Charlotte Hardouin

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BIRS, 17-22 september 2017

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# Some transcendence results for special functions

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### Proposition

The function  $\Gamma(x)$  satisfying  $\Gamma(x + 1) = x\Gamma(x)$  is transcendental over  $C_1(x)$  where  $C_1$  is the field of 1-periodic meromorphic functions over  $\mathbb{C}$ .

#### heorem

Let  $b_1, \ldots, b_r \in \mathbb{C}(x)$  and  $f_1, \ldots, f_r$  meromorphic functions over  $\mathbb{C}$  solutions of

$$f_i(x+1) = f_i(x) + b_i(x)$$
 for all  $i = 1, ..., r$ 

The  $f_i$ 's are algebraically dependent over  $C_1(x)$  if and only if there exist  $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$  not all zero and  $g \in \mathbb{C}(x)$  such that

 $\gamma_1 b_1(x) + \cdots + \gamma_r b_r(x) = g(x+1) - g(x).$ 

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### Theorem (Roques 2007)

For  $q \in \mathbb{C}$  with |q| > 1. Let  $y_1(x), y_2(x)$  two linearly independent solutions of

$$y(q^{2}x) - \frac{2ax - 2}{a^{2}x - 1}y(qx) - \frac{x - 1}{a^{2}x - q^{2}x}y(x) = 0$$

with  $a \notin q^{\mathbb{Z}}$  and  $a^2 \in q^{\mathbb{Z}}$ . Then,  $y_1(x), y_2(x), y_1(qx)$  are algebraically independent.

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### These results come from a Galois theory for linear discrete equations and from the comprehension of the associated linear algebraic groups

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### Suppose to the contrary that $\Gamma(x)$ is algebraic.

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 $\Gamma(x)^{n} + a_{n-1}(x)\Gamma(x)^{n-1} + \ldots + a_{0}(x) = 0, \qquad (1.1)$ minimal relation with  $a_{n}(x) \in \mathbb{C}(x)$  and  $a_{n}(x) \neq 0$ 

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Use  $\Gamma(x + 1) = x\Gamma(x)$  in (1.2) and eliminate  $\Gamma(x)^n$  between (1.1) and (1.2), we find

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# Set $K = C_1(x)$ and $K_{\Gamma} = C_1(x)(\Gamma) \subset \operatorname{Mer}(\mathbb{C})$ .

These fields are closed under  $\sigma(f(x)) = f(x+1)$  and

 $\sigma(\Gamma) = x\Gamma$ 

$$K_{\Gamma}^{\sigma} := \{ f \in K_{\Gamma} | \sigma(f) = f \} = C_1 = K^{\sigma}$$
  
Consider

 $\mathsf{Gal}(K_{\Gamma}|K) = \{\tau \in \mathsf{Aut}(K_{\Gamma}) \ \tau|_{K} = id_{K} \ \tau \circ \sigma = \sigma \circ \tau \}.$ 

Let  $\tau \in \operatorname{Gal}(K_{\Gamma}|K)$ . Then

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Thus, there exists  $c_{\tau} \in C_1^*$  such that  $\tau(\Gamma) = c_{\tau}\Gamma$ .

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Picard Vessiot pseudofields : Set  $K = C_1(x)$  and  $K_{\Gamma} = C_1(x)(\Gamma) \subset \operatorname{Mer}(\mathbb{C})$ . These fields are closed under  $\sigma(f(x)) = f(x+1)$  and

 $\sigma(\Gamma) = x\Gamma$ 

• 
$$K_{\Gamma}^{\sigma} := \{f \in K_{\Gamma} | \sigma(f) = f\} = C_1 = K^{\sigma}$$
  
• Consider

 $\mathsf{Gal}(K_{\Gamma}|K) = \{ \tau \in \mathsf{Aut}(K_{\Gamma}) \ \tau|_{K} = id_{K} \ \tau \circ \sigma = \sigma \circ \tau \}.$ 

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Picard Vession pseudofields :

### First, some Galoisian facts

•  $Gal(K_{\Gamma}|K) = \{c_{\tau}\}$  is an algebraic subgroup of  $(C_1, *)$ 

 $f \in K_{\Gamma} \text{ fixed by } \mathsf{Gal}(K_{\Gamma}|K) \Leftrightarrow f \in K$ 

