Quantum groups and Whittaker functions

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(based on joint work with Buciumas, Bump, and Friedberg)

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Big Picture

Quantum groups are certain Hopf algebras, including deformations of universal enveloping algebras, that provide solutions to Yang-Baxter equations.

Whittaker functions are certain matrix coefficients for representations of reductive algebraic groups over local fields, like $\mathrm{GL}_r(\mathbb{R})$ or $\mathrm{GL}_r(\mathbb{Q}_p)$, or their covers.

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GOAL: Explain these terms in greater detail, and describe connections between them. From work in the '70's by Kazhdan-Kostant over $\mathbb{R}$ to recent work for metaplectic covers of groups over non-archimedean fields (B.-Buciumas-Bump (2016), B^3-Friedberg (2017), and more in Daniel Bump's talk).

Basics of quantum groups in two slides (1 of 2)

Here's an example of a presentation of a quantum group:

$$U_q(\mathfrak{sl}_2) := \left\langle E, F, K, K^{-1} \;\middle|\; \begin{array}{c} KK^{-1} = K^{-1}K = 1 \\ KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F \\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \end{array} \right\rangle$$

- A similar presentation is possible for $U_q(\mathfrak{g})$ for any complex semisimple Lie algebra $\mathfrak{g}$, using Cartan matrix and q-binomial coefficients (so has a PBW-type basis).

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- If q is not a root of unity, representation theory of $U_q(\mathfrak{g})$ closely resembles that of $\mathfrak{g}$ (semisimplicity, highest weight theory).
- More generally, quantum groups B are quasi-triangular Hopf algebras, so there is a co-algebra structure $\Delta : B \longrightarrow B \otimes B$ (and so tensor products of B-modules are B-modules).

Quantum groups and the Yang-Baxter equation

co-algebra structure $\Delta : B \longrightarrow B \otimes B$

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- Drinfeld (ICM86) demonstrated such an R for version of $U = U_q(\mathfrak{sl}_2)$ and showed

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (*)$$

  on $U \otimes U \otimes U$, where R_{ij} denotes R on the i-th and j-th copy of U.

Relation $(*)$ is known as the quantum Yang-Baxter equation.

Utility of Quantum Groups

- History really in reverse – Drinfeld, Jimbo defined these structures to provide instances of qYBE
- Extremely rigid structure – canonical bases with structure constants in positive integers arise "at $q = 0$" (Kashiwara, Lusztig)
- Jimbo: Generalized Schur-Weyl duality in which (S_r, GL_n) on $V^{\otimes r}$ is replaced by the pair $(\mathcal{H}_r, U_q(\mathfrak{gl}_n))$, with $\mathcal{H}_r$ the finite Hecke algebra
- Jones: Restrict this to GL_2 and to reps of S_r with Young diagram at most two rows to obtain Temperley-Lieb algebra in place of $\mathcal{H}_r$.
- In this talk, discuss how the matrix R associated to particular quantum group appears in a wholly new context in Whittaker functions of covering groups

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- Jacquet ('67) defined them on groups over local fields F. Let ψ be a non-degenerate character of the unipotent radical $U(F)$ of a Borel sg. Then a Whittaker function $W(g)$ is a function satisfying

$$W(ug) = \psi(u)W(g) \quad u \in U(F), g \in G(F), \quad (**)$$

  and appearing in an irreducible subspace under the G-action by right translation on functions satisfying $(**)$.

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Theorem (Gelfand-Graev, Jacquet-Langlands, P-S, Shalika, Rodier)

F : finite or local. An irreducible representation (π, V) of $G(F)$ has at most one Whittaker model (space of Whittaker functions isomorphic to π).

Whittaker functions and number theory

Whittaker functions over local fields are a basic tool in the theory of automorphic forms and the construction of automorphic L-functions:

- They give the local contributions of Fourier coefficients of non-holomorphic Eisenstein series, so feature in the Langlands-Shahidi method and are fundamental to the Rankin-Selberg method.
- As we'll see in the next few slides, they appear in important structure theorems in local theory of automorphic forms.

