Interaction energy between vortices of vector fields on Riemannian surfaces

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We consider 3 related problems for vector fields on 2-dimensounal Riemannian manifolds:

Problem 1: Intrinsic

Let (S, g) be a compact 2-dimensional Riemannian manifold. Consider tangent vector fields u, and minimize the *intrinsic* energy

$$I_\varepsilon^{in}(u) = \frac{1}{2}\int_S \left[|Du|_g^2 + \frac{1}{\varepsilon^2}\left|1 - |u|_g^2\right| \right] \mathrm{vol}_g.$$

Here

$$|Du|_g^2(x) := |D_{\tau_1} u|_g^2(x) + |D_{\tau_2} u|_g^2(x)$$

where D_v denotes covariant differentiation and $\{\tau_1, \tau_2\}$ are any orthonormal basis for $T_x S$.

Problem 2: extrinsic, tangent vector fields

Let (S, g) be a compact, connected, oriented 2-dimensional Riemannian manifold isometrically embedded in $\mathbb{R}^3$. Consider sections m of the tangent bundle of S, and minimize the *extrinsic* energy

$$I_\varepsilon^{ex}(m) \;=\; \frac{1}{2}\int_S \left[|\bar{D}m|^2 + \frac{1}{\varepsilon^2}\left|1-|m|^2\right|\right] d\mathcal{H}^2$$

Here $m \in H^1(S; \mathbb{R}^3)$, with

$$m(x) \in T_x S \text{ for every } x \in S,$$

and $|\bar{D}m|^2 := |\bar{D}_{\tau_1}\bar{m}|^2 + \bar{D}_{\tau_2}\bar{m}|^2$, where

- $\bar{m}$ is an extension of m to a neighborhood of S,
- $\{\tau_1(x), \tau_2(x)\}$ form a basis for $T_x S$,
- $\bar{D}_v$ denotes covariant derivative in $\mathbb{R}^3$.

well known: $|\bar{D}m|^2$ is independent of the choice of extension $\bar{m}$.

Problem 3: extrinsic, S^2 constraint

Let (S, g) be compact a 2-dimensional Riemannian manifold isometrically embedded in $\mathbb{R}^3$. Consider maps $M : S \to \mathbb{S}^2$, and minimize

$$I_\varepsilon^{S^2}(M) \;=\; \frac{1}{2}\int_S \left[|\nabla M|^2 + \frac{1}{\varepsilon^2}(M\cdot\nu)^2 \right] d\mathcal{H}^2$$

Here $|\nabla M|^2 := |\tau_1 \cdot \nabla \bar{M}|^2 + |\tau_2 \cdot \nabla \bar{M}|^2$, where $\bar{M}$ is an extension of S to a neighborhood of S and $\{\tau_1(x), \tau_2(x)\}$ form a basis for $T_x S$. As usual, $|\nabla M|^2$ is independent of the choice of extension $\bar{M}$

Remark 1: If $M = 1$ and m denotes the tangential part of M, then

$$(M\cdot\nu)^2 = 1 - |m|^2 = \Big|1 - |m|^2\Big|.$$

Remark 2 :

Let $S \subset \mathbb{R}^3$ be a fixed smooth surface isometrically embedded in $\mathbb{R}^3$. A curved magnetic shell is considered occupying the domain

$$\Omega_h := \{x' + sN(x') \,:\, s \in (0,h),\ x' \in S\}.$$

The magnetization $m : \Omega_h \to \mathbb{S}^2$ is a stable state of the energy functional

$$E^{3D}(m) = \varepsilon^2 \int_{\Omega_h} |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla U|^2 \, dx,$$

