Spectral analysis of a complex Schrödinger operator in the semiclassical limit

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BIRS - May 2017

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$$\partial_t M - \Delta M - M \times B = 0$$

- controllability of Kolmogorov type equations

$$\partial_t f + v^\gamma \partial_x f - \partial_v^2 f = u(t, x, v)$$

periodic ($\gamma = 1, 2$)

1D analysis

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Discreteness of the spectrum $\Rightarrow \sigma(\mathcal{L}) = \emptyset$

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$$\|e^{-t\mathcal{L}}\| \leq Ce^{-t^3/12} \Rightarrow \sigma(\mathcal{L}) = \emptyset$$

Davies (2007)

Semi-infinite 1D problem

Dirichlet

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Transmission $u = (u_+, u_-)$

$$D(\mathcal{L}_+^T) = \{u \in H^2(\mathbb{R}_+) \times H^2(\mathbb{R}_-) \,|\, xu \in L^2(\mathbb{R}) \\ u_+'(0) = u_-'(0) = \kappa[u_+(0) - u_-(0)]\}$$

Grebenkov, Helffer & Henry (2016)

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$$|\nabla V| \leq [2+\alpha_2^2]^{1/2}\sqrt{1+V^2}$$

$$\Downarrow$$

$$\|\Delta u\|_2^2 + \|Vu\|_2^2 \leq C(\|\mathcal{P}^D u\|_2^2 + \|u\|_2^2)\,.$$

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2. $\exists C_0 > 0$:

$$\max_{|\beta|=r+1} |D_x^\beta V(x)| \leq C_0 \, m(V, r, x) \quad \forall x \in \mathbb{R}^n$$

$$m := m(V, r, x) = \sqrt{\sum_{|\alpha| \leq r} |D_x^\alpha V(x)|^2 + 1}\,.$$

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Proposition (Almog, Grebenkov & Helffer)

$$\|m^{\frac{2}{2^{r+1}-1}}u\|^2 + \|m^{-2\frac{2^{r-1}-1}{2^{r+1}-1}}Vu\|^2 \leq C\left(\|\mathcal{P}_V u\|^2 + \|u\|^2\right), \ \forall u \in C_0^\infty(\mathbb{R}^n)$$

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$$m \xrightarrow[|x|\to\infty]{} \infty \Rightarrow (\mathcal{P} - \lambda)^{-1} \text{compact}$$

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$(\mathcal{P}^\# - \lambda)^{-1}$ compact for all $\lambda \in \rho(\mathcal{P})$.

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Theorem (Almog (2008), Henry (2015), AGH (2017))

$$\inf \Re\sigma(\mathcal{A}_h) \geq \lambda^{\#}[J_m h]^{2/3} + o(h^{2/3}) \quad \text{as } h \to 0$$

Limit operator (Dirichlet, Neumann, Robin)

$x = (x', x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}_+$ $\mathcal{L}_+^\# = -\partial_{x_n}^2 + iJ_n x_n$
$\mathcal{L}_\| = -\partial_{x_{n-1}}^2 + iJ_\| x_{n-1}$

$$\mathcal{L} = -\Delta_{x'} + \mathcal{L}_+^\# + \mathcal{L}_\| \quad D(\mathcal{L}_h) \subset H^2(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n; x_n^2 dx) \cap S^\#$$

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$$e^{-t\mathcal{L}} = e^{t\Delta_{x'}} \otimes e^{-t\mathcal{L}_+^{\#}} \otimes e^{-t\mathcal{L}_\parallel} \leq Ce^{-t^3 J_\parallel^2/12}\,.$$

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$J_\| = 0$ Continuous spectrum

$$\sigma(\mathcal{L}) = \bigcup_{r \geq 0} \{\sigma(\mathcal{L}_+^\#) + r\}$$

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$$\leq \int_0^\infty \|e^{t\Delta}\| \, \|e^{-t(\mathcal{L}_+^\# - \lambda)}\| \, dt \leq \frac{C}{\Re(\lambda^\# - \lambda)}$$

Upper bound

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Theorem (Almog & Henry (2016) $n = 2 \;\; \# = D$)