Archimedean Whittaker functions ($F = \mathbb{R}$ or $\mathbb{C}$)

Often, we seek formulas for the Whittaker function of a K-fixed vector in the representation (K – maximal compact). "spherical Whittaker function"
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Skipping lots of interesting results evaluating archimedean Whittaker functions (Stade, Givental, GLO). See Lam's article (arXiv:1308.5451) for a nice summary.

Geometric Satake Equivalence

Theorem (Satake Isomorphism)

For a non-archimedean local field F with ring of integers $\mathcal{O}$,

$$C_c[G(F)/G(\mathcal{O})]^{G(\mathcal{O})} \simeq \mathbb{C}[X_*(T)]^W \simeq \mathbb{C} \otimes K_0(Rep(G^\vee))$$

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Lusztig, Ginzburg, and Mirkovic-Vilonen ('07) demonstrated an equivalence between $\text{Rep}(G^\vee)$ and a category of perverse sheaves (D-modules) on $\text{Gr}(G) := G((t))/G[[t]]$.

This offers a construction of $G^\vee$ without using root data.

Quantizing the Geometric Satake Equivalence

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Theorem (Gaitsgory, Lurie)

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This "Whittaker category" was studied in earlier papers of Frenkel, Gaitsgory, Kazhdan, and Vilonen, resulting in a geometric proof of the Casselman-Shalika formula for spherical Whittaker function over local field.

Whittaker functions over non-archimedean local fields

For unramified principal series of $G(F)$: $(\chi : T(F)/T(\mathcal{O}) \longrightarrow \mathbb{C})$

$$i(\chi) := \mathrm{Ind}_B^G(\delta^{1/2}\chi) = \{f \in C^\infty(G) \mid f(bg) = \delta^{1/2}\chi(b)f(g), \forall b \in B, g \in G\}$$

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Theorem (Shintani, Kato, Casselman-Shalika)

Let χ have Langlands parameters $\boldsymbol{z} = (z_1, \ldots, z_r)$. Then for $t_\lambda \in T$ with λ dominant,

$$W_\chi(f^\circ)(t_\lambda) = \delta^{1/2}(t_\lambda) \prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha) s_\lambda(\boldsymbol{z})$$

s_λ : the character of the irreducible rep of $G^\vee(\mathbb{C})$ of highest weight λ.
q : cardinality of the residue field of $G(\mathcal{O})$

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- Suffices to compute this "magic factor" for a simple reflection $s \in W$.

Depending on lots of choices, the "magic factor" is roughly of the form:

$$\frac{\boldsymbol{z}^{\alpha^\vee} - q^{-1}}{1 - \boldsymbol{z}^{\alpha^\vee}}$$

Whittaker functions for metaplectic covers

Suppose that $\mu_n \subset F$. Construct a central extension:

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \xrightarrow{\pi} G(F) \longrightarrow 1$$

We can ask about spherical Whittaker functions, but everything is MORE COMPLICATED!

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For example, $\tilde{T} = \pi^{-1}(T(F))$ is not abelian, but we can still construct a principal series with $G(\mathcal{O})$ fixed vectors. Whittaker models are generally not unique (Kazhdan-Patterson ('84), Savin ('88,'04), McNamara ('12))

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For example, if $G = \mathrm{GL}_r$ it is a complicated though sparse square matrix of size n^r and its entries contain n-th order Gauss sums.

Results on spherical Whittaker functions for $\tilde{G}$

- Lots of (non-canonical) descriptions as generating functions parametrized by representation-theoretic data on "dual group" depending on cover degree (B-Bump-Friedberg, McNamara, B-Friedberg, Friedberg-Zhang), or

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- Descriptions as average of metaplectic version of Weyl group action or Hecke algebra action (Chinta-Offen, McNamara, Patnaik-Puskás)
- We'd like an algebraic characterization of the result.