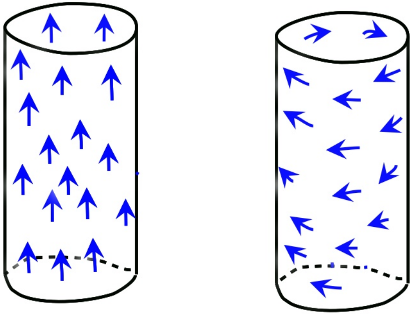
where $U : \mathbb{R}^3 \to \mathbb{R}$ solves the static Maxwell equation

$$\Delta U = \nabla \cdot \left(m \mathbf{1}_\Omega \right) \quad \text{in} \quad \mathbb{R}^3.$$

Carbou (2001) shows that $I_\varepsilon^{S^2}$ arises as the Γ-limit of E^{3D} with ε fixed and $h \searrow 0$.

This is the original motivation for our study.

example



$$I_\varepsilon^{in}(u_{left}) = 0, \qquad I_\varepsilon^{in}(u_{right}) = 0,$$
$$I_\varepsilon^{ex}(u_{left}) = 0, \qquad I_\varepsilon^{ex}(u_{right}) > 0,$$
$$I_\varepsilon^{S^2}(u_{left}) = 0, \qquad I_\varepsilon^{S^2}(u_{right}) > 0.$$

related problems

simplified Ginzburg-Landau on a manifold

(S, g) abstract manifold, $\psi \in H^1(S; \mathbb{C})$,

$$I_\varepsilon(\psi) := \frac{1}{2}\int_S |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}(1 - |\psi|^2)^2 d\,\mathrm{vol}_g \ .$$

See Baraket (1996). (Compare Bethuel-Brezis-Hélein (1994) for Euclidean case)

Ginzburg-Landau on a complex line bundle

ψ a section of a complex line bundle E over a Riemann surface S.
A a connection on E.

$$G_\varepsilon(\psi, A) := \frac{1}{2}\int_S |D_A\psi|^2 + |F_A|^2 + \frac{1}{2\varepsilon^2}(1 - |\psi|^2)^2 d\mathcal{H}^2 \ .$$

See Orlandi (1996), Qing (1997). (Compare Bethuel-Rivière (1994) for Euclidean case)

Ginzburg-Landau on thin shells

(S, g) isometrically embedded in $\mathbb{R}^3$, $\psi \in H^1(S; \mathbb{C})$,

$$I_\varepsilon(\psi) := \frac{1}{2}\int_S |(\nabla - i(A^e)^\tau)\psi|^2 + \frac{1}{2\varepsilon^2}(1 - |\psi|^2)^2 d\mathcal{H}^2 .$$

See Contreras-Sternberg (2010), Contreras (2011). Related work Alama-Bronsard-Galvao-Sousa (2010, 2013) (Compare Sandier-Serfaty (late 90s) for Euclidean case)

discrete-to-continuum limit

(S, g) isometrically embedded in $\mathbb{R}^3$, $\mathcal{T}_\varepsilon := \varepsilon$- triangulation of S,

$$I_\varepsilon^{disc}(\psi) := \frac{1}{2}\sum_{i \neq j \in \mathcal{T}_\varepsilon} \kappa_\varepsilon^{ij} |\psi_\varepsilon(i) - \psi_\varepsilon(j)|^2,$$

where $\psi_\varepsilon(i) \in T_i S$, $|\psi_\varepsilon(i)| = 1$ for all i. Canevari-Segatti (2017)

prior results:

- Euler characteristic nonzero $\Rightarrow \liminf_{\varepsilon\searrow 0} I_\varepsilon^{\square} = +\infty$.

