Minimal Solutions to Some Variational Inequalities

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Introduction

Ω is a bounded open set of $\mathbb{R}^n$.

K is a closed convex subset of $W_0^{1,p}(\Omega)$, $p > 1$ such that

$$\begin{cases} 0 \in K, \\ u, v \in K \Rightarrow u \vee v, u \wedge v \in K. \end{cases} \tag{1}$$

Examples :

$K = W_0^{1,p}(\Omega)$ the V.I. will be an equation.

$K = \{v \in W_0^{1,p}(\Omega) \mid v(x) \geq \Psi(x) \;\; a.e.\}$ where $\Psi \leq 0$ on $\partial\Omega$ - Obstacle problem.

$K = \{v \in W_0^{1,p}(\Omega) \mid |\nabla v(x)| \leq C \;\; a.e.\}$ - Elastic-plastic torsion problem.

Introduction

$a_i(x,\xi)$, $(x,\xi) \in \Omega \times \mathbb{R}^{n+1}$, $i = 0, \cdots, n$

Carathéodory functions such that :

coerciveness : $\sum_{i=0}^{n} a_i(x,\xi)\xi_i \geq \alpha \sum_{i=1}^{n} |\xi_i|^p$, $a.e.\ x, \forall \xi$,

monotonicity : $\sum_{i=0}^{n}(a_i(x,\xi) - a_i(x,\zeta))(\xi_i - \zeta_i) \geq 0$, $a.e.\ x, \forall \xi, \zeta$,

growth condition : for some $\nu \in L^p(\Omega)$,

$$|a_i(x,\xi)| \leq \nu(x) + \beta |\xi|^{p-1}, \ a.e.\ x, \forall \xi, \forall i = 0, \cdots, n.$$

One can then set

$$\langle Au, v \rangle = \int_\Omega a_i(x,u,\nabla u)\partial_i v + a_0(x,u,\nabla u) v dx \quad \forall u, v \in W_0^{1,p}(\Omega).$$

Introduction

For $f \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$ there exists u solution to

$$\begin{cases} u \in K, \\ \langle Au, v - u \rangle \geq \int_\Omega f(v-u)dx \quad \forall v \in K. \end{cases} \tag{2}$$

One has :

Theorem

Suppose that $f \geq 0$. Then there exists a minimal solution to (2) i.e.

$$u(x) = \min\{v(x) \mid v \text{ is solution to } (2)\}$$

is solution to (2). Moreover, one has a comparison principle between the minimal solutions.

$$\Omega_\ell = (-\ell, \ell) \times \Omega.$$

$(y, x) \in \Omega_\ell$ will denote the points in Ω_ℓ.

$$K_\ell = \{v \in W_0^{1,p}(\Omega_\ell) \mid v(y, \cdot) \in K \text{ a.e. } y \in (-\ell, \ell)\}.$$

There exists a unique solution to

$$\begin{cases} u_\ell \in K_\ell, \\ \int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (v - u_\ell) dydx + \\ \qquad \int_{-\ell}^{\ell} \langle Au_\ell, v - u_\ell \rangle dy \geq \int_{\Omega_\ell} f(v - u_\ell) dydx \quad \forall v \in K_\ell. \end{cases} \tag{3}$$

A "$\ell \to +\infty$" approach

Lemma

Under the assumptions above ($f \geq 0$)

(i) u_ℓ is a non decreasing "sequence" bounded by any solution to (2).

(ii) $\forall \ell_0 > 0$ there exists a constant $C(\ell_0)$ independent of ℓ such that

$$|u_\ell|_{W_0^{1,p}(\Omega_{\ell_0})} \leq C(\ell_0),$$

($|\ \cdot\ |_{W_0^{1,p}(\Omega_{\ell_0})}$ denotes a usual $W_0^{1,p}(\Omega_{\ell_0})$-norm.

Proof : a) $u_\ell \geq 0$.

