Towards the tunneling effect for the semiclassical magnetic Laplacian in an ellipse

Virginie Bonnaillie-Noël

DMA, CNRS, ENS Paris

Joint work with F. Hérau and N. Raymond

Phase Transitions Models

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Schrödinger operator

Notation

Ω open, smooth, bounded and simply connected domain of $\mathbb{R}^2$

$\mathbf{B}$ magnetic field
$\mathbf{A}$ magnetic potential s.t. $\operatorname{curl} \mathbf{A} = \mathbf{B}$

$\hbar > 0$ semi-classical parameter

$$\mathbf{A} = \frac{1}{2}(x_2, -x_1)$$

Semiclassical Magnetic Laplacian

$\mathcal{L}_\hbar := (-i\hbar\nabla + \mathbf{A})^2$ on Ω

with magnetic Neumann boundary condition on $\partial\Omega$

Some motivations

Aim: analysis the spectrum of $\mathcal{L}_\hbar$ in the semiclassical limit $\hbar \to 0$

Let $\lambda_n(\hbar)$ be the n-th eigenvalue of $\mathcal{L}_\hbar$

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Let $\lambda_n(\hbar)$ be the n-th eigenvalue of $\mathcal{L}_\hbar$

Motivations:

- The lowest eigenvalue $\lambda_1(\hbar)$ of the magnetic Laplacian is involved in the theories of superconductivity and liquid crystals
- Simplicty of $\lambda_1(\hbar)$?
- Estimate of the gap $\lambda_2(\hbar) - \lambda_1(\hbar)$?

Pure electric potential

A partially semiclassical electric Laplacian

Approximation of the eigenpairs of operators with electrical potential

$$-h^2\Delta_s - \Delta_t + V(s,t)$$

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Born-Oppenheimer strategy

- replace, for fixed s, $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$
- consider the reduced operator

$$-h^2\Delta_s + \mu_1(s)$$

- apply the semiclassical techniques *à la* Helffer-Sjöstrand
- μ_1 symmetric $\Rightarrow$ tunneling effect

cf. [Helffer-Sjöstrand 84-85,Combes-Duclos-Seiler 81, Martinez 89, Klein-Martinez-Seiler-Wang 92]

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Is there an analog in the pure magnetic case?

Schrödinger operator

Some references

Many results about the asymptotic expansions of $\lambda_n(\hbar)$

Constant magnetic field

[Erdos, Baumann-Phillips-Tang] (2D, disk)

[Bernoff-Sternberg, del Pino-Felmer-Sternberg, Helffer-Morame] (2D, smooth Ω)

[Helffer-Morame] (3D, smooth Ω)

[Fournais-Persson] (3D, ball)

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[Fournais-Persson] (3D, ball)
[Jadallah, BN] (2D, corners)

Variable magnetic field

[Lu-Pan, Raymond] (2D, smooth Ω, non vanishing field)
[Montgomery, Helffer-Morame, Pan-Kwek, Helffer-Kordyukov] (2D, smooth Ω, vanishing field)
[Lu-Pan, Raymond, Helffer-Kordyukov] (3D, smooth Ω, non vanishing field)
[BN, BN-Dauge, BN-Dauge-Martin-Vial] (2D, non vanishing field, corners)
[BN-Dauge-Popoff] (3D, corners)

Schrödinger operator

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Many results about the asymptotic expansions of $\lambda_n(\hbar)$

Essentially variational analysis based on

1. a construction of appropriate test functions for the Rayleigh quotients
2. a reduction, through a space partition of unity
   and estimates of Agmon type,
   to local models whose spectrum is known

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Asymptotic expansion of the first eigenfunctions?

Do the magnetic eigenfunctions admit WKB expansions?

Smooth case

Non symmetric case

Assumption

The algebraic curvature κ of the boundary of $\partial\Omega$ has a unique and non-degenerate maximum at $s = 0$

Let $k_2 = -\kappa''(0)$

Theorem

We have the asymptotic expansion

$$\lambda_n(\hbar) = \Theta_0\hbar - C_1\kappa_{\max}\hbar^{3/2} + (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}\hbar^{7/4} + o(\hbar^{7/4})$$

Θ_0 *and* $C_1 > 0$*: constants related to the De Gennes operator*

cf. [Helffer-Morame, Fournais-Helffer]

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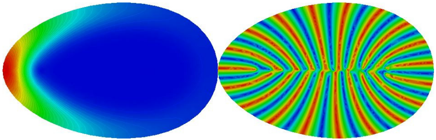

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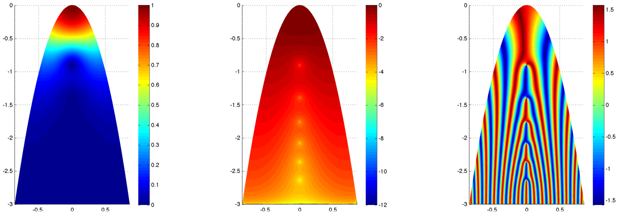
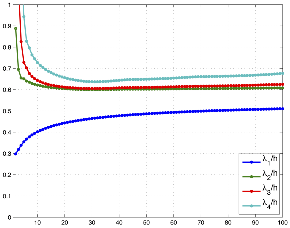


Modulus, log, and phase of the first eigenfuntion, $\hbar = \frac{1}{20}$



Smooth case

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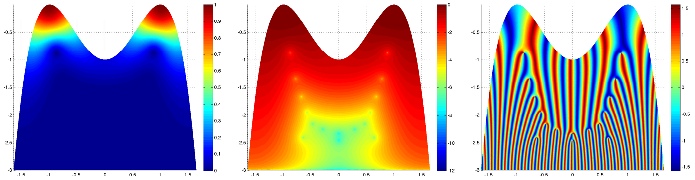
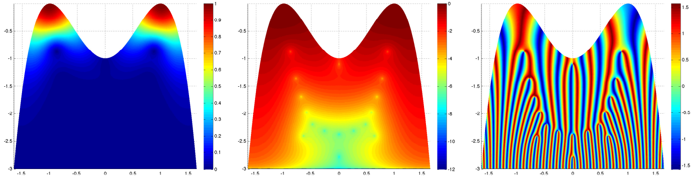