  Canevari, Segatti, Veneroni (2015), Segatti, Snarski, Veneroni (2016)

- study of variational problem when Euler characteristic $= 0$. Segatti, Snarski, Veneroni (2016)

New results (Ignat - J 2017)

- For Problems 1-3, *canonical unit vector fields* and *renormalized energy* for prescribed singularities and fluxes, as in Bethuel-Brezis-Hélein (1994).
- in every case, “second-order Γ-convergence".
- Extrinsic Problem 2 (tangent constraint) and Problem 3 (S^2 constraint with penalization) have essentially the same asymptotics.
- Problem 1 (intrinsic) admits a “lifting” to a linear problem (with topological considerations).
- Problems 2 and 3 seem to be inescapably nonlinear.
- *intrinsic* canonical harmonic unit vector field provides Coulomb gauge for the more nonlinear Problems 2,3.

general set-up for intrinsic problem

- always assume S is oriented
- can then define $i : TS \to TS$ such that
  - i isometry of $T_x S$ to itself for every x, and
  - $\{v, iv\}$ properly oriented orthonormal basis of $T_x S$, or every unit $v \in T_x S$.
- given any vector field u, define 1-form $j(u)$ by

$$j(u)(v) = (D_v u, iu)_g$$

- define *vorticity* associated to u by

$$\omega(u) = dj(u) + \kappa \, \mathrm{vol}_g, \qquad \kappa = \text{Gaussian curvature.}$$

Remark: if u is a smooth unit vector field in an open set O, then $dj(u) = -\kappa \, \mathrm{vol}_g$ and thus $\omega(u) = 0$ in O.

canonical harmonic unit vector field

Theorem

For any $a_1, \ldots, a_k \in S$ and $d_1, \ldots, d_k \in \mathbb{Z}$ such that $\sum d_k = \chi(S)$, there exists unit vector field u^* in $W^{1,p}$ for all $p < 2$, such that

$$\omega(u^*) = 2\pi \sum d_i \delta_{a_i}, \qquad d^* j(u^*) = 0.$$

- *If $\mathfrak{g} := genus(S) = 0$, then u^* is unique up to a global phase.*
- *If $\mathfrak{g} := genus(S) > 0$, then u^* is unique up to a global phase, once $2\mathfrak{g}$ "flux integrals" $\Phi_\ell, \ell = 1, \ldots, 2\mathfrak{g}$ are specified.*
- *Finally, $\mathcal{L}(a, d) := \{\text{admissible values of } (\Phi_1, \ldots, \Phi_{2\mathfrak{g}})\}$ are quantized and depend smoothly on $\sum d_i \delta_{a_i}$*

(All results: Ignat - J, 2017)

outline of proof

Given a_i, d_i as above

1. Find 1-form j^* as described below
2. find unit vector field u^* such that $j(u^*) = j^*$. If genus> 0 need to pay attention to topology.

Construction of j^*. if $j(u^*) = j^*$, then equations for u^* become

$$\boxed{dj^* = -\kappa\, \mathrm{vol}_g + 2\pi \sum d_i \delta_{a_i}, \qquad d^* j^* = 0}.$$

In fact

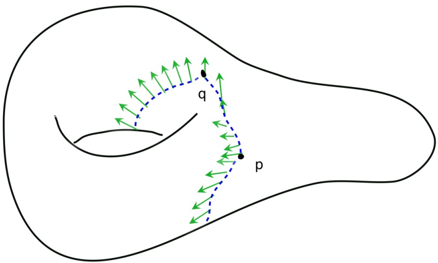
$$\boxed{j^* = d^*\psi^* + \text{ harmonic 1-form, if } \mathfrak{g} > 0}$$

where

$$\boxed{-\Delta\psi^* = -\kappa\, \mathrm{vol}_g + 2\pi \sum d_i \delta_{a_i}}$$

The condition $j^* = j(\text{unit vector field})$ implies constraints on the harmonic 1-form.