•  $\operatorname{trdeg}(K_{\Gamma}|K) = \operatorname{dim}(\operatorname{Gal}(K_{\Gamma}|K)|C_1)$ 

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 $\blacksquare \exists n | \operatorname{Gal}(K_{\Gamma}|K) = \{c_{\tau}\} \subset \mu_n, \text{ i.e.}$ 

 $c_{\tau}^n = 1$ 

# $\tau(\Gamma^n) = (\tau(\Gamma))^n = (c_{\tau}\Gamma)^n = \Gamma^n \text{ for all } \tau \in \text{Gal}(K_{\Gamma}|K)$ $\Gamma^n \in C_1(x)$

There exists  $g(x) = \Gamma(x)^n \in C_1(x)$  such that  $g(x+1) = x^n g(x)$ 

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Picard Vession pseudofields :

### Galois theory for difference fields

- Application to transcendence and differential transcendence
- General framework : Galois theory for difference rings

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Picard Vessiot pseudofields : A difference field  $(K, \sigma)$  is a field K together with  $\sigma : K \to K$ an automorphism.

### xamples (Endomorphism of the complex variable)

 $K = \mathbb{C}(x), \ \sigma(f(x)) = f(x+1);$ 

•  $K = \mathbb{C}(x), \ \sigma(f(x)) = f(qx) \text{ for } |q| > 1$ 

 $K = \mathbb{C}(x), \sigma(f(x)) = f(x^p)$  with  $p \in \mathbb{N}$ ; This is not surjective! Replace K by  $\hat{K} = \bigcup_{n=0}^{\infty} \mathbb{C}(x^{1/p^n})$  and set  $\sigma(x^{1/p^n}) = x^{1/p^{n-1}}$ .

Algebraic notions and compatibility with  $\sigma$  :  $\sigma$ -ring,  $\sigma$ -morphism,  $\sigma$ -subfield,  $\sigma$ -field extension etc

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### Examples (Automorphisms of an elliptic curve)

Let (E, ⊕) ⊂ P<sup>2</sup>C be an elliptic curve and let Ω ∈ E.
 Let C<sub>E</sub> be the field of elliptic functions.
 Then, (C<sub>E</sub>, σ : C<sub>E</sub> → C<sub>E</sub>, f(P) ↦ f(P ⊕ Ω)) is σ-field.

Let  $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$  a lattice such that  $\mathbb{C}/\Lambda \simeq E$ The application  $\mathbb{C} \to E, \omega \mapsto (x(\omega), y(\omega))$  identifies

•  $C_E = \{(\omega_1, \omega_2) \text{ -periodic functions }\} \subset \mathcal{M}er(\mathbb{C})$ 

• the action of  $\sigma$  to  $\sigma(f(\omega)) = f(\omega + \omega_3)$  with  $\omega_3 \in \mathbb{C}$  such that  $\Omega = (x(\omega_3), y(\omega_3))$ .

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 Let Λ = Zω<sub>1</sub> + Zω<sub>2</sub> ⊂ C a lattice such that C/Λ ≃ E.

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• Let  $(E, \oplus) \subset \mathbb{P}^2 \mathbb{C}$  be an elliptic curve and let  $\Omega \in E$ . Let  $C_E$  be the field of elliptic functions. Then,  $(C_E, \sigma : C_E \to C_E, f(P) \mapsto f(P \oplus \Omega))$  is  $\sigma$ -field.

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Picard Vession pseudofields : Let  $(K, \sigma)$  be a difference field. The field of constants is  $k = K^{\sigma} = \{a \in K | \sigma(a) = a\}.$ 

### zxamples

K = C(x), σ(x) = x + 1. Then k = C;
K = Mer(C), σ(x) = x + 1. Then k = C<sub>1</sub>;
K = C<sub>E</sub> with σ(ω) = ω + ω<sub>3</sub> and for all n ∈ Z\*, nω<sub>3</sub> ∉ Zω<sub>1</sub> + Zω<sub>2</sub>. Then k = C.
K = Mer(C), σ(ω) = ω + ω<sub>3</sub>. Then k is the field of ω<sub>3</sub>-periodic functions.