Take $v = u_\ell^+ = u_\ell \vee 0 \in K_\ell$ in (3). Note that $u_\ell^+ - u_\ell = u_\ell^-$

$$-\int_{\Omega_\ell} |\partial_y u_\ell^-|^{p-2} \partial_y u_\ell^- \partial_y u_\ell^- \, dydx$$

$$-\int_{-\ell}^{\ell} \langle A(-u_\ell^-), (-u_\ell^-) \rangle dy \geq \int_{\Omega_\ell} f \; u_\ell^- \, dydx \geq 0.$$

Changing the signs and using the coerciveness of the operator we get

$$\int_{\Omega_\ell} |\partial_y u_\ell^-|^p dydx + \alpha \int_{\Omega_\ell} |\nabla u_\ell^-|^p dydx \leq 0.$$

Hence $u_\ell \geq 0$.

b) $u_\ell \nearrow$ with $\ell \nearrow$.

$\ell' > \ell$. Take $v = u_\ell - (u_\ell - u_{\ell'})^+ = u_\ell \wedge u_{\ell'} \in K_\ell$ in (3) :

$$-\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (u_\ell - u_{\ell'})^+ dydx$$

$$-\int_{-\ell}^{\ell} \langle A(u_\ell), (u_\ell - u_{\ell'})^+) \rangle dy \geq -\int_{\Omega_\ell} f\ (u_\ell - u_{\ell'})^+ dydx.$$

Take $v = u_{\ell'} + (u_\ell - u_{\ell'})^+ = u_\ell \vee u_{\ell'} \in K_{\ell'}$ in (3) corresponding to ℓ' :

$$\int_{\Omega_\ell} |\partial_y u_{\ell'}|^{p-2} \partial_y u_{\ell'} \partial_y (u_\ell - u_{\ell'})^+ dydx$$

$$+\int_{-\ell}^{\ell} \langle A(u_{\ell'}), (u_\ell - u_{\ell'})^+) \rangle dy \geq \int_{\Omega_\ell} f\ (u_\ell - u_{\ell'})^+ dydx.$$

A "$\ell \to +\infty$" approach

Adding these two inequalities leads to

$$\int_{\Omega_\ell} \{|\partial_y u_\ell|^{p-2}\partial_y u_\ell - |\partial_y u_{\ell'}|^{p-2}\partial_y u_{\ell'}\}\partial_y (u_\ell - u_{\ell'})^+ \, dydx$$

$$+ \int_{-\ell}^{\ell} \langle A(u_\ell) - A(u_{\ell'}), (u_\ell - u_{\ell'})^+)\rangle dy \leq 0.$$

(Note that $(u_\ell - u_{\ell'})^+$ vanishes outside Ω_ℓ).
From the monotonicity property of the operator we get

$$\int_{\Omega_\ell} \{|\partial_y u_\ell|^{p-2}\partial_y u_\ell - |\partial_y u_{\ell'}|^{p-2}\partial_y u_{\ell'}\}\partial_y (u_\ell - u_{\ell'})^+ \, dydx \leq 0$$

Since

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq C(|a| + |b|)^{p-2}|a-b|^2$$

we derive

$$\partial_y (u_\ell - u_{\ell'})^+ = 0.$$

Thus

$$(u_\ell - u_{\ell'})^+ = 0$$

and

$$u_\ell \leq u_{\ell'}.$$

c) u_ℓ is bounded by any solution u to (2)

$u + (u_\ell(y,\cdot) - u)^+ = (u_\ell(y,\cdot) \vee u \in K$ a.e. $y \in (-\ell,\ell)$. From (2)

$$\langle Au, (u_\ell(y,\cdot) - u)^+ \rangle \geq \int_\Omega f \, (u_\ell(y,\cdot) - u)^+ dx \ \text{ a.e. } y \in (-\ell,\ell).$$

Integrating in y

$$\int_{-\ell}^{\ell} \langle Au, (u_\ell(y,\cdot) - u)^+ \rangle dy \geq \int_{\Omega_\ell} f \ (u_\ell(y,\cdot) - u)^+ dydx.$$

$u_\ell - (u_\ell - u)^+ = u_\ell \wedge u \in K_\ell$ and by (3)

$$-\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (u_\ell - u)^+ dydx$$

$$- \int_{-\ell}^{\ell} \langle A(u_\ell), (u_\ell - u)^+) \rangle dy \geq - \int_{\Omega_\ell} f \ (u_\ell - u)^+ dydx.$$