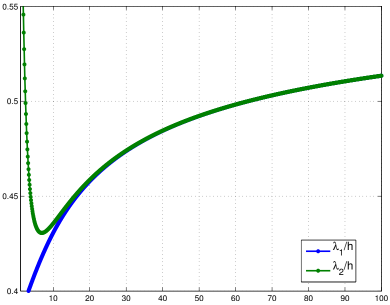
Modulus and phase of the first eigenfunction, $\hbar = \frac{1}{20}$



Modulus and phase of the second eigenfunction, $\hbar = \frac{1}{20}$

Smooth case

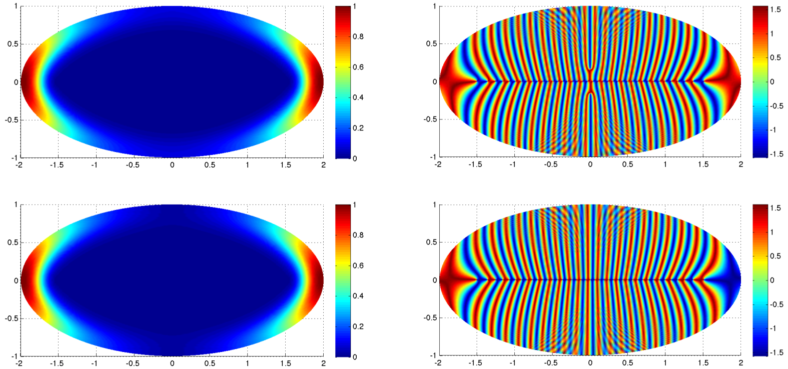
Symmetric case



$$1/\hbar \mapsto \hbar^{-1}\lambda_j(\hbar)$$

Smooth case

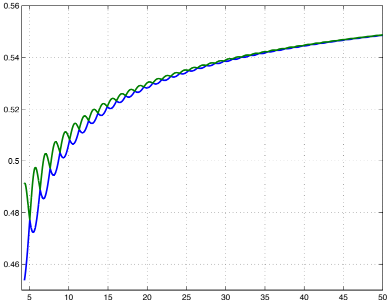
Symmetric case



First two eigenfunctions, $\hbar = \frac{1}{50}$

Smooth case

Symmetric case



$\hbar^{-1}\lambda_n(\hbar)$ *vs.* $\hbar^{-1}$

De Gennes operator

For $\zeta \in \mathbb{R}$,

$$\mathcal{H}_\zeta = D_\tau^2 + (\tau - \zeta)^2$$

defined on $\mathrm{L}^2(\mathbb{R}^+)$ with Neumann conditions on the boundary

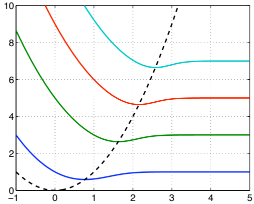
$\mu(\zeta)$: first eigenvalue of $\mathcal{H}_\zeta$
u_ζ: corresponding positive L^2-normalized eigenfunction

Proposition

$\zeta \mapsto \mu(\zeta)$ and $\zeta \mapsto u_\zeta$ are real analytic w.r.t. ζ

There exists $\zeta_0 > 0$ such that

μ is decreasing on $(-\infty, \zeta_0)$
and increasing on $(\zeta_0, +\infty)$



$$\Theta_0 := \mu(\zeta_0) = \zeta_0^2, \qquad \mu'(\zeta_0) = 0, \qquad |u_{\zeta_0}(0)|^2 = \frac{\mu''(\zeta_0)}{2\zeta_0}$$

cf. [Dauge-Helffer, Fournais-Helffer]

De Gennes operator

Momenta

$$M_k(\zeta) = \int_0^{\infty} (\tau - \zeta)^k |u_\zeta(\tau)|^2 \,\mathrm{d}\tau, \qquad \forall k \in \mathbb{N}$$

Proposition

With $C_1 = \frac{u_{\zeta_0}^2(0)}{3}$, *we have*

$$\int_{\mathbb{R}_+} (\zeta_0 - \tau) u_{\zeta_0}^2(\tau) \,\mathrm{d}\tau = 0$$

$$\int_{\mathbb{R}_+} (\partial_\zeta u)_{\zeta_0}(\tau) u_{\zeta_0}(\tau) \,\mathrm{d}\tau = 0$$

$$2 \int_{\mathbb{R}_+} (\zeta_0 - \tau) (\partial_\zeta u)_{\zeta_0}(\tau) u_{\zeta_0}(\tau) \,\mathrm{d}\tau = \frac{\mu''(\zeta_0)}{2} - 1$$

$$\int_{\mathbb{R}_+} \left(\partial_\tau + 2\tau(\zeta_0 - \tau)^2 + \tau^2(\zeta_0 - \tau)\right) u_{\zeta_0} u_{\zeta_0} \,\mathrm{d}\tau = -C_1$$

Change of variables

Boundary coordinates

$\ell = |\partial\Omega|$
$\gamma : \mathbb{R}/\ell\mathbb{Z} \to \partial\Omega$ a parametrization of the boundary with $|\gamma'| = 1$
$\nu(s)$ inward unit vector at $\gamma(s)$

$\kappa(s)$ curvature s.t.

$$\gamma''(s) = \kappa(s)\nu(s)$$

Let us consider the diffeomorphism

$$\begin{array}{rcl} \mathsf{F} : \mathbb{R}/\ell\mathbb{Z} & \to & \Omega_{t_0} \\ (s,t) & \mapsto & \gamma(s) + t\nu(s) \end{array}$$

with, for $t_0 > 0$,

$$\Omega_{t_0} = \{x \in \Omega,\ \mathrm{dist}(x, \partial\Omega) < t_0\}$$

Change of variables

Potential

$$\widetilde{\mathbf{A}}(s,t) = \begin{pmatrix} (1-t\kappa(s))\mathbf{A}(\mathrm{F}(s,t))\cdot\gamma'(s) \\ \mathbf{A}(\mathrm{F}(s,t))\cdot\nu(s) \end{pmatrix}$$

Change of gauge: there exists a gauge function φ s.t.