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$$K = \mathcal{M}er(\mathbb{C}), \ \sigma(x) = x + 1.$$
 Then  $k = C_1$ ;

•  $K = C_E$  with  $\sigma(\omega) = \omega + \omega_3$  and for all  $n \in \mathbb{Z}^*$ ,  $n\omega_3 \notin \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ . Then  $k = \mathbb{C}$ .

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Let 
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 be a  $\sigma$ -field and  $g$  a solution of  
 $\sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \dots + a_0y = 0$  ( $\mathcal{L}$ ),  
with  $a_0 \neq 0$ ,  $a_i \in K$ . Then,  $Z := \begin{pmatrix} g \\ \sigma(g) \\ \vdots \\ \sigma^{n-1}(g) \end{pmatrix}$  is solution of  
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with



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 $\sigma(Y) = A_{\mathcal{L}}Y$ 

with

$$A_{\mathcal{L}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in \operatorname{GL}_n(\mathcal{K}). \quad (2.1)$$

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# Let $(K, \sigma)$ be a $\sigma$ -field. An equation $\sigma(Y) = AY$ with $A \in GL_n(K)$ is called difference system.

From now on, we will always consider  $\sigma$ -fields with non periodic element.

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Let  $(K, \sigma)$  be a  $\sigma$ -field and  $A \in GL_n(K)$ . Let L be a  $\sigma$ -field extension of K. An matrix  $U \in GL_n(L)$  such that  $\sigma(U) = AU$  is called fundamental solution matrix of  $\sigma(Y) = AY$ 

Let  $U_1, U_2 \in \operatorname{GL}_n(L)$  two fundamental solution matrices for  $\sigma(Y) = AY$  then there exists  $D \in \operatorname{GL}_n(L^{\sigma})$  such that

 $U_1=U_2\mathbf{D}.$ 

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# Consider the difference field extension $(\mathbb{C}(x), \sigma(f(x)) = f(x+1)) \subset (Mer(\mathbb{C}), \sigma(f(x)) = f(x+1)).$

Then  $\Gamma \in \operatorname{Mer}(\mathbb{C})^*$  is a fundamental solution matrix for  $\sigma(y) = xy$ .

Let  $\psi(x)$  be the digamma function  $\frac{\Gamma'}{\Gamma}$ . Then,  $\sigma(\psi(x)) = \psi(x) + \frac{1}{x}$ . This correspond to the difference system

$$\sigma(Y) = \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} Y$$

with fundamental solution matrix  $U = \begin{pmatrix} 1 & \psi(x) \\ 0 & 1 \end{pmatrix} \in Gl_2(\mathcal{M}er(\mathbb{C}))$ 

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More generally, the difference equations  $\sigma(y_i) = y_i + \frac{(-1)^{i-1}}{\sqrt{i}}$  for An overview of difference  $i = 1, \ldots, r$  are encoded by the difference system Galois theory Charlotte  $\sigma(Y) = \begin{pmatrix} 1 & \frac{1}{x} & 0 & \dots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & \frac{(-1)^{r-1}}{x^r} \\ 0 & 0 & 0 & 1 \end{pmatrix} Y$ Hardouin Difference equations and systems with  $U = \begin{pmatrix} 1 & \psi(x) & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & \frac{d^{r}}{dx^{r}}(\psi(x)) \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \operatorname{GL}_{2(r+1)}(\mathcal{M}er(\mathbb{C}))$ イロト イヨト イヨト イヨト

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Picard-Vessiot Field extensions

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Let  $(K, \sigma)$  be a  $\sigma$ -field and let  $A \in GL_n(K)$ . A  $\sigma$ -field extension  $K_A|K$  is called a Picard-Vessiot field extension of  $\sigma(Y) = AY$  over K if

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•  $K^{\sigma}_{\Delta} = K^{\sigma}$ ;

• there exists  $U \in GL_n(K_A)$  fundamental solution matrix such that  $K_A = K(U)$ .

The K- $\sigma$ -algebra  $R_A = K[U, \frac{1}{\det(U)}] \subset K_A$  is called a **PV-** ring.