Adding leads to

A "$\ell \to +\infty$" approach

$$\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (u_\ell - u)^+ dydx \le 0$$

i.e.

$$\int_{\Omega_\ell} \{|\partial_y u_\ell|^{p-2}\partial_y u_\ell - |\partial_y u|^{p-2}\partial_y u\}\partial_y(u_\ell - u)^+ dydx \le 0$$

and us above

$$(u_\ell - u)^+ = 0 \iff u_\ell \le u$$

d) Bound for $|u_\ell|_{W_0^{1,p}(\Omega_{\ell_0})}$

Let $\rho \in \mathcal{D}(-2\ell_0, 2\ell_0)$ such that $0 \le \rho \le 1, \ \rho = 1$ on $(-\ell_0, \ell_0)$.

$$u_\ell - \rho^p(u_\ell - u) \in \mathcal{K}_\ell$$

From (3) we derive

$$-\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2}\partial_y u_\ell \partial_y \rho^p(u_\ell - u)dydx$$

$$-\int_{-\ell}^{\ell} \langle A(u_\ell), \rho^p(u_\ell - u))\rangle dy \geq -\int_{\Omega_\ell} f\rho^p(u_\ell - u)dydx \geq 0.$$

We derive (recall that u is independent of y) :

$$\int_{\Omega_\ell} \rho^p|\partial_y u_\ell|^p dydx + \int_{-\ell}^{\ell}\langle A(u_\ell), u_\ell\rangle \rho^p dy$$

$$\leq p\int_{\Omega_\ell} \rho^{p-1}|\partial_y u_\ell|^{p-2}\partial_y u_\ell(u_\ell - u)\partial_y \rho dydx + \int_{-\ell}^{\ell}\langle A(u_\ell), u\rangle \rho^p dy.$$

A "$\ell \to +\infty$" approach

Hence

$$\int_{\Omega_\ell} \rho^p |\partial_y u_\ell|^p dydx + \int_{-\ell}^{\ell} \langle A(u_\ell), u_\ell \rangle \rho^p dy \leq$$

$$\leq C \int_{\Omega_\ell} \rho^{p-1} |\partial_y u_\ell|^{p-1} |(u - u_\ell)| dydx + \int_{-\ell}^{\ell} \langle A(u_\ell), u \rangle \rho^p dy$$

$$\leq C \int_{\Omega_\ell} \rho^{p-1} |\partial_y u_\ell|^{p-1} u dydx + C \int_{-\ell}^{\ell} \rho^p (|u_\ell|_{1,p}^{p-1} + 1) |u|_{1,p} dy.$$

From the ellipticity condition and the young inequality we obtain

$$\int_{\Omega_\ell} \rho^p |\partial_y u_\ell|^p dydx + \alpha \int_{-\ell}^{\ell} \rho^p |u_\ell|_{1,p} dy \leq$$

$$\epsilon \{ \int_{\Omega_\ell} \rho^p |\partial_y u_\ell|^p dydx + \int_{-\ell}^{\ell} \rho^p |u_\ell|_{1,p} dy \}$$

$$+ C_\epsilon \{ \int_{\Omega_{2\ell_0}} |u|^p dxdy + \int_{-\ell}^{\ell} \rho^p (|u|_{1,p}^p + 1) dy) \}$$

Since $\rho = 1$ on$(-\ell_0, \ell_0)$, choosing ε small enough we get

$$|u_\ell|_{W_0^{1,p}(\Omega_{\ell_0})} \leq C(\ell_0).$$

This achieves the proof of Lemma.

Next we have :

Lemma

The solution u_ℓ of (3) converges to $\tilde{u}$, as ℓ goes to $+\infty$, a solution of (2).

Proof : We start by applying the previous Lemma. It follows that u_ℓ is converging towards some function $\tilde{u}$.

(i) $\tilde{u}$ is independent of y.