- *Local change of variables on Ω_{t_0} :*

$$\widehat{\mathbf{A}}(s,t) = \widetilde{\mathbf{A}} - \nabla\varphi = \begin{pmatrix} \gamma_0 - t + \frac{t^2}{2}\kappa(s) \\ 0 \end{pmatrix} \quad \text{with} \quad \gamma_0 = \frac{1}{\ell}\int_\Omega \operatorname{curl}\mathbf{A}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

cf. [Fournais-Helffer]

Change of variables

Potential

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- *Local change of variables on $\tilde{\Omega}_{t_0} = (s_1, s_2) \times (0, t_0)$:*

$$\widehat{\mathbf{A}}(s,t) = \begin{pmatrix} \hbar^{\frac{1}{2}}\zeta_0 - t + \frac{t^2}{2}\kappa(s) \\ 0 \end{pmatrix}$$

cf. [Fournais-Helffer]

Change of variables

New operator

$$\mathcal{L}_\hbar = (-i\hbar\nabla + \mathbf{A})^2 \sim \widetilde{\mathcal{L}}_\hbar$$

with

$$\widetilde{\mathcal{L}}_\hbar = m(s,t)^{-1}\hbar D_t m(s,t)\hbar D_t$$
$$+m(s,t)^{-1}\left(\hbar D_s + \hbar^{\frac{1}{2}}\zeta_0 - t + \frac{t^2}{2}\kappa(s)\right) m(s,t)^{-1}\left(\hbar D_s + \hbar^{\frac{1}{2}}\zeta_0 - t + \frac{t^2}{2}\kappa(s)\right)$$

$$m(s,t) = 1 - t\kappa(s)$$

Change of variables

New operator

$$\mathcal{L}_\hbar = (-i\hbar\nabla + \mathbf{A})^2 \sim \widetilde{\mathcal{L}}_\hbar \sim h^2\mathfrak{L}_h, \qquad h = \hbar^{\frac{1}{2}}$$

with

$$\widetilde{\mathcal{L}}_\hbar = m(s,t)^{-1}\hbar D_t m(s,t)\hbar D_t$$
$$+m(s,t)^{-1}\left(\hbar D_s + \hbar^{\frac{1}{2}}\zeta_0 - t + \frac{t^2}{2}\kappa(s)\right)m(s,t)^{-1}\left(\hbar D_s + \hbar^{\frac{1}{2}}\zeta_0 - t + \frac{t^2}{2}\kappa(s)\right)$$

Scaling $(s,t) = (\sigma, h\tau)$, $h = \hbar^{\frac{1}{2}}$ $\qquad m(\sigma, h\tau) = 1 - h\tau\kappa(\sigma)$

$$\mathfrak{L}_h = m(\sigma,h\tau)^{-1}D_\tau m(\sigma,h\tau)D_\tau$$
$$+m(\sigma,h\tau)^{-1}\left(hD_\sigma + \zeta_0 - \tau + h\frac{\tau^2}{2}\kappa(\sigma)\right)m(\sigma,h\tau)^{-1}\left(hD_\sigma + \zeta_0 - \tau + h\frac{\tau^2}{2}\kappa(\sigma)\right)$$

$\lambda_n(h)$: n-th eigenvalue of $\mathfrak{L}_h$

WKB constructions in the simple well case

Theorem

There exist a sequence of real numbers $(\lambda_{n,j})_{j\geq 0}$ s.t.

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j} h^{\frac{j}{2}}$$

Besides there exists a formal series of smooth functions on $\mathcal{V}$

$$a_n \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j} h^{\frac{j}{2}}$$

such that

$$(\mathfrak{L}_h - \lambda_n(h))\left(a_n e^{-\Phi/h^{\frac{1}{2}}}\right) = \mathcal{O}(h^\infty)\, e^{-\Phi/h^{\frac{1}{2}}}$$

with

$$\Phi : \sigma \mapsto \Phi(\sigma) = \left(\frac{2C_1}{\mu''(\zeta_0)}\right)^{1/2} \left|\int_0^\sigma (\kappa(0) - \kappa(\varsigma))^{1/2}\, d\varsigma\right|$$

defined in a neighborhood $\mathcal{V}$ of $(0,0)$ such that $\operatorname{Re}\Phi''(0) > 0$

WKB constructions in the simple well case

Theorem

We also have that

$$\lambda_{n,0} = \Theta_0, \quad \lambda_{n,1} = 0, \quad \lambda_{n,2} = -C_1\kappa_{\max}, \quad \lambda_{n,3} = (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$$