Construction of u^*: To solve find u^* such that $j(u^*) = j^*$:



- choose $p \in S$ and $v \in T_pS$, and set $u^*(p) := v$.
- given $q \in S$, consider $\gamma : [0,1] \to S$ with $\gamma(0) = p, \gamma(1) = q$.
- Let $U(s) \in T_{\gamma(s)}S$ solve

$$D_{\gamma'(s)}U(s) = j(\gamma'(s))\ iU(s), \qquad U(0) = v \in T_pS.$$

- Define $u^*(q) = U(1)$.
- **check that this is well-defined**. This determines $\mathcal{L}(a,d)$.

intrinsic renormalized energy

Theorem

Given a_i, d_i as above, let u^* be the canonical harmonic unit vector field with flux integrals $\{\Phi_k\}$ Then

$$\lim_{r\to 0}\left[\int_{S\setminus\cup B(r,a_i)} \frac{1}{2}|Du^*|_g^2 - (\sum d_i^2)\pi \log\frac{1}{r}\right] = W^{in}(a,d,\Phi)$$

where

$$W^{in}(a,d,\Phi) = 4\pi^2 \sum_{l\neq k} d_l d_k G(a_l,a_k) + 2\pi \sum_{k=1}^{n}\left[\pi d_k^2 H(a_k,a_k) + d_k\psi_0(a_k)\right] + \frac{1}{2}|\Phi|^2 + C_S\ ,$$

where $G(\cdot,\cdot)$ is the Green's function for the Laplacian with regular part $H(\cdot,\cdot)$, and

$$-\Delta\psi_0 = -\kappa + \bar{\kappa}$$

extrinsic renormalized energy

Theorem

Suppose (S, g) *is isometrically embedded in* $\mathbb{R}^3$*.*
Let a_i, d_i *be given such that* $\sum d_i = \chi(S)$*, and fix* $u^* = u^*(a, d, \Phi)$*.*
Suppose that

$$u = e^{i\alpha} u^* \qquad \text{for some } \alpha \in H^1(S; \mathbb{R}).$$

Then for the extrinsic *Dirichlet energy,*

$$W^{ex}(a, d, \Phi) := \lim_{r \to 0} \left[\int_{S \setminus \cup B(r, a_i)} \frac{1}{2} |\bar{D}u|_g^2 - (\sum d_i^2) \pi \log \frac{1}{r} \right]$$

$$= W^{in}(a, d, \Phi) + \int_S \left(\frac{1}{2} |\nabla \alpha|_g^2 + Q_{u^*}(\cos\alpha, \sin\alpha) \right) vol_g$$

Here

$$Q_{u^*}(\cos\alpha, \sin\alpha) = |A(e^{i\alpha} u^*)|^2, \qquad A = \text{2nd fundamental form}$$

is an explicit quadratic function of $\cos\alpha, \sin\alpha$.

Theorem (Γ-convergence)

1) *(Compactness) Let* $(u_\varepsilon)_{\varepsilon\downarrow 0}$ *be a family of vector fields on* S *satisfying*

$$I_\varepsilon^{\square}(u_\varepsilon) \leq N\pi|\log\varepsilon| + C, \qquad \square = {}^{in}, {}^{ex}, {}^{S^2}.$$

Then there exists a sequence $\varepsilon \downarrow 0$ *such that*

$$\omega(u_\varepsilon) \longrightarrow 2\pi\sum_{k=1}^{n} d_k\delta_{a_k} \quad in \;\; W^{-1,1}, \;\; as \;\; \varepsilon \to 0,$$

where $\{a_k\}_{k=1}^n$ *are distinct points in* S *and* $\{d_k\}_{k=1}^n$ *are nonzero integers satisfying* $\sum d_k = \chi(S)$ *and* $\sum |d_k| \leq N$.
Moreover, if $\sum_{k=1}^n |d_k| = N$, *then* $n = N$, $|d_k| = 1$ *for every* $k = 1, \ldots, n$ *and for a subsequence, there exists* $\Phi \in \mathcal{L}(a, d)$ *such that*

$$\Phi(u_\varepsilon) := \left(\int_S (j(u_\varepsilon), \eta_1)_g \, vol_g, \ldots, \int_S (j(u_\varepsilon), \eta_{2\mathfrak{g}})_g \, vol_g\right) \to \Phi$$

as $\varepsilon \to 0$. *(in particular,* $n = \chi(S)$ *modulo 2).*

Theorem (Γ-convergence, continued)