A "$\ell \to +\infty$" approach

Let $h \in \mathbb{R}$.
The function $\mathcal{T}_h u_\ell (y,x) = u_\ell (y+h,x)$ is supported in the closure of

$$\Omega_\ell^h := (-\ell - h, \ell - h) \times \Omega.$$

From (3), we have by a change of variable

$$\int_{\Omega_\ell^h} |\partial_y \mathcal{T}_h u_\ell|^{p-2} \partial_y \mathcal{T}_h u_\ell \partial_y (v - \mathcal{T}_h u_\ell) \, dxdy + \int_{-\ell-h}^{\ell-h} \langle A \mathcal{T}_h u_\ell, v - \mathcal{T}_h u_\ell \rangle dy$$
$$\geq \int_{\Omega_\ell^h} f(x) (v - \mathcal{T}_h u_\ell) \, dxdy, \quad \forall v \in \mathcal{K}_{\ell,h}, \tag{4}$$

where

$$\mathcal{K}_{\ell,h} := \{\mathcal{T}_h v \mid v \in \mathcal{K}_\ell\}$$
$$= \left\{ v \in W_0^{1,p}\left(\Omega_\ell^h\right) \mid v(y,.) \in \mathcal{K} \text{ a.e. in } (-\ell - h, \ell - h) \right\}.$$

A "$\ell \to +\infty$" approach

Choosing $v = \mathcal{T}_h u_\ell - (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \in \mathcal{K}_{\ell,h}$ in (4) we get

$$\int_{\Omega_\ell^h} |\partial_y \mathcal{T}_h u_\ell|^{p-2} \partial_y \mathcal{T}_h u_\ell \partial_y (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \, dxdy$$

$$+ \int_{-\ell-h}^{\ell-h} \langle A \mathcal{T}_h u_\ell, (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \rangle dy$$

$$\leq \int_{\Omega_\ell^h} f(x) (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \, dxdy. \qquad (5)$$

$$v = u_{\ell+h} + (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \in \mathcal{K}_{\ell+h}.$$

(Note that the support of $(\mathcal{T}_h u_\ell - u_{\ell+h})^+$ is contained in Ω_ℓ^h).

From (3) written for $u_{\ell+h}$ we obtain :

$$
\begin{aligned}
&- \int_{\Omega_\ell^h} |\partial_y u_{\ell+h}|^{p-2} \partial_y u_{\ell+h} \partial_y \left(\mathcal{T}_h u_\ell - u_{\ell+h}\right)^+ dxdy \\
&\qquad - \int_{-\ell-h}^{\ell-h} \langle A u_{\ell+h}, (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \rangle dy \\
&\qquad\qquad \le - \int_{\Omega_\ell^h} f(x) \left(\mathcal{T}_h u_\ell - u_{\ell+h}\right)^+ dxdy. \qquad (6)
\end{aligned}
$$

Adding (5), (6) we get

$$
\begin{aligned}
&\int_{\Omega_\ell^h} \left(|\partial_y \mathcal{T}_h u_\ell|^{p-2} \partial_y \mathcal{T}_h u_\ell - |\partial_y u_{\ell+h}|^{p-2} \partial_y u_{\ell+h}\right) \partial_y \left(\mathcal{T}_h u_\ell - u_{\ell+h}\right)^+ dxdy \\
&\qquad + \int_{-\ell-h}^{\ell-h} \langle A\mathcal{T}_h u_\ell - A u_{\ell+h}, (\mathcal{T}_h u_\ell - u_{\ell+h})^+ \rangle dy \le 0.
\end{aligned}
$$

And by the monotonicity condition the first integral above is non positive.

This implies

$$u_\ell (y + h, x) \leq u_{\ell+h} (y, x) .$$

Passing to the limit as $\ell \to \infty$, we get

$$\tilde{u} (y + h, x) \leq \tilde{u} (y, x) .$$

Since h is arbitrary we derive

$$\tilde{u} (y, x) = \tilde{u} (x) .$$

(ii) $\tilde{u}$ is solution to (2).