The main term in the Ansatz is in the form

$$a_{n,0}(\sigma,\tau) = f_{n,0}(\sigma)u_{\xi_0}(\tau)$$

For all $n \geq 1$, there exist $h_0 > 0$, $c > 0$ s.t. for all $h \in (0, h_0)$, we have

$$\mathcal{B}\left(\lambda_{n,0} + \lambda_{n,2}h + \lambda_{n,3}h^{\frac{3}{2}}, ch^{\frac{3}{2}}\right) \cap \mathrm{sp}\left(\mathfrak{L}_h\right) = \{\lambda_n(h)\}$$

and $\lambda_n(h)$ is a simple eigenvalue

WKB constructions in the simple well case

Sketch of proof – Conjugation via a weight function $\Phi = \Phi(s)$

$$\mathfrak{L}_h = m(\sigma, h\tau)^{-1} D_\tau m(\sigma, h\tau) D_\tau + m(\sigma, h\tau)^{-1} \mathcal{P}_h m(\sigma, h\tau)^{-1} \mathcal{P}_h$$

with

$$\mathcal{P}_h = h\left(D_\sigma + \frac{\tau^2}{2}\kappa(\sigma)\right) + (\zeta_0 - \tau) \qquad \text{and} \qquad m(\sigma, h\tau) = 1 - h\tau\kappa(\sigma)$$

WKB constructions in the simple well case

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Conjugate operator

$\Phi = \Phi(\sigma)$: phase function defined in a neighborhood of $\sigma = 0$

$$\mathfrak{L}_h^{\mathrm{wg}} = \mathrm{e}^{\Phi(\sigma)/h^{\frac{1}{2}}} \mathfrak{L}_h \mathrm{e}^{-\Phi(\sigma)/h^{\frac{1}{2}}}$$

WKB constructions in the simple well case

Sketch of proof – Conjugation via a weight function $\Phi = \Phi(s)$

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Conjugate operator

$$\mathfrak{L}_h^{\mathrm{wg}} = e^{\Phi(\sigma)/h^{\frac{1}{2}}}\mathfrak{L}_h e^{-\Phi(\sigma)/h^{\frac{1}{2}}} \sim \mathfrak{L}_0 + h^{\frac{1}{2}}\mathfrak{L}_1 + h\mathfrak{L}_2 + h^{\frac{3}{2}}\mathfrak{L}_3 + \dots$$

where

$$\mathfrak{L}_0 = D_\tau^2 + (\zeta_0 - \tau)^2$$

$$\mathfrak{L}_1 = 2(\zeta_0 - \tau)i\Phi'(\sigma)$$

$$\mathfrak{L}_2 = \kappa(\sigma)\partial_\tau + 2\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(\zeta_0 - \tau) - \Phi'(\sigma)^2 + 2\kappa(\sigma)(\zeta_0 - \tau)^2\tau$$

$$\mathfrak{L}_3 = \left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(i\Phi'(\sigma)) + (i\Phi'(\sigma))\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right) + 4i\Phi'(\sigma)\tau\kappa(\sigma)(\zeta_0 - \tau)$$

WKB constructions in the simple well case

Sketch of proof – Formal series

Look for a formal series solution on the form

$$\lambda \sim \sum_{j\geq 0} \lambda_j h^{\frac{j}{2}} \qquad a \sim \sum_{j\geq 0} h^{\frac{j}{2}} a_j$$

such that, *in the sense of formal series*,

$$\mathfrak{L}_h^{\mathrm{wg}} a \sim \lambda a$$

Let us now solve the formal system

i.e. we cancel each power of $h^{\frac{1}{2}}$ step by step

WKB constructions in the simple well case

Sketch of proof

- First equation in h^0

$$\mathfrak{L}_0 a_0 = \lambda_0 a_0$$

$$\Rightarrow \lambda_0 = \Theta_0, \qquad a_0(\sigma,\tau) = f_0(\sigma) u_{\zeta_0}(\tau)$$

where f_0 has to be determined

WKB constructions in the simple well case

Sketch of proof

- Second equation in $h^{\frac{1}{2}}$

$$(\mathfrak{L}_0 - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}_1)a_0 = (\lambda_1 - 2(\zeta_0 - \tau)i\Phi'(\sigma))u_{\zeta_0}(\tau)f_0(\sigma)$$

Fredholm alternative

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ a_1(\sigma,\tau) = i\Phi'(\sigma)f_0(\sigma)\left(\partial_\zeta u\right)_{\zeta_0}(\tau) + f_1(\sigma)u_{\zeta_0}(\tau) \end{cases}$$

where f_1 is to be determined in a next step

WKB constructions in the simple well case

Sketch of proof

- Third equation in h

$$(\mathfrak{L}_0-\lambda_0)a_2=(\lambda_2-\mathfrak{L}_2)a_0-\mathfrak{L}_1 a_1$$

The right hand side can be explicitly computed

and the equation becomes

$$(\mathfrak{L}_0-\lambda_0)\tilde{a}_2=\lambda_2 u_{\zeta_0} f_0+\frac{\mu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} f_0+\kappa f_0(-\partial_\tau u_{\zeta_0}-2(\zeta_0-\tau)^2\tau u_{\zeta_0}-\tau^2(\zeta_0-\tau)u_{\zeta_0})$$

where

$$\tilde{a}_2=a_2-(\partial_\zeta u)_{\zeta_0}(i\Phi' f_1-i\partial_\sigma f_0)+\frac{1}{2}(\partial_\zeta^2 u)_{\zeta_0}\Phi'^2 f_0$$

WKB constructions in the simple well case

Sketch of proof

- Third equation in h

Using properties of u_{ζ_0},

$$\lambda_2 + \frac{\mu''(\zeta_0)}{2}\Phi'^2(\sigma) + C_1\kappa(\sigma) = 0, \qquad \text{with} \qquad C_1 = \frac{u_{\zeta_0}^2(0)}{3}$$

eikonal equation of a pure electric 1D-problem with potential $C_1\kappa(\sigma)$
Thus

$$\lambda_2 = -C_1\kappa(0)$$

and

$$\Phi(\sigma) = \left(\frac{2C_1}{\mu''(\zeta_0)}\right)^{1/2}\left|\int_0^\sigma (\kappa(0)-\kappa(\varsigma))^{1/2}\,\mathrm{d}\varsigma\right|$$

