(\sigma)$$

Agmon distance between the wells

We let

$$\mathsf{S} = \min\left(\mathsf{S}_{\mathsf{u}}, \mathsf{S}_{\mathsf{d}}\right)\,, \quad \mathsf{S}_{\mathsf{u}} = \int_{[s_{\mathsf{ri}},s_{\mathsf{le}}]} \sqrt{\mathfrak{v}(s)}ds\,, \quad \mathsf{S}_{\mathsf{d}} = \int_{[s_{\mathsf{le}},s_{\mathsf{ri}}]} \sqrt{\mathfrak{v}(s)}ds\,,$$

where $[p,q]$ denotes the arc joining p and q in $\partial\Omega$ counter-clockwise.

Heuristics: tunneling formula

The splitting formula for the operator $\mathcal{M}_{\hbar}^{\mathrm{circ}}$ is obtained by adding the "upper" and "lower" contributions and reads

$$\lambda_2(\hbar) - \lambda_1(\hbar) = 4\hbar^{\frac{1}{2}}\pi^{-\frac{1}{2}}\gamma^{\frac{1}{2}}\left(\mathsf{A}_{\mathsf{u}}\sqrt{\mathfrak{v}(0)}e^{-\frac{\mathsf{S}_{\mathsf{u}}}{\hbar}} + \mathsf{A}_{\mathsf{d}}\sqrt{\mathfrak{v}(L)}e^{-\frac{\mathsf{S}_{\mathsf{d}}}{\hbar}}\right) + \mathcal{O}(\hbar^{\frac{3}{2}}e^{-\frac{\mathsf{S}}{\hbar}}).$$

Then, for the particular model $\mathcal{M}_h^{\mathsf{eff}}$, we easily notice that

$$\mu_2^{\mathsf{eff}}(h) - \mu_1^{\mathsf{eff}}(h) = h^{\frac{3}{2}}(\lambda_2(\hbar) - \lambda_1(\hbar)) ,$$

so that, under Assumption 2, we have

$$\begin{aligned}&\mu_2^{\mathsf{eff}}(h) - \mu_1^{\mathsf{eff}}(h)\\ &\quad = 4h^{\frac{13}{8}}\pi^{-\frac{1}{2}}\gamma^{\frac{1}{2}}\left(\mathsf{A_u}\sqrt{\mathfrak{v}(0)}\exp-\frac{\mathsf{S_u}}{h^{\frac{1}{4}}} + \mathsf{A_d}\sqrt{\mathfrak{v}(L)}\exp-\frac{\mathsf{S_d}}{h^{\frac{1}{4}}}\right)\\ &\qquad + \mathcal{O}\left(h^{\frac{13}{8}+\frac{1}{4}}\exp-\frac{\mathsf{S}}{h^{\frac{1}{4}}}\right) .\end{aligned}$$

A tunneling estimate

Theorem

Under Assumptions 1 and 2, we have

$$\mu_2(h) - \mu_1(h) \underset{h\to 0}{\sim} \mu_2^{\text{eff}}(h) - \mu_1^{\text{eff}}(h) \,.$$

The tubular operator

We divide the operator by h. We let $\hbar = h^{\frac{1}{4}}$ and we are reduced to study the operator acting on $L^2(\widehat{a}\mathrm{d}\Gamma\mathrm{d}\tau)$ and expressed in the coordinates (σ, τ).

$$\widehat{\mathcal{V}}_T = \{(\sigma,\tau) \ : \ \sigma \in \Gamma \text{ and } 0 < \tau < T\}\,,$$
$$\widehat{V}_T = \{u \in H^1(\widehat{\mathcal{V}}_T) \ : \ u(\sigma, T) = 0\}\,,$$
$$\widehat{\mathcal{D}}_T = \{u \in H^2(\widehat{\mathcal{V}}_T) \cap \widehat{V}_T \ : \ \partial_\tau u(\sigma,0) = -u(\sigma,0)\}\,,$$
$$\widehat{\mathcal{Q}}^T_\hbar(u) = \int_{\widehat{\mathcal{V}}_T} \left(\hbar^4 \langle \nabla_\sigma u, \widehat{g}^{-1}\nabla_\sigma u\rangle + |\partial_\tau u|^2\right)\widehat{a}\mathrm{d}\Gamma\mathrm{d}\tau - \int_\Gamma |u(\sigma,0)|^2\mathrm{d}\Gamma\,,$$
$$\widehat{\mathcal{L}}^T_\hbar = -\hbar^4 \widehat{a}^{-1}\nabla_\sigma(\widehat{a}\widehat{g}^{-1}\nabla_\sigma) - \widehat{a}^{-1}\partial_\tau \widehat{a}\partial_\tau\,.$$

Here $T = D\hbar^{-1}$, with $D > \mathsf{S}$.