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•  $(\mathcal{M}er(\mathbb{C}), \sigma(f(x)) = f(x+1))$  with  $\mathcal{M}er(\mathbb{C})^{\sigma} = C_1$ . For any  $A \in Gl_n(\mathcal{M}er(\mathbb{C}))$ , there exists  $U \in GL_n(\mathcal{M}er(\mathbb{C}))$  such that  $\sigma(U) = AU$  (Praagman). Then,  $K_A = C_1(x)(U) \subset Mer(\mathbb{C})$  is a PV-field extension for  $\sigma(Y) = AY$  over  $K = C_1(x)$ .

• Similar result for  $\mathcal{M}er(\mathbb{C}^*)$  and  $\sigma(f(x)) = f(qx)$ 

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•  $\left(\mathcal{M}er(\mathbb{C}), \sigma(f(x)) = f(x+1)\right)$  with  $\mathcal{M}er(\mathbb{C})^{\sigma} = C_1$ . For any  $A \in Gl_n(\mathcal{M}er(\mathbb{C}))$ , there exists  $U \in GL_n(\mathcal{M}er(\mathbb{C}))$  such that  $\sigma(U) = AU$  (Praagman). Then,  $K_A = C_1(x)(U) \subset Mer(\mathbb{C})$  is a PV-field extension for  $\sigma(Y) = AY$  over  $K = C_1(x)$ .

Similar result for  $\mathcal{M}er(\mathbb{C}^*)$  and  $\sigma(f(x)) = f(qx)$ 

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### Group representation

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### Theorem

### The application

$$\rho_U: \quad \operatorname{Gal}(K_A|K) \longrightarrow \operatorname{GL}_n(K^{\sigma})$$

$$\tau \longrightarrow [\tau]_U$$

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where  $\tau(U) = U[\tau]_U$  identifies  $Gal(K_A|K)$  with an algebraic subgroup of  $GL_n(K^{\sigma})$ .

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### H is an algebraic subgroup of $GL_n(k)$ if

• *H* subgroup of  $GL_n(k)$ 

•  $H = \{M | P(M) = 0 \text{ for all } P \in S\}$  with  $S \subset k[X, \frac{1}{\det(X)}]$ 

#### xamples

■  $\mu_n = \{x_{1,1}^n = 1\} \subset Gl_1(k)$ ■  $Sl_n(k) = \{X = (x_{i,j})_{i,j=1,...,n} | \det(X) = 1\}$ 

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 $\rho_U: \qquad \operatorname{Gal}(K_A|K) \longrightarrow (C_1^r, +)$ 

 $\longrightarrow (c_{\tau}^{1}, \ldots, c_{\tau}^{r})$ 

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where  $\tau(u_i) = u_i + c_{\tau}^i$ .

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$$(K, \sigma) = (C_1(x), \sigma(f(x)) = f(x+1)) \text{ and } b_1, \dots, b_r \in C_1(x).$$
  

$$\sigma(y_1) = y_1 + b_1$$
  

$$\sigma(y_2) = y_2 + b_2$$
  

$$\vdots = \vdots$$
  

$$\sigma(y_r) = y_r + b_r$$

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### • Algebraic subgroups of $(C_1^r, +)$ are $C_1$ -vector spaces

# $\operatorname{Gal}(K_A|K) \begin{cases} = (C_1^r, +) \\ \subset \{(c_i)|\gamma_1 c_1 + \dots + \gamma_r c_r = 0\} \text{ for } \gamma_i \in C_1 \end{cases}$

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$$\mathsf{Gal}(\mathcal{K}_{\mathcal{A}}|\mathcal{K}) \begin{cases} = (\mathcal{C}_{1}^{r}, +) \\ \subset \{(c_{i})|\gamma_{1}c_{1} + \dots + \gamma_{r}c_{r} = 0\} \text{ for } \gamma_{i} \in \mathcal{C}_{1} \end{cases}$$

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### Theorem

Let  $(K, \sigma)$  be a  $\sigma$ -field. Let  $A \in GL_n(K)$  and  $K_A|K$  a PV-field extension for  $\sigma(Y) = Y$ . Then,

# $K_A^{\mathsf{Gal}(K_A|K)} := \{ f \in K_A | \tau(f) = f \text{ for all } \tau \in \mathsf{Gal}(K_A|K) \} = K$

 $\operatorname{degtr}(K_A|K) = \operatorname{dim}_k(\operatorname{Gal}(K_A|K))$ 

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### Algebraic relations between solutions