In particular we have

$$\Phi''(0) = \left(\frac{k_2 C_1}{\mu''(\zeta_0)}\right)^{1/2}, \qquad \text{with} \qquad k_2 = -\kappa''(0) > 0$$

WKB constructions in the simple well case

Sketch of proof

- Third equation in h

This leads to take

$$a_2 = f_0 \hat{a}_2 + (\partial_\zeta u)_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) - \frac{1}{2}(\partial_\eta^2 u)_{\zeta_0}\Phi'^2 f_0 + f_2 u_{\zeta_0}$$

where $\hat{a}_2$ is the unique solution, orthogonal to u_{ζ_0} for all σ, of

$$(\mathfrak{L}_0 - \lambda_0)\hat{a}_2 = \lambda_2 u_{\zeta_0} + \frac{\mu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} + \kappa\left(-\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2 \tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}\right)$$

and f_2 has to be determined

WKB constructions in the simple well case

Sketch of proof

- Fourth equation in h^2

$$(\mathfrak{L}_0 - \lambda_0)a_3 = (\lambda_3 - \mathfrak{L}_3)a_0 + (\lambda_2 - \mathfrak{L}_2)a_1 - \mathfrak{L}_1 a_2$$

Fredholm condition

$$\Rightarrow \langle \mathfrak{L}_3 a_0 + (\mathfrak{L}_2 - \lambda_2)a_1 + \mathfrak{L}_1 a_2, u_{\zeta_0}\rangle_{\mathrm{L}^2(\mathbb{R}_+,\,\mathrm{d}\tau)} = \lambda_3 f_0$$

This equation in σ-variable writes

$$\frac{\mu''(\zeta_0)}{2}\left(\Phi'(\sigma)\partial_\sigma + \partial_\sigma\Phi'(\sigma)\right) f_0 + F(\sigma)f_0 = \lambda_3 f_0$$

where F is a smooth function which vanishes at $\sigma = 0$

WKB constructions in the simple well case

Sketch of proof

- Fourth equation in h^2

Linearized equation at $\sigma = 0$:

$$\Phi''(0)\frac{\mu''(\zeta_0)}{2}\left(\sigma\partial_\sigma + \partial_\sigma\sigma\right) f_0 = \lambda_3 f_0$$

WKB constructions in the simple well case

Sketch of proof

- Fourth equation in h^2

Linearized equation at $\sigma = 0$:

$$C_1 \Theta_0^{1/4} \sqrt{\frac{3k_2}{2}} \left(\sigma\partial_\sigma + \partial_\sigma \sigma\right) f_0 = \lambda_3 f_0$$

Choose λ_3 in the spectrum of this *transport equation*

$$\left\{(2n-1) C_1 \Theta_0^{1/4} \sqrt{\frac{3k_2}{2}}, \quad n \geq 1\right\}$$

$\Rightarrow$ Local resolution of the transport equation

The operator symbol

Global change of variables

$$\mathfrak{L}_h = m(\sigma, h\tau)^{-1} D_\tau m(\sigma, h\tau) D_\tau$$
$$+ m(\sigma, h\tau)^{-1} \left(hD_\sigma + \frac{\gamma_0}{h} - \tau + h\frac{\tau^2}{2}\kappa(\sigma) \right) m(\sigma, h\tau)^{-1} \left(hD_\sigma + \frac{\gamma_0}{h} - \tau + h\frac{\tau^2}{2}\kappa(\sigma) \right)$$

seen as an operator valued operator:

$$hD_\sigma + \frac{\gamma_0}{h} \quad \leftrightarrow \quad \zeta$$

$\Rightarrow$ 1D-operator in the τ-variable

$$\mathcal{H}_{\sigma,\zeta,h} = -m(\sigma, h\tau)^{-1} \partial_\tau m(\sigma, h\tau) \partial_\tau$$
$$+ m(\sigma, h\tau)^{-1} \left(\zeta - \tau + h\frac{\tau^2}{2}\kappa(\sigma) \right) m(\sigma, h\tau)^{-1} \left(\zeta - \tau + h\frac{\tau^2}{2}\kappa(\sigma) \right) + \mathcal{O}(h^2)$$

The operator symbol

Asymptotics expansion

Since $m(\sigma, h\tau) = 1 - h\tau\kappa(\sigma)$, then

$$m(\sigma, h\tau)^{-1} = 1 + h\tau\kappa(\sigma) + \mathcal{O}(h^2)$$

Thus

$$\begin{aligned}\mathcal{H}_{\sigma,\zeta,h} = &-\left(1 + h\tau\kappa(\sigma) + \mathcal{O}(h^2)\right)\partial_\tau\left(1 - h\tau\kappa(\sigma)\right)\partial_\tau \\ &+ \left(1 + h\tau\kappa(\sigma) + \mathcal{O}(h^2)\right)^2\left(\zeta - \tau + h\frac{\tau^2}{2}\kappa(\sigma)\right)^2 + \mathcal{O}(h^2)\end{aligned}$$

The operator symbol

Asymptotics expansion

Since $m(\sigma, h\tau) = 1 - h\tau\kappa(\sigma)$, then

$$m(\sigma, h\tau)^{-1} = 1 + h\tau\kappa(\sigma) + \mathcal{O}(h^2)$$

Thus

$$\mathcal{H}_{\sigma,\zeta,h} = \mathcal{H}_\zeta + h\kappa(\sigma)\left(\partial_\tau + 2\tau(\zeta - \tau)^2 + \tau^2(\zeta - \tau)\right) + \mathcal{O}(h^2)$$

on $\mathbb{R}/\ell\mathbb{Z} \times \mathbb{R}_+$

The lowest eigenvalue $\nu(\sigma, \zeta, h)$ of operator $\mathcal{H}_{\sigma,\zeta,h}$ is simple and isolated

Born-Oppenheimer strategy

Computation...