Let ω be an (open) interval in the circle of length $2L$ identified with the interval $(-L, L]$. We assume that ω contains a unique point s_ω of maximum curvature (i.e. $\kappa(s_\omega) = \kappa_{\max}$) that is non degenerate.

$$\begin{aligned}
\widehat{\mathcal{V}}_\omega &= \omega \times (0, T)\,,\\
\widehat{V}_\omega &= \{u \in H^1(\widehat{\mathcal{V}}_\omega)\ :\ u = 0 \text{ on } \tau = T \text{ and } \partial\omega \times (0, T)\}\,,\\
\widehat{\mathcal{D}}_\omega &= \{u \in H^2(\widehat{\mathcal{V}}_\omega) \cap \widehat{V}_\omega\ :\ \partial_\tau u = -u \text{ on } \tau = 0\}\,.
\end{aligned}$$

The operator $\widehat{\mathcal{L}}_\omega$ is the self-adjoint operator on $L^2(\widehat{\mathcal{V}}_\omega; \widehat{a} d\sigma d\tau)$ with domain $\widehat{\mathcal{D}}_\omega$ and

$$\widehat{\mathcal{L}}_\omega = -\hbar^4\, \widehat{a}^{-1}(\partial_\sigma \widehat{a}^{-1})\partial_\sigma - \widehat{a}^{-1}(\partial_\tau \widehat{a})\partial_\tau\,.$$

We denote by $\mu_\omega(\hbar)$ its lowest eigenvalue. The corresponding positive and L^2-normalized eigenfunction is denoted by $\phi_{\hbar,\omega}$.

Complex WKB expansion

Proposition

There exists a sequence of smooth functions (a_j) such that the following holds. We consider the formal series (or a smooth realization)

$$\Psi_{\hbar,\omega}(\sigma,\tau) \sim \hbar^{-\frac{1}{4}} e^{-\Phi_\omega(\sigma)/\hbar} \sum_{j\geq 0} \hbar^j a_j(\sigma,\tau) ,$$

Proposition (continued)

Φ_ω is the Agmon distance to the well at $\sigma = s_\omega$ of the effective potential

$$\mathfrak{v}(\sigma) = \kappa_{\mathsf{max}} - \kappa(\sigma)$$

and defined by the formula

$$\Phi_\omega(\sigma) = \int_{[s_\omega,\sigma]} \sqrt{\mathfrak{v}(\tilde{\sigma})}\,d\tilde{\sigma}\,,$$

Proposition (continued)

a_0 is in the form $a_0(\sigma,\tau) = \xi_{0,\omega}(\sigma) u_0(\tau)$ where

$$u_0(\tau) = \sqrt{2} e^{-\tau}\,,$$

and

$$\xi_{0,\omega}(\sigma) = \xi_0(\sigma) = \left(\frac{\gamma}{\pi}\right)^{\frac{1}{4}} \exp\left(-\int_{s_\omega}^{\sigma} \frac{\Phi_\omega'' - \gamma}{2\Phi_\omega'} d\tilde{\sigma}\right)$$

is the solution of the transport equation of the effective Hamiltonian

$$\Phi_\omega' \partial_\sigma \xi_0 + \partial_\sigma(\Phi_\omega' \xi_0) = \gamma \xi_0\,, \qquad \textit{with } \gamma = \sqrt{\frac{-\kappa''(s_\omega)}{2}}\,.$$

For $j \geq 1$, $a_j(\sigma,\tau)$ is a linear combination of functions of the form

$$f_{j,k}(\sigma) g_{j,k}(\tau)\,, \quad f_{j,k} \in \mathcal{C}^\infty(\omega) \text{ and } g_{j,k} \in \mathcal{S}(\overline{\mathbb{R}_+})\,.$$