2) *(Γ-liminf inequality) Assume that the vector fields* $u_\varepsilon \in \mathcal{X}^{1,2}(S)$ *satisfy*

$$\begin{aligned} &\omega(u_\varepsilon) \longrightarrow 2\pi \sum_{k=1}^{n} d_k \delta_{a_k} \quad in\ W^{-1,1},\ as\ \varepsilon \to 0, \\ &\Phi(u_\varepsilon) \to \Phi \in \mathcal{L}(a,d) \end{aligned} \tag{1}$$

for n distinct points $\{a_k\}_{k=1}^n \in S^n$ *with* $|d_k| = 1$*. Then*

$$\liminf_{\varepsilon\to 0} \left[I_\varepsilon^\square(u_\varepsilon) - n\pi|\log\varepsilon| \right] \ \geq\ W^\square(a,d,\Phi) + n\gamma_F.$$

3) *(Γ-limsup inequality) For every n distinct points* $a_1, \ldots, a_n \in S$ *and* $d_1, \ldots, d_n \in \{\pm 1\}$ *satisfying* $\sum d_k = \chi(S)$ *and every* $\Phi \in \mathcal{L}(a,d)$ *there exists a sequence of vector fields* u_ε *on S such that* (1) *holds and*

$$I_\varepsilon^\square(u_\varepsilon) - n\pi|\log\varepsilon| \longrightarrow W^\square(a,d,\Phi) + n\gamma_F \quad as\ \varepsilon \to 0.$$

Proofs use

- vortex ball construction
- indirect method in the Calculus of Variations: optimality/lower bounds follow (essentially) from equations that characterize u^*:

$$\omega(u^*) = dj(u^*) + \kappa\,\mathrm{vol}_g = 2\pi \sum_{k=1}^{n} d_k \delta_{a_k}$$

$$d^* j(u) = 0.$$

- careful accounting involving flux integrals Φ.

in Euclidean case, derivation of renormalized energy via 2nd-order Γ convergence: Colliander-J 1999, L i n - X i n 1999, Alicandro-Ponsiglione 2014.

“the indirect method"

- From elementary algebra,

$$\int_{S_{r_\varepsilon}} e_\varepsilon^{in}(u_\varepsilon) = \int_{S_{r_\varepsilon}} \frac{1}{2}|j_\varepsilon^*|_g^2 + \frac{1}{2}\left|\frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*\right|_g^2 + 2(j_\varepsilon^*, \frac{j(u_\varepsilon)}{|u_\varepsilon|_g} - j_\varepsilon^*)_g + e_\varepsilon^{in}(|u_\varepsilon|_g)$$

  Here $S_{r_\varepsilon} = S \setminus \cup B(a_{k,\varepsilon}, r_\varepsilon)$.
  In fact the integrands are pointwise equal.

- In addition, as $\varepsilon \to 0$,

$$\frac{1}{2}\int_{S_{r_\varepsilon}} |j_\varepsilon^*|^2 \,\text{vol} = \pi(\sum_k d_k^2) \log \frac{1}{r_\varepsilon} + W(a^\varepsilon, d^\varepsilon, \Phi^\varepsilon) + O(\sqrt{r_\varepsilon}) + O(r_\varepsilon^2 |\Phi^\varepsilon|^2).$$

- So we only need to estimate $\int_{S_{r_\varepsilon}} (j_\varepsilon^*, \frac{j(u)}{|u|_g} - j_\varepsilon^*)_g \,\text{vol}_g$.
- Equations for j_ε^* (with vortex ball construction) imply

$$j_\varepsilon^* = d^*\psi_\varepsilon + \sum_{k=1}^{2\mathfrak{g}} \Phi_{\varepsilon,k}\eta_k$$

$$d(j(u) - j_\varepsilon^*) \quad \text{is small}$$

thank you for your attention!

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