For each $\sigma \in \mathbb{R}/\ell\mathbb{Z}$ and $\zeta \in \mathbb{R}$, compute the integral

$$\int_0^\infty \mathcal{H}_{\sigma,\zeta,h} u_\zeta(\tau) u_\zeta(\tau)\, \mathrm{d}\tau$$

We get, as $h \to 0$, $\sigma \to 0$ and $\zeta \to \zeta_0$,

$$\int_0^\infty \mathcal{H}_{\sigma,\zeta,h} u_\zeta(\tau) u_\zeta(\tau)\, \mathrm{d}\tau = \Theta_0 + \frac{\mu''(\zeta_0)}{2}(\zeta - \zeta_0 + \alpha_0 h)^2 - C_1 h\kappa(\sigma)$$
$$+ \mathcal{O}(h^2) + \mathcal{O}(h\sigma^2(\zeta - \zeta_0)) + \mathcal{O}(h(\zeta - \zeta_0)^2) + \mathcal{O}((\zeta - \zeta_0)^3)$$

with α_0 defined by $\mu''(\zeta_0)\alpha_0 = C_2\kappa_{\max}$

Born-Oppenheimer strategy

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with α_0 defined by $\mu''(\zeta_0)\alpha_0 = C_2\kappa_{\max}$

To determine the effective operator

$$\zeta \quad \leftrightarrow \quad hD_\sigma + \frac{\gamma_0}{h}$$

Effective operator

Maxima

$\mathcal{M} = \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$: set of all curvilinear abcissa where the maximal curvature $\kappa_{\max}$ is attained
In the previous asymptotics, replace $\sigma \to 0$ by $\mathrm{dist}(\sigma, \mathcal{M}) \to 0$
$\mathrm{dist}(\sigma, \mathcal{M})$: curvilinear distance between σ and the set $\mathcal{M}$

At a formal level, and coming back to operators in variable σ,
one expects that the low lying spectrum of the operator $\mathfrak{L}_h$ should be asymptotically the same as the one of

$$\mathfrak{L}_h^{\mathrm{eff}} = \Theta_0 + \frac{\mu''(\zeta_0)}{2}\Big(hD_\sigma + \frac{\gamma_0}{h} - \zeta_0 + \alpha_0 h\Big)^2 - C_1\kappa(\sigma)h$$

acting on $\mathrm{L}^2(\mathbb{R}/\ell\mathbb{Z}, \mathrm{d}\sigma)$
and up to operators with symbol

$$\mathcal{O}\big(h^2\big), \quad \mathcal{O}\Big(h\big(\mathrm{dist}(\sigma,\mathcal{M})\big)^2(\zeta-\zeta_0)\Big), \quad \mathcal{O}\big(h(\zeta-\zeta_0)^2\big) \quad \text{and} \quad \mathcal{O}\big((\zeta-\zeta_0)^3\big)$$

Effective operator

Scaling

$$\mathcal{L}_{\hbar}^{\mathrm{eff}} = \Theta_0 \hbar + \frac{\mu''(\zeta_0)}{2}(\hbar D_s + \gamma_0 - \zeta_0 \hbar^{\frac{1}{2}} + \alpha_0 \hbar)^2 - C_1 \kappa(s) \hbar^{\frac{3}{2}}$$

with

$$\gamma_0 = \frac{|\Omega|}{\ell}$$

Effective operator

Scaling

$$\mathcal{L}_\hbar^{\mathrm{eff}} = \Theta_0\hbar + \frac{\mu''(\zeta_0)}{2}(\hbar D_s + \gamma_0 - \zeta_0\hbar^{\frac{1}{2}} + \alpha_0\hbar)^2 - C_1\kappa(s)\hbar^{\frac{3}{2}}$$

with

$$\gamma_0 = \frac{|\Omega|}{\ell}$$

One expects

$$\lambda_n(\hbar) \simeq \lambda_n(\mathcal{L}_\hbar^{\mathrm{eff}})$$

Tunneling effect for the ellipse

Effective operator

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\}, \quad a > b > 0$$

$\ell = |\partial\Omega|$, $k_2 = -\kappa''(0)$

Effective operator

$$\mathcal{L}_\hbar^{\mathrm{eff}} = \Theta_0 \hbar + \frac{\mu''(\zeta_0)}{2} (\hbar D_s + \gamma_0 - \zeta_0 \hbar^{\frac{1}{2}} + \alpha_0 \hbar)^2 - C_1 \kappa(s) \hbar^{\frac{3}{2}}$$

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- Shift by $\Theta_0 \hbar - C_1 \kappa(0) \hbar^{\frac{3}{2}}$
- rescaling $s = \frac{\ell}{2\pi} x$

$$\Rightarrow \qquad \frac{\mu''(\zeta_0)}{2} \hbar^{\frac{3}{2}} \left[(\varepsilon D_x + \xi_0)^2 + V(x)\right] \qquad \text{on} \qquad \mathrm{L}^2(\mathbb{R}/2\pi\mathbb{Z})$$

$$\varepsilon = \frac{2\pi \hbar^{\frac{1}{4}}}{\ell}, \qquad \xi_0 = \frac{\gamma_0}{\hbar^{\frac{3}{4}}} - \frac{\zeta_0}{\hbar^{\frac{1}{4}}} + \alpha_0 \hbar^{\frac{1}{4}}, \qquad V(x) = \frac{2C_1}{\mu''(\zeta_0)}\left(\kappa(0) - \kappa\left(\frac{\ell x}{2\pi}\right)\right)$$