Proposition (continued)

The formal series $\Psi_{\hbar,\omega}$ *satisfies*

$$e^{\Phi_\omega/\hbar}\left(\widehat{\mathcal{L}}_\omega - \mu\right)\Psi_{\hbar,\omega} \sim 0\,,$$

where μ *is an asymptotic series in the form*

$$\mu \sim -1 - \kappa_{\max}\hbar^2 + \gamma\hbar^3 + \sum_{j\geq 4}\mu_j\hbar^j\,.$$

This series is the Taylor series of the first eigenvalue $\mu_\omega(\hbar)$*.*

About the proof

The proof appears in a previous paper by **Helffer-Kachmar**, see also the talk by Virginie...

Lemma

There exist constants $C > 0$ and $\hbar_0 \in (0,1)$ such that, for all $\hbar \in (0, \hbar_0)$ and $u \in \widehat{V}_\omega$,

$$\widehat{\mathcal{Q}}_\omega(u) \geq \int_{\widehat{\mathcal{V}}_\omega} \widehat{a}^{-2} \hbar^4 |\partial_\sigma u|^2 \,\widehat{a} d\sigma d\tau + \int_{\widehat{\mathcal{V}}_\omega} \left(-1 - \kappa_{\max}\hbar^2 + \hbar^2 \mathfrak{v}(\sigma) - C\hbar^4\right) |u|^2 \,\widehat{a} d\sigma d\tau \,.$$

WKB approximations

Proposition

Let K be a compact set in ω. There holds

$$e^{\Phi_\omega/\hbar}(\Psi^\sharp_{\hbar,\omega} - \Pi_\omega \Psi^\sharp_{\hbar,\omega}) = \mathcal{O}(\hbar^\infty)\,,$$
$$e^{\Phi_\omega/\hbar}\partial_\sigma(\Psi^\sharp_{\hbar,\omega} - \Pi_\omega \Psi^\sharp_{\hbar,\omega}) = \mathcal{O}(\hbar^\infty)\,,$$

in $\mathcal{C}(K; L^2(0,T))$ and where we have let $\Psi^\sharp_{\hbar,\omega} = \chi(T^{-1}\tau)\Psi_{\hbar,\omega}$.

Let us introduce the two quasimodes

$$f_{\hbar,\mathrm{ri}} = \chi_{\mathrm{ri}}\phi_{\hbar,\mathrm{ri}}\,, \qquad f_{\hbar,\mathrm{le}} = \chi_{\mathrm{le}}\phi_{\hbar,\mathrm{le}}\,,$$

Proposition

The splitting between the first two eigenvalues of $\widehat{\mathcal{L}}_\hbar$ *is given by*

$$\hat{\lambda}_2(\hbar) - \hat{\lambda}_1(\hbar) = 2|w_{\mathrm{le,ri}}(\hbar)| + \tilde{\mathcal{O}}(e^{-2S/\hbar}),$$

where

$$w_{\mathrm{le,ri}}(\hbar) = \langle (\widehat{\mathcal{L}}_\hbar - \mu(\hbar)) f_{\hbar,\mathrm{le}}, f_{\hbar,\mathrm{ri}} \rangle = \langle [\widehat{\mathcal{L}}_\hbar, \chi_{\mathrm{le}}] \phi_{\hbar,\mathrm{le}}, \chi_{\mathrm{ri}} \phi_{\hbar,\mathrm{ri}} \rangle .$$

Then, we perform some integrations by parts and replace by the WKB approximations...

This ends the linear part of this talk.

Semiclassical Sobolev constants
for the electro-magnetic Robin Laplacian.
with **L. Le Treust**, **S. Fournais** and **J. Van Schaftingen**.
Journal of the Mathematical Society of Japan (2017)

Let us introduce

i. the electro-magnetic potential $(V, \mathbf{A}) \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R} \times \mathbb{R}^d)$,

ii. the variable Robin coefficient $\gamma \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$.

We let

$$\mathcal{G} = (\Omega, \mathsf{Id}, V, \mathbf{A}, \gamma) ,$$

where Id stands for the Euclidean metrics.
For notational convenience, we will constantly consider quintuples gathering the Robin electro-magnetic geometry

$$\mathsf{G} = (U, \mathsf{R}, \mathsf{V}, \mathsf{A}, \mathsf{c}) .$$

The magnetic field

If we write

$$\mathsf{A} = \sum_{j=1}^{d} A_j \mathrm{d}x_j \,,$$

the magnetic field is the 2-form

$$\mathsf{B} = \mathrm{d}\mathsf{A} \,,$$

that may be identified with a skew-symmetric matrix.