If $V_t(x_0)V_{ss}(x_0) > 0$, $J_m = \inf_{x \in \partial\Omega_\perp} |\nabla V(x)|$, *then*

$$\exists \lambda \in \sigma(\mathcal{A}_h) : \left|\lambda - iV(x_0) - e^{i\pi/3}|\nu_1|(J_m h)^{2/3} - \sqrt{\alpha_1/2}\, e^{i\frac{\pi}{4}} h\right| \sim o(h)\,.$$

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Theorem (Almog, Grebenkov, Helffer (2017))

Let $\mu_1 = \sum_{j=1}^{n-1}|\alpha_j|^{1/2}e^{i\pi/4\,\mathrm{sign}\,\alpha_j}$.

$$\exists\lambda \in \sigma(\mathcal{A}_h) : \left|\lambda - iV(x_0) - \Lambda_m^\#(\kappa)h^{2/3} - \mu_1(x_0)h\right| \sim o(h)\,.$$

Quasimode construction

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Eigenfunctions

$$\{u_{nk}\}_{n,k=1}^\infty = \{A_i(\tau + \nu_n)h_k(e^{\pm i\frac{\pi}{8}}\xi)\}_{n,k=1}^\infty \quad \text{complete}$$

h_k- Hermite functions

quasimode

$$U(s,\rho) = u_{11}(\xi(s),\tau(\rho)) \quad \Lambda = [J_m h^2]^{2/3}\lambda_{11}$$

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$$\lambda^* \in \partial B(\Lambda, rh) \subset \rho(\mathcal{A}_h) \quad h^{1/6} \ll r \ll 1.$$

$$\langle \eta_{h^{1/3}} U, (\mathcal{A}_h - \lambda^*)^{-1}(\eta_{h^{1/3}} U)\rangle = -\frac{1}{\lambda^* - \Lambda}[1 - \langle \eta_{h^{1/3}} U, (\mathcal{A}_h - \lambda^*)^{-1} f\rangle]$$

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Cauchy Theorem + bound on $\|(\mathcal{A}_h - \lambda^*)^{-1} f\|_2$

$$\Downarrow$$

$$\sigma(A_h) \cap B(\Lambda, rh) \neq \emptyset$$

resolvent estimates

$$\mathcal{B}_\varepsilon = \mathcal{L}_\tau^{\#} + \varepsilon^{1/2}\mathcal{L}_\xi$$

$$\mathcal{L}_\tau = -\frac{d^2}{d\tau^2} + i\tau \quad \mathcal{L}_\xi = -\frac{d^2}{d\xi^2} \pm \frac{i}{2}\xi^2$$

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Wild spectral behaviour u_n - eigenfunction

$$\langle \bar{u}_n, u_m \rangle = \delta_{nm} \quad \|u_n\|_2 \geq Ce^{\gamma n}$$

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$$\lambda_0 = e^{i\pi/3}|\nu_1| \quad ; \quad \lambda_2 = \sqrt{2}e^{\pm i\pi/4}$$

Lemma

$\exists r_0 > 0,\ \varepsilon_0 > 0$ *and* $C > 0 : \forall r \in (0, r_0),$

$$|\lambda - \lambda_0 - \varepsilon^{1/2}\lambda_2| = r\varepsilon^{1/2} \Rightarrow \|(\mathcal{B}_\varepsilon - \lambda)^{-1}\| \leq \frac{C}{r}\varepsilon^{-1/2} \quad \forall 0 < \varepsilon < \varepsilon_0 \,.$$

$$\mathcal{L}_\tau^{\#} v_k = \nu_k v_k \quad (\mathcal{B}_\varepsilon - \lambda) w = g$$

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$$\Pi_1(w) = \varepsilon^{-1/2} (\mathcal{L}_\xi - \lambda_2 - re^{i\alpha})^{-1} \Pi_1(g) \,.$$

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$$\Pi_k = \langle \bar{v}_k, \cdot \rangle v_k \quad \langle \bar{v}_m, v_k \rangle = \delta_{km}$$

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$$\|(\mathcal{B}_\varepsilon - \lambda)^{-1}\| \leq \|(\mathcal{B}_\varepsilon - \lambda)^{-1}\Pi_1\| + \|(\mathcal{B}_\varepsilon - \lambda)^{-1}(I - \Pi_1)\| \leq \frac{C}{r}\varepsilon^{-1/2}$$