Let $\ell_0 \in \mathbb{R}$, for ℓ large enough, from our preceding Lemma :

$$|\partial_y u_\ell|^{p-2} \partial_y u_\ell \text{ and } \{a_i (x, u_\ell, \nabla u_\ell)\}_{i=0,\cdots,n}$$

are bounded in $L^q (\Omega_{\ell_0})$. Therefore

$$
\begin{aligned}
&u_\ell \to \tilde{u}, \quad \nabla u_\ell \rightharpoonup \nabla \tilde{u} \ \text{ in } L^p\left(\Omega_{\ell_0}\right), \\
&\left|\partial_y u_\ell\right|^{p-2} \partial_y u_\ell \rightharpoonup d, \quad a_i\left(x, u_\ell, \nabla u_\ell\right) \rightharpoonup d_i \text{ in } L^q\left(\Omega_{\ell_0}\right).
\end{aligned} \tag{7}
$$

The two first convergences hold for the whole sequence since $(u_\ell)_{\ell>0}$ is nondecreasing. (Once the limit are uniquely identified, the previous convergences will take place for the whole sequence). Let ϕ be a nonnegative function in $\mathcal{D}\left(-\ell_0, \ell_0\right)$, up to subsequence :

$$
\lim_{\ell \to +\infty} \int_{-\ell_0}^{\ell_0} \phi \langle A u_\ell, u_\ell \rangle dy = \int_{\Omega_{\ell_0}} \phi \sum_{0 \leqslant i \leqslant n} d_i \partial_{x_i} \tilde{u} dxdy, \tag{8}
$$

$(\partial_{x_0} \tilde{u} = \tilde{u})$.

$$
\lim_{\ell \to +\infty} \int_{\Omega_{\ell_0}} \phi \left|\partial_y u_\ell\right|^p dxdy = 0. \tag{9}
$$

The last limit means that $d = 0$.

Indeed, using the monotonicity condition

$$\langle Au_\ell, u_\ell \rangle \geq \langle Au_\ell, \tilde{u} \rangle + \langle A\tilde{u}, u_\ell - \tilde{u} \rangle.$$

Thus one easily derives

$$\liminf_{\ell \to +\infty} \int_{-\ell_0}^{\ell_0} \phi \langle Au_\ell, u_\ell \rangle dy \geq \liminf_{\ell \to +\infty} \int_{-\ell_0}^{\ell_0} \phi \langle Au_\ell, \tilde{u} \rangle dy =$$

$$\liminf_{\ell \to +\infty} \int_{-\ell_0}^{\ell_0} \phi \sum_{i=0}^{n} a_i(x, u_\ell, \nabla u_\ell) \partial_{x_i} \tilde{u} dy = \int_{\Omega_{\ell_0}} \phi \sum_{0 \leqslant i \leqslant n} d_i \partial_{x_i} \tilde{u} dxdy. \tag{10}$$

On the other hand, since $u_\ell - \frac{\phi}{|\phi|_\infty}(u_\ell - \tilde{u}) \in \mathcal{K}_\ell$, from (3)

$$\int_{\Omega_{\ell_0}} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y \{\phi (u_\ell - \tilde{u})\} dxdy$$

$$+ \int_{-\ell_0}^{\ell_0} \phi \langle Au_\ell, u_\ell - \tilde{u} \rangle dy \leq \int_{\Omega_{\ell_0}} \phi f (u_\ell - \tilde{u}) dxdy \leq 0.$$

Thus

$$\int_{\Omega_{\ell_0}} \phi \, |\partial_y u_\ell|^p \, dxdy + \int_{-\ell_0}^{\ell_0} \phi \langle A u_\ell, u_\ell \rangle dy$$

$$\leq \int_{\Omega_{\ell_0}} |\partial_y u_\ell|^{p-2} \, \partial_y u_\ell \partial_y \phi \, (\tilde{u} - u_\ell) \, dxdy + \int_{-\ell_0}^{\ell_0} \phi \langle A u_\ell, \tilde{u} \rangle dy.$$

Passing to the $\limsup$ as $\ell \to \infty$, we get

$$\limsup_{\ell \to +\infty} \left[\int_{\Omega_{\ell_0}} \phi \, |\partial_y u_\ell|^p \, dxdy + \int_{-\ell_0}^{\ell_0} \phi \langle A u_\ell, u_\ell \rangle dy \right]$$

$$\leq \int_{\Omega_{\ell_0}} \phi \sum_{0 \leqslant i \leqslant n} d_i \partial_{x_i} \tilde{u} dxdy.$$

Combining this with (10) we end up with (8) and (9).