Homogeneous geometries

We will meet the following homogeneous Euclidean geometries:

i. if $\mathbf{x}_0 \in \Omega$, we consider

$$\mathcal{G}_{\mathbf{x}_0} = \left(\mathbf{x}_0 + \mathbb{R}^d, \mathsf{Id}, V(\mathbf{x}_0), \mathcal{A}^{\mathsf{L}}_{\mathbf{x}_0}, 0\right),$$

ii. if $\mathbf{x}_0 \in \partial\Omega$, we consider

$$\mathcal{G}_{\mathbf{x}_0} = \left(\mathbf{x}_0 + \mathsf{T}_{\mathbf{x}_0}(\partial\Omega) + \mathbb{R}_+\mathbf{n}(\mathbf{x}_0), \mathsf{Id}, V(\mathbf{x}_0), \mathcal{A}^{\mathsf{L}}_{\mathbf{x}_0}, \gamma(\mathbf{x}_0)\right),$$

where $\mathsf{T}_{\mathbf{x}_0}(\partial\Omega)$ is the linear tangent space of $\partial\Omega$ at $\mathbf{x}_0$.

$\mathcal{A}^{\mathsf{L}}_{\mathbf{x}_0}$ is a linear vector potential with uniform magnetic field $\mathbf{B}(\mathbf{x}_0)$.

Homogeneous geometry at infinity

When $\mathsf{G} = (U, \mathrm{Id}, \mathsf{V}, \mathsf{A}, \mathsf{c})$ is an homogeneous geometry on a half-space, we let

$$\underline{\mathsf{G}} = (\mathbb{R}^d, \mathrm{Id}, \mathsf{V}, \mathsf{A}, 0)\,.$$

Let $p \in [2, 2^*)$, with $2^* = \frac{2d}{d-2}$. We are mainly interested in the following "optimal Sobolev constant", in the case of a Euclidean geometry[3],

$$\lambda(\mathsf{G}, h, p) = \inf_{\substack{\psi \in \mathsf{H}^1_{\mathsf{A}}(U), \\ \psi \neq 0}} \frac{\mathfrak{Q}_{\mathsf{G},h}(\psi)}{\|\psi\|^2_{\mathsf{L}^p(U)}},$$

where

$$\mathsf{H}^1_{\mathsf{A}}(U) = \{\psi \in \mathsf{L}^2(U) : (-ih\nabla + \mathsf{A})u \in \mathsf{L}^2(U)\}$$

and for all $\psi \in \mathsf{H}^1_{\mathsf{A}}(U)$,

$$\mathfrak{Q}_{\mathsf{G},h}(\psi) = \int_U |(-ih\nabla + \mathsf{A})\psi|^2 + h\mathsf{V}|\psi|^2 \mathrm{d}\mathbf{x} + h^{\frac{3}{2}} \int_{\partial U} \mathsf{c}|\psi|^2 \mathrm{d}s.$$

[3] see the talk by Hynek...

NLS equation with Robin condition

$$\begin{cases} (-ih\nabla + \mathrm{A})^2\psi + h\mathrm{V}\psi = \lambda(\mathrm{G}, h, p)|\psi|^{p-2}\psi\,, \\ \qquad (-ih\nabla + \mathrm{A})\psi\cdot\mathbf{n} = -ih^{\frac{1}{2}}\mathrm{c}\psi, \text{ on } \partial U\,. \end{cases}$$

Previous results

The p-eigenvalues in presence of pure and constant magnetic fields in $\mathbb{R}^d$ have been investigated in the seminal paper:

Stationary solutions of nonlinear Schrödinger equations with an external magnetic field
M. Esteban and **P-L. Lions**
Nonlinear Differential Equations Appl. (1989)

In this case, they prove, by means of a concentration-compactness analysis, that $\lambda(\mathrm{G}, 1, p)$ is a minimum.

Previous results

The semiclassical estimates of the p-eigenvalues is tackled in the following papers.