Magnetic flux effect on the circle

Framework

Let $\mathfrak{P}_\varepsilon$ the electro-magnetic Laplacian

$$\mathfrak{P}_\varepsilon = (\varepsilon D_x + a(x))^2 + V(x) \qquad \text{on } \mathrm{L}^2_{2\pi-per}(\mathbb{R}, \mathrm{d}x)$$

with a and V smooth, 2π-periodic

Gauge transform:

$$\mathfrak{P}_\varepsilon \sim \mathcal{P}_\varepsilon = (\varepsilon D_x + \xi_0)^2 + V(x), \qquad \xi_0 = \int_{-\pi}^{\pi} a(x)\,\mathrm{d}x$$

eigenvalues : $\rho_k(\varepsilon)$

Magnetic flux effect on the circle

Gap between the first two eigenvalues

Theorem

- *exactly two non-degenerate minima at* 0 *and* π *with* $V(0) = V(\pi) = 0$
- $V(x) = V(\pi - x) = V(-x)$

$$\rho_{1,2}(\varepsilon) = V_2\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \qquad \text{with } V_2 = \sqrt{\frac{V''(0)}{2}}$$

Let us define the positive Agmon distance and the constant A *by*

$$\mathsf{S} = \int_{[0,\pi]} \sqrt{V(x)}\,\mathrm{d}x, \qquad \text{and} \qquad \mathsf{A} = \exp\left(-\int_{[0,\frac{\pi}{2}]} \frac{\partial_x \sqrt{V} - V_2}{\sqrt{V}}\,\mathrm{d}x\right)$$

Then we have the spectral gap estimate

$$\rho_2(\varepsilon) - \rho_1(\varepsilon) = 8\varepsilon^{1/2}\mathsf{A}\sqrt{V\left(\frac{\pi}{2}\right)}\sqrt{\frac{V_2}{\pi}}\left|\cos\left(\frac{\xi_0\pi}{\varepsilon}\right)\right| \mathrm{e}^{-\mathsf{S}/\varepsilon} + \varepsilon^{3/2}\mathcal{O}\left(\mathrm{e}^{-\mathsf{S}/\varepsilon}\right)$$

cf. [Outassourt87, BN-Hérau-Raymond]

Tunneling effect for the ellipse

Spectral gap for the effective operator

Proposition

The spectral gap of the effective operator $\mathcal{L}_h^{\mathrm{eff}}$ *is given by*

$$\lambda_2^{\mathrm{eff}}(\hbar) - \lambda_1^{\mathrm{eff}}(\hbar) \underset{\hbar \to 0}{\sim} \hbar^{\frac{13}{8}} \mathsf{A} \frac{2^{\frac{5}{2}} C_1^{\frac{3}{4}}}{\sqrt{\pi}} \left|\kappa''(0)\mu''(\zeta_0)\right|^{\frac{1}{4}} \left(\kappa(0) - \kappa\left(\frac{\ell}{4}\right)\right)^{\frac{1}{2}} \times \left|\cos\left(\frac{\ell}{2}\left(\frac{\gamma_0}{\hbar} - \frac{\zeta_0}{\hbar^{\frac{1}{2}}} + \alpha_0\right)\right)\right| e^{-\mathsf{S}/\hbar^{\frac{1}{4}}}$$

where

$$\mathsf{S} = \sqrt{\frac{2C_1}{\mu''(\zeta_0)}} \int_0^{\frac{\ell}{2}} \sqrt{\kappa(0) - \kappa(s)}\, \mathrm{d}s,$$

$$\mathsf{A} = \exp\left(-\int_{[0,\frac{\ell}{4}]} \frac{\partial_s \sqrt{\kappa(0) - \kappa(s)} - \sqrt{-\frac{\kappa''(0)}{2}}}{\sqrt{\kappa(0) - \kappa(s)}}\, \mathrm{d}s\right)$$

Tunneling effect for the ellipse

Spectral gap for the operator

Conjecture

The spectral gap of the operator $\mathcal{L}_h$ is given by

$$\lambda_2(\hbar) - \lambda_1(\hbar) \underset{\hbar\to 0}{\sim} \hbar^{\frac{13}{8}} \mathsf{A} \frac{2^{\frac{5}{2}} C_1^{\frac{3}{4}}}{\sqrt{\pi}} \left|\kappa''(0)\mu''(\zeta_0)\right|^{\frac{1}{4}} \left(\kappa(0) - \kappa\left(\frac{\ell}{4}\right)\right)^{\frac{1}{2}} \times \left|\cos\left(\frac{\ell}{2}\left(\frac{\gamma_0}{\hbar} - \frac{\zeta_0}{\hbar^{\frac{1}{2}}} + \alpha_0\right)\right)\right| e^{-\mathsf{S}/\hbar^{\frac{1}{4}}}$$

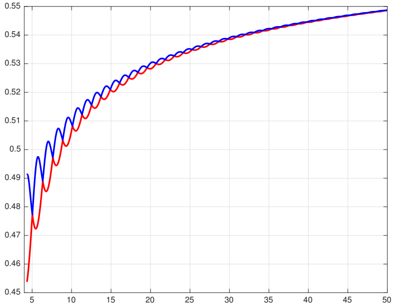
where

$$\mathsf{S} = \sqrt{\frac{2C_1}{\mu''(\zeta_0)}} \int_0^{\frac{\ell}{2}} \sqrt{\kappa(0) - \kappa(s)}\, \mathrm{d}s,$$

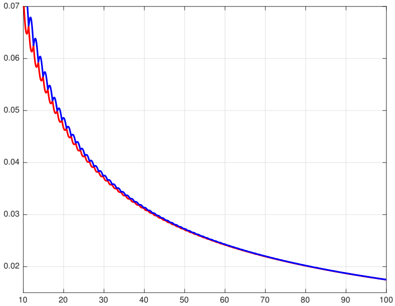
$$\mathsf{A} = \exp\left(-\int_{[0,\frac{\ell}{4}]} \frac{\partial_s \sqrt{\kappa(0) - \kappa(s)} - \sqrt{-\frac{\kappa''(0)}{2}}}{\sqrt{\kappa(0) - \kappa(s)}}\, \mathrm{d}s\right)$$