A "$\ell \to +\infty$" approach

For $\psi \in \mathcal{K}$ and $\phi \in \mathcal{D}\left(-\frac{\ell_0}{2}, \frac{\ell_0}{2}\right)$ with $\phi \geq 0$, $\phi \not\equiv 0$,

$$u_\ell + \frac{\phi}{|\phi|_\infty}(\psi - u_\ell) \in \mathcal{K}_\ell.$$

From (3)

$$\int_{\Omega_{\frac{\ell_0}{2}}} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y \{\phi(\psi - u_\ell)\} \, dxdy + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \phi \langle Au_\ell, \psi - u_\ell \rangle dy$$

$$\geq \int_{\Omega_{\frac{\ell_0}{2}}} f\phi(\psi - u_\ell)\, dxdy \Rightarrow \text{ (monotonicity)}$$

$$\int_{\Omega_{\frac{\ell_0}{2}}} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y \{\phi(\psi - u_\ell)\} \, dxdy + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \phi \langle A\psi, \psi - u_\ell \rangle dy$$

$$\geq \int_{\Omega_{\frac{\ell_0}{2}}} f\phi(\psi - u_\ell)\, dxdy.$$

A "$\ell \to +\infty$" approach

From (9) $\partial_y u_\ell \to 0$ in $L^p\left(\Omega_{\frac{\ell_0}{2}}\right)$. Passing to the limit in the above inequality as $\ell \to \infty$ yields

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \phi\langle A\psi, \psi - \tilde{u}\rangle dy \geq \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \phi \int_\Omega f\,(\psi - \tilde{u})\, dxdy.$$

This implies

$$\langle A\psi, \psi - \tilde{u}\rangle \geq \int_\Omega f\,(\psi - \tilde{u})\, dx, \;\; \forall \psi \in \mathcal{K}.$$

Choosing $\psi = \tilde{u} + t\,(v - \tilde{u})$, where $0 < t < 1$ and $v \in \mathcal{K}$ we deduce

$$\langle A\,(\tilde{u} + t\,(v - \tilde{u}))\,, v - \tilde{u}\rangle \geq \int_\Omega f\,(v - \tilde{u})\, dx, \;\; \forall v \in \mathcal{K}.$$

Passing to the limit as $t \to 0$, taking into account the fact that the operator is defined with Carathéodory functions we get

$$\langle A\tilde{u}, v - \tilde{u}\rangle \geq \int_\Omega f\,(v - \tilde{u})\,dx,\ \forall v \in \mathcal{K}.$$

Then the Lemma is proved.

By using the above lemmas, we can now turn to the proof of our first theorem which we rephrase as

Theorem

Suppose that $f \in L^q(\Omega), f \geq 0$. Then, under the assumptions above there exists a minimal solution of (2) i.e.

$$\tilde{u}\,(x) = \min\left\{u\,(x)\,,\ u \textit{ solution to } (2)\right\}, \quad \tilde{u} \in \mathcal{K}$$

is solution to (2). Moreover, if $\tilde{u}_1$ and $\tilde{u}_2$ are the minimal solutions of (2) obtained by replacing f with f_1 and f_2 respectively, then, if $f_1 \leq f_2$, we have $\tilde{u}_1 \leq \tilde{u}_2$.

Proof : Let u be an arbitrary solution of the problem (2) and u_ℓ be the solution to (3). Then from the Lemma above we have

$$u_\ell(y,x) \leq u(x) \text{ for a.e. } (y,x) \in \Omega_\ell.$$

Passing to the limit as $\ell \to \infty$, we derive from that $u_\ell(y,.)$ converges towards some $\tilde{u} \in \mathcal{K}$ solution to (2). Thus

$$\tilde{u} \leq u \text{ a.e. in } \Omega.$$

This means that $\tilde{u}$ is the minimal solution of the problem (2).

Let $u_{\ell,1}$ and $u_{\ell,2}$ be the solutions of (3), obtained if we replace f by f_1 and f_2 respectively.