Semiclassical stationary states for nonlinear Schrödinger equations with a strong external magnetic field
J. Di Cosmo and **J. Van Schaftingen**
J. Differential Equations (2015)

Estimate of $\lambda(\mathcal{G}, h, p)$ ($\Omega = \mathbb{R}^d$ with an electro-magnetic field) in the semiclassical limit, up to subsequences extractions of the semiclassical parameter.

Previous results

Optimal magnetic Sobolev Constants in the Semiclassical Limit

S. Fournais and **N. Raymond**

Ann. Inst. H. Poincaré Anal. Non Linéaire (2016)

Estimate of $\lambda(\mathcal{G}, h, p)$ ($\Omega \subset \mathbb{R}^2$ bounded with Dirichlet b.c. and pure magnetic field) with quantitative remainder when h goes to 0.

Let us now state our main assumption which is of spectral nature: we assume that the 2-eigenvalue is not degenerate.

Assumption

We assume that

i. $\Omega \ni \mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, 2) = \mathrm{Tr}^{+}\,\mathbf{B}(\mathbf{x}) + V(\mathbf{x})$ *does not vanish,*

ii. $\partial\Omega \ni \mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, 2)$ *is bounded from below by a positive constant.*

Lemma

There exist $h_0, C > 0$ *such that, for all* $h \in (0, h_0)$, *we have*

$$\lambda(\mathcal{G}, h, 2) \geq h \inf_{\mathbf{x}\in\overline{\Omega}} \lambda(\mathcal{G}_{\mathbf{x}}, 1, 2) - C h^{\frac{5}{4}} > 0\,,$$

and the infimum defining $\lambda(\mathsf{G}, h, p)$ *for* $\mathsf{G} = \mathcal{G}$ *is a minimum.*

Proposition

The function $\mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, p)$ *is lower semi-continuous on* $\overline{\Omega}$ *for* $p > 2$. *In particular, it has a minimum.*

Theorem

There exist $h_0 > 0, C > 0$ such that, for all $h \in (0, h_0)$,

$$h^{\frac{d}{2}-\frac{d}{p}} h(1 - C h^{\frac{1}{6}}) \inf_{\mathbf{x}\in\overline{\Omega}} \lambda(\mathcal{G}_{\mathbf{x}}, 1, p) \leq \lambda(\mathcal{G}, h, p) ,$$

$$\lambda(\mathcal{G}, h, p) \leq h^{\frac{d}{2}-\frac{d}{p}} h(1 + C h^{\frac{1}{2}} |\log h|) \inf_{\mathbf{x}\in\overline{\Omega}} \lambda(\mathcal{G}_{\mathbf{x}}, 1, p) .$$

We may consider $\mathcal{M} \subset \overline{\Omega}$ the set of the minimizers of the concentration function $\mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, p)$. In relation with the last theorem, we can deduce the following (exponential) decay estimate of the minimizers away from $\mathcal{M}$.

Theorem

For all $\varepsilon > 0$ we define

$$\mathcal{M}_\varepsilon = \mathcal{M} + D(0, \varepsilon)\,.$$

Then, for all $\varepsilon > 0$ and $\rho \in \left(0, \frac{1}{2}\right)$, there exists $h_0 > 0$, $C > 0$ such that, for all $h \in (0, h_0)$ and all L^p-normalized minimizers ψ_h,

$$\|\psi_h\|_{\mathsf{L}^p(\complement \mathcal{M}_\varepsilon)} \leq Ce^{-\varepsilon h^{-\rho}}\,.$$

Theorem

Let us consider $p \in (2, 2^*)$. *We have the following two existence results.*

i. *If* G *is a homogeneous geometry with* *U being an half-space* *and such that* $\lambda(\mathrm{G}, 1, 2)$ *is positive and*

$$\lambda(\mathrm{G}, 1, p) < \lambda(\underline{\mathrm{G}}, 1, p)\,, \tag{1}$$

then the infimum $\lambda(\mathrm{G}, 1, p)$ *is attained.*

ii. *The condition* (1) *is always satisfied (for a given electro-magnetic field) as soon as* $\gamma \in (-\infty, c_0]$ *with* $c_0 > 0$ *small enough.*

(Extremely condensed) proof

Consider a minimizing sequence $(\psi_j)_{j\in\mathbb{N}}$ with $\|\psi_j\|_{\mathsf{L}^p(\mathbb{R}^d_+)} = 1$.

i. exclude the boundary vanishing (use the energy gap): up to a magnetic translation parallel to the boundary and a subsequence extraction, we may assume that (ψ_j) converges to $\psi \neq 0$ weakly in $\mathsf{H}^1_{\mathbf{A}}(\mathbb{R}^d_+)$,

ii. exclude the dichotomy ($p > 2$): prove that $\|\psi\|_{\mathsf{L}^p(\mathbb{R}^d_+)} = 1$,

iii. we are reduced to the precompact case.