Numerical simulations

Eigenvalues



(a) $\lambda_j(\hbar)/\hbar$ vs. $1/\hbar$



(b) $\lambda_j(\hbar)/\hbar - \Theta_0 + C_1\kappa_{\max}\hbar^{1/2}$ vs. $1/\hbar$

Figure: Behavior of the first two eigenvalues vs. $1/\hbar$

Numerical simulations

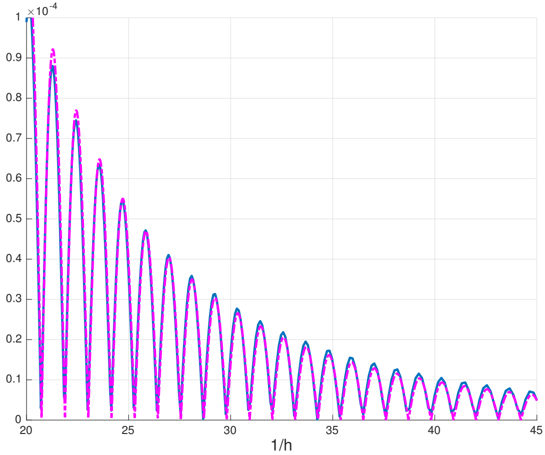
Computations of the parameters

$$
\begin{aligned}
|\Omega| &= \pi ab \simeq 6.283 \\
\ell &= 4\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2 \sin^2 t}\,\mathrm{d}t \simeq 9.688 \\
e &= \sqrt{1-\frac{b^2}{a^2}} = \frac{\sqrt{3}}{2} \\
\kappa(s) &= \frac{b}{a^2}\left(1-\mathrm{e}^2\cos^2\left(\frac{2\pi s}{\ell}\right)\right)^{-3/2} = \frac{1}{4}\left(1-\frac{3}{4}\cos^2\left(\frac{2\pi s}{\ell}\right)\right)^{-3/2} \\
\kappa_{\max} &= \kappa(0) = 2 \\
\kappa(\frac{\ell}{4}) &= 1/4
\end{aligned}
$$

$$
\mathrm{S} = \sqrt{\frac{2C_1}{\mu''(\zeta_0)}\frac{\ell}{2\pi}\frac{\sqrt{b}}{a}\int_0^{\pi}\sqrt{(1-e^2)^{-3/2}-(1-e^2\cos^2 \mathrm{s})^{-3/2}}\,\mathrm{ds}} \simeq 3.357
$$

Numerical simulations

Validity of the conjecture



Domains with corners

Stratification of $\overline{\Omega}$ (Ω bounded simply connected in $\mathbb{R}^2$)

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{e}\in\mathfrak{E}} \mathbf{e} \Big) \cup \Big(\bigcup_{\mathbf{v}\in\mathfrak{V}} \mathbf{v} \Big)$$

with $\mathfrak{E}$ and $\mathfrak{V}$: the set of edges $\mathbf{e}$ and vertices $\mathbf{v}$ of Ω

For any $\mathbf{x} \in \overline{\Omega}$, let $\Pi_{\mathbf{x}}$ be its tangent cone

Dimension	$\mathbf{x} \in \overline{\Omega}$	Model geometry for $\Pi_{\mathbf{x}}$
2D	Ω	plane $\mathbb{R}^2$
	$\mathbf{e}$	half-plane $\mathbb{R}^2_+$
	$\mathbf{v}$	angular sector $\mathcal{S}_\alpha$

cf. [BN-Dauge-Popoff]

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For any $\mathbf{x} \in \overline{\Omega}$, let $\Pi_{\mathbf{x}}$ be its tangent cone

Local ground energy
$E(\mathbf{B}, \Pi_{\mathbf{x}})$: bottom of the spectrum of the tangent operator $(-i\nabla + \mathbf{A})^2$ on $\Pi_{\mathbf{x}}$

cf. [BN-Dauge-Popoff]

Domains with corners

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Lowest local energy

$$\mathscr{E}(\mathbf{B}, \Omega) := \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}, \Pi_{\mathbf{x}})$$

cf. [BN-Dauge-Popoff]

Domains with corners

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Theorem

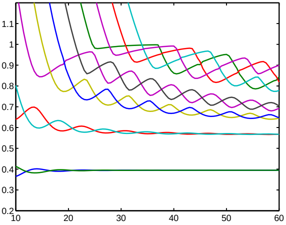
There exists $C > 0$, $\kappa > 1$ such that

$$\Big|\lambda_1(\hbar) - \hbar\mathscr{E}(\mathbf{B}, \Omega)\Big| \leq C\,\hbar^{\kappa} \qquad \text{as } \hbar \to 0$$

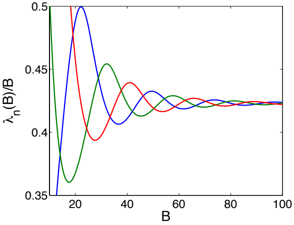
cf. [BN-Dauge-Popoff]

Asymptotics

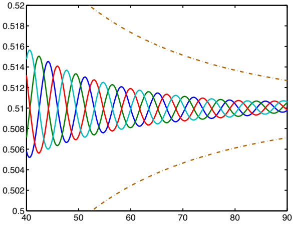
Numerical simulations



Rhombus



equilateral triangle



square

$\hbar^{-1}\lambda_n(\hbar)$ vs. $\hbar^{-1}$

cf. [BN-Dauge-Martin-Vial]