Take $v = u_{\ell,1} - (u_{\ell,1} - u_{\ell,2})^+$ and $v = u_{\ell,2} + (u_{\ell,1} - u_{\ell,2})^+$ in (3) for f_1 and f_2 respectively, we get

A "$\ell \to +\infty$" approach

$$\int_{\Omega_\ell} \left(|\partial_y u_{\ell,1}|^{p-2}\,\partial_y u_{\ell,1} - |\partial_y u_{\ell,2}|^{p-2}\,\partial_y u_{\ell,2}\right) \partial_y \left(u_{\ell,1} - u_{\ell,2}\right)^+ dxdy$$

$$+ \int_{-\ell}^{\ell} \langle A u_{\ell,1} - A u_{\ell,2}, (u_{\ell,1} - u_{\ell,2})^+ \rangle dy$$

$$\le \int_{\Omega_\ell} (f_1 - f_2)\,(u_{\ell,1} - u_{\ell,2})^+ \,dxdy \le 0.$$

By our monotonicity condition we obtain

$$\int_{\Omega_\ell} \left(|\partial_y u_{\ell,1}|^{p-2}\,\partial_y u_{\ell,1} - |\partial_y u_{\ell,2}|^{p-2}\,\partial_y u_{\ell,2}\right) \partial_y \left(u_{\ell,1} - u_{\ell,2}\right)^+ dxdy \le 0.$$

This implies

$$u_{\ell,1} \le u_{\ell,2} \text{ in } \Omega_\ell.$$

Passing to the limit as $\ell \to \infty$, using the above argument we get

$$\tilde{u}_1 \leq \tilde{u}_2 \text{ in } \Omega.$$

This completes the proof of our Theorem.

An Example

For $n = 1$ and $p = 2$, let $v \in H_0^1(0,1)$ be the nonnegative function defined by

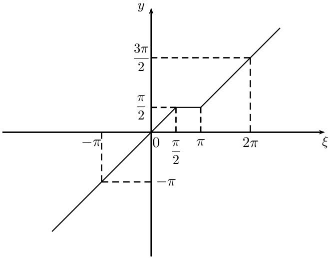
$$v(x) = \frac{3\sqrt{3}}{2} x \chi_{(0,\frac{1}{3})} + \sin(\pi x)\, \chi_{(\frac{1}{3},1)},$$

where χ_A denotes the characteristic function of the set A. Consider

An Example

$$\mathcal{K} = \left\{ w \in H_0^1(0,1) : w \geq v \text{ a.e. in } (0,1) \right\},$$

$a_0 = 0$ and $a : \mathbb{R} \to \mathbb{R}$ is a single-valued function whose graph is depicted in the following figure



An example

It is easy to see (by taking $\alpha = \frac{1}{2}$, $\beta = 1$ and $\vartheta = \frac{\pi}{2}$) that A is monotone and satisfies the coerciveness and growth conditions above. Hence, the solution to (2) exists and moreover it is not necessary unique. Indeed, it is enough to check that the functions

$$u_\lambda = \lambda \sin(\pi x) + (1-\lambda)\, v, \quad \forall \lambda \in [0,1]$$

satisfy (2) for

$$f(x) = \pi^2 \sin(\pi x)\, \chi_{\left(\frac{1}{3},1\right)}.$$

Indeed, it is clear that $u_\lambda \in \mathcal{K}$, for all $\lambda \in [0,1]$. Now, since the derivatives of these functions belong to $\left(\frac{\pi}{2}, \pi\right)$ for every $x \in \left(0, \frac{1}{3}\right)$ and $u'_\lambda = \pi \cos(\pi x)$ on $\left(\frac{1}{3}, 1\right)$, it follows that

$$a\left(u'_\lambda\right) = \frac{\pi}{2}\chi_{\left(0,\frac{1}{3}\right)} + \pi \cos(\pi x)\, \chi_{\left(\frac{1}{3},1\right)}.$$

An example

This implies

$$Au_\lambda(x) := -\frac{d}{dx}a\left(\frac{d}{dx}u_\lambda\right) = f,$$

which means that u_λ is the solution to (2) and moreover $u_0 = v$ is the minimal solution.

References

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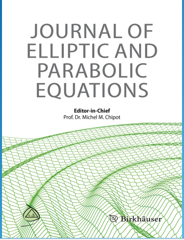

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