A one dimensional model

Find a solution of the following ODE on $\mathbb{R}_+$:

$$-u'' + u = \lambda|u|^{p-2}u\,, \qquad u'(0) = cu(0)\,,$$

where $\lambda = \lambda((\mathbb{R}_+, \mathsf{Id}, 1, 0, c), 1, p)$, $u \in \mathsf{H}^1(\mathbb{R}_+, \mathbb{R})$, $\|u\|_{\mathsf{L}^p(\mathbb{R}_+)} = 1$, $p > 2$ and $c \in \mathbb{R}$.

Proposition

The previous system has a unique solution for $c \in (-1, 1)$ and no solution for $|c| \geq 1$.

Proposition

Here we consider a constant geometry $\mathcal{G}_{\mathbf{x}}$ with $\mathbf{x} \in \overline{\Omega}$ and let us consider a subcritical $p \in (2, 2^*)$. We have the following continuity properties.

i. *The function $\mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, p)$ is continuous on Ω.*
ii. *The function $\mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, p)$ is continuous on $\partial\Omega$.*
iii. *The function $\mathbf{x} \mapsto \lambda(\mathcal{G}_{\mathbf{x}}, 1, p)$ is lower semi-continuous on $\overline{\Omega}$.*

For the first point, see
Properties of groundstates of nonlinear Schrödinger equations under a constant magnetic field
D. Bonheure, **M. Nys** and **J. Van Schaftingen**

(Extremely condensed) proof

Let us consider $\mathbf{x}_* \in \Omega$ and a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \Omega$ such that $\mathbf{x}_n \to \mathbf{x}_*$ when n goes to infinity. We denote $\mathfrak{Q}_{\mathcal{G}_{\mathbf{x}_n},1} = \mathfrak{Q}_n$ and

$$\lambda(\mathcal{G}_{\mathbf{x}_n}, 1, p) =: \lambda_n\,, \qquad \liminf_{n\to+\infty} \lambda_n =: \lambda_*\,,.$$

We want to show that

$$\lambda_* \geq \lambda(\mathcal{G}_{\mathbf{x}_*}, 1, p)\,.$$

(Extremely condensed) proof

Let ψ_n be an L^p-normalized function such that

$$\mathfrak{Q}_n(\psi_n) = \lambda_n \,.$$

The sequence (ψ_n) is bounded in $L^2(\mathbb{R}^d)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. By diamagnetism, we infer that $(|\psi_n|)$ is also bounded in $H^1(\mathbb{R}^d)$.

(Extremely condensed) proof

i. Excluding the vanishing: up to extraction and magnetic translations, we may assume that (ψ_n) weakly converges in $\mathsf{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and in $\mathsf{L}^p(\mathbb{R}^d)$ to some $\psi_* \neq 0$.

ii. Excluding the dichotomy: we show that

$$\liminf_{n\to+\infty} \lambda_n \geq \liminf_{n\to+\infty} \lambda_n \left(\alpha^{\frac{2}{p}} + (1-\alpha)^{\frac{2}{p}}\right) .$$

with $\alpha = \|\psi_*\|^p_{\mathsf{L}^p(\mathbb{R}^d)} \in (0,1]$ and thus $\alpha = 1$.

iii. Precompact case: (ψ_n) converges in $\mathsf{L}^p(\mathbb{R}^d)$.