Whittaker functions on covers of $\mathrm{GL}_r(F)$

Theorem (B-Buciumas-Bump, 2016)

For a simple reflection s_i, the Kazhdan-Patterson scattering matrix for $\mathrm{GL}_2^{(n)}(F)$ is a Drinfeld twist of the R-matrix for the standard module of $U_{\sqrt{q^{-1}}}(\widehat{\mathfrak{gl}}_n)$.

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- The Yang-Baxter equation for the model is actually the R-matrix of a twist of the standard module for $U_{\sqrt{q^{-1}}}(\widehat{\mathfrak{gl}}(n|1))$, but only a piece of the R-matrix appears in the K-P scattering matrix.

Further directions for Whittaker functions on covers

- One can ask about other groups. We expect to be able to handle classical groups similarly. (Progress on this in type C by N. Gray)
- In very recent work (BBBF, '17), we give a formalism for creating Hecke algebra actions from R-matrices. The resulting Hecke action recovers metaplectic Demazure operators in Chinta-Gunnells-Puskas.
- The R matrix associated to s_i acts on $V_{z_i} \otimes V_{z_{i+1}}$, with each V_{z_i} a copy of the standard module of $U_{\sqrt{q^{-1}}}(\mathfrak{gl}_n)$. So in total, scattering matrices act on an r-fold tensor of V_{z_i}'s.
- The Hecke actions constructed in BBBF also agree with those of Ginzburg-Reshetikhin-Vasserot ('94) and Kashiwara-Miwa-Stern ('95) in the context of quantum affine versions of Schur-Weyl duality.

The theorem again...

Theorem (B-Buciumas-Bump (arXiv:1604.02206))

There is an isomorphism $\theta_{\mathbf{z}}$ of the space $\mathcal{W}_{\mathbf{z}}$ of spherical Whittaker functions to the vector space $V(z_1) \otimes \cdots V(z_r)$, which takes the vectors $v_{a_1}(z_1) \otimes \cdots \otimes v_{a_r}(z_r)$ into the basis of $\mathcal{W}_{\mathbf{z}}$ given in KP84. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W}_{\mathbf{z}} & \xrightarrow{\theta_{\mathbf{z}}} & V(z_1) \otimes \cdots \otimes V(z_i) \otimes V(z_{i+1}) \otimes \cdots \otimes V(z_r) \\ \downarrow \bar{\mathcal{A}}^*_{s_i} & & \downarrow I_{V_+(z_1)} \otimes \cdots \otimes \tau R_{z_i, z_{i+1}} \otimes \cdots \otimes I_{V(z_r)} \\ \mathcal{W}_{\mathbf{s_i z}} & \xrightarrow{\theta_{\mathbf{s_i z}}} & V(z_1) \otimes \cdots \otimes V(z_{i+1}) \otimes V(z_i) \otimes \cdots \otimes V(z_r) \end{array}$$

where $\bar{\mathcal{A}}^*_{s_i}$ denotes the map obtained by $W^\chi_b \mapsto W^\chi_b \circ \bar{\mathcal{A}}_{s_i}$ with appropriately normalized intertwining operator $\bar{\mathcal{A}}_{s_i}$.

So what to make of all this?

Spherical Whittaker functions for the local field $F = \ldots$

- (Kazhdan-Kostant) $\ldots \mathbb{R}$ are eigenfunctions of quantum Toda lattice, whose R-matrix is that of a standard module on $U_q(\mathfrak{g})$. (no dual gp)
- (Gaitsgory-Lurie-Lysenko) $\ldots \mathbb{F}_q((t))$ are evaluated using facts about the Whittaker category, which appears in the quantized geometric Satake equivalence with $U_q(\mathfrak{g}^\vee)$. (dual gp)
- (B-Buciumas-Bump) $\ldots$ non-archimedean, metaplectic n-covers of GL_r have scattering matrices which are R-matrices for standard modules on $U_q(\widehat{\mathfrak{gl}}_n)$.

What, if anything, connects these points of view?