Proposition

Let $\mathbf{x}_0 \in \Omega$. *There exist* $h_0 > 0$, $C > 0$ *such that, for all* $h \in (0, h_0)$,

$$\lambda(\mathcal{G}, h, p) \leq h^{\frac{d}{2}-\frac{d}{p}} h \left(\lambda(\mathcal{G}_{\mathbf{x}_0}, 1, p) + C h^{\frac{1}{2}}\right) .$$

Proposition

Let $\mathbf{x}_0 \in \partial\Omega$. *There exist* $h_0 > 0$, $C > 0$ *such that, for all* $h \in (0, h_0)$,

$$\lambda(\mathcal{G}, h, p) \leq h^{\frac{d}{2}-\frac{d}{p}} h \left(\lambda(\mathcal{G}_{\mathbf{x}_0}, 1, p) + C h^{\frac{1}{2}} |\log h|\right) .$$

Lemma

Let us consider $E = \{(\alpha, \rho, h, \mathbf{k}) \in (\mathbb{R}_+)^3 \times \mathbb{Z}^d : \alpha \geq \rho\}$*. There exists a family of smooth cutoff functions* $(\chi^{[\mathbf{k}]}_{\alpha,\rho,h})_{(\alpha,\rho,h,\mathbf{k})\in E}$ *on* $\mathbb{R}^d$*, with*

$$\chi^{[\mathbf{k}]}_{\alpha,\rho,h}(\mathbf{x}) = \chi^{[0]}_{\alpha,\rho,h}(\mathbf{x} - (2h^\rho + h^\alpha)\mathbf{k}),$$

such that $0 \leq \chi^{[\mathbf{k}]}_{\alpha,\rho,h} \leq 1$,

$$\chi^{[\mathbf{k}]}_{\alpha,\rho,h} = 1, \quad \text{on} \quad |\mathbf{x} - (2h^\rho + h^\alpha)\mathbf{k}|_\infty \leq h^\rho\,,$$
$$\chi^{[\mathbf{k}]}_{\alpha,\rho,h} = 0, \quad \text{on} \quad |\mathbf{x} - (2h^\rho + h^\alpha)\mathbf{k}|_\infty \geq h^\rho + h^\alpha\,,$$

and such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\chi^{[\mathbf{k}]}_{\alpha,\rho,h}\right)^2 = 1\,.$$

Lemma (continued)

There exists also $D > 0$ such that, for all $h > 0$,

$$\int_{\mathbb{R}^d} |\nabla \chi^{[0]}_{\alpha,\rho,h}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y} \leq D h^{\rho d} h^{-\alpha-\rho} \, .$$

Lemma (Reconstruction of the L^p norm)

Let $p \geq 2$. Let us consider the partition of unity $(\chi^{[\mathbf{k}]}_{\alpha,\rho,h})$, with $\alpha \geq \rho > 0$. There exist $C > 0$ and $h_0 > 0$ such that for all $\psi \in \mathsf{L}^p(\Omega)$ and $h \in (0, h_0)$, there exists $\tau_{\alpha,\rho,h,\psi} = \tau \in \mathbb{R}^2$ such that

$$\int_\Omega |\psi(\mathbf{x})|^p \mathrm{d}\mathbf{x} \leq (1 + Ch^{\alpha-\rho}) \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_\Omega |\tilde{\chi}^{[\mathbf{k}]}_{\alpha,\rho,h}\psi(\mathbf{x})|^p \mathrm{d}\mathbf{x},$$

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \mathfrak{Q}_{\mathcal{G},h}(\tilde{\chi}^{[\mathbf{k}]}_{\alpha,\rho,h}\psi) - \tilde{D}h^{2-\rho-\alpha}\|\psi\|^2_{\mathsf{L}^2(\Omega)} \leq \mathfrak{Q}_{\mathcal{G},h}(\psi),$$

with $\tilde{\chi}^{[\mathbf{k}]}_{\alpha,\rho,h}(\mathbf{x}) = \tilde{\chi}^{[\mathbf{k}]}_{\alpha,\rho,h}(\mathbf{x} - \tau)$.

As a gift from the previous step, we get the following.

Proposition

For all $\varepsilon > 0$*, there exist* $h_0, C > 0$ *such that for all* $h \in (0, h_0)$ *and all* L^p*-normalized minimizer* ψ_h*, we have*

$$\|\psi_h\|_{\mathsf{L}^p(\complement\mathcal{M}_\varepsilon)} \leq C h^{\frac{1}{6p}}\,.$$

We get an a priori control of the non-linearity and this implies an exponential estimate.

Thanks for your attention!