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### Independence of discrete logarithms

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Picard Vession pseudofields :

Set 
$$(K, \sigma) = (C_1(x), \sigma(f(x) = f(x + 1)))$$
.  
Let  $b_1, \ldots, b_r \in \mathbb{C}(x)$  and  $u_1, \ldots, u_r \in \mathcal{M}er(\mathbb{C})$  solutions of

 $\sigma(y_1) = y_1 + b_1$   $\sigma(y_2) = y_2 + b_2$   $\vdots = \vdots$  $\sigma(y_r) = y_r + b_r$ 

If the  $u_i$  are algebraically dependent over K then there exists  $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$  not all zero and  $g \in \mathbb{C}(x)$  such that

$$\gamma_1 b_1 + \dots + \gamma_r b_r = \sigma(g) - g. \tag{5.1}$$

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s  $\Gamma$  differentially transcendental  $\mathbb{C}(x)$ , i.e.

 $\Gamma, \frac{d}{dx}(\Gamma), \ldots, \frac{d^r}{dx^r}(\Gamma), \ldots$  are alg. independent over  $\mathbb{C}(x)$ .

Vith  $\psi(x) = \frac{a}{dx}(\Gamma)$ 

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Picard Vessiot pseudofields : Deriving the functional equation for  $\Gamma$  and using  $\frac{d}{dx}\circ\sigma=\sigma\circ\frac{d}{dx}$  we find

$$\sigma(\psi(x)) = \psi(x) + \frac{1}{x}$$
$$\sigma(\frac{d}{dx}(\psi(x))) = \frac{d}{dx}(\psi(x)) + \frac{d}{dx}(\frac{1}{x})$$
$$\vdots = \vdots$$
$$\sigma(\frac{d^{r}}{dx^{r}}(\psi(x))) = \frac{d^{r}}{dx^{r}}(\psi(x)) + \frac{d^{r}}{dx^{r}}(\frac{1}{x}).$$

Thus  $\Gamma(x)$  diff.alg iff  $\exists r \ \psi(x), \dots, \frac{d^r}{dx^r}(\psi(x))$  algebraically dependent  $\Rightarrow \exists r, \gamma_1, \dots, \gamma_{r+1} \in \mathbb{C}$ , not all zero  $g \in \mathbb{C}(x)$  such that

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$$\gamma_1 \frac{1}{x} + \dots + \gamma_{r+1} \frac{d^r}{dx^r} (\frac{1}{x}) = g(x+1) - g(x).$$

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$$\sigma(y_1) = y_1 + \frac{d}{dx}(\frac{1}{x})$$
  
$$\sigma(y_2) = y_2 + \frac{d^2}{dx^2}(\frac{1}{x})$$
  
$$\vdots = \vdots$$

$$\sigma(y_{r+1})=y_{r+1}+\frac{d^r}{dx^r}(\frac{1}{x}).$$

Thus  $\Gamma(x)$  diff.alg iff  $\exists r \ \psi(x), \dots, \frac{d^r}{dx^r}(\psi(x))$  algebraically dependent

 $\Rightarrow \exists r, \gamma_1, \dots, \gamma_{r+1} \in \mathbb{C}, ext{ not all zero } g \in \mathbb{C}(x)$  such that

$$\gamma_1 \frac{1}{x} + \dots + \gamma_{r+1} \frac{d^r}{dx^r} (\frac{1}{x}) = g(x+1) - g(x).$$

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### Theorem

Let  $(K, \sigma)$  a  $\sigma$ -field with a derivation  $\delta$  commuting with  $\sigma$  and let  $b \in K$ . Let  $f \in L$ , a  $\sigma$ - $\delta$ -field extension of K, such that  $\sigma(f) = f + b$ .

Assume that  $K^{\sigma}$  is algebraically closed. Then, f is differentially transcendental over K if there are no  $r \in \mathbb{N}, \gamma_1, \ldots, \gamma_{r+1} \in k$  and  $g \in K$  such that

 $\gamma_1 b + \ldots + \gamma_{r+1} \delta^r(b) = \sigma(g) - g.$ 

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# Examples

 $(K, \sigma) = (\mathbb{C}(x), \sigma(f(x)) = f(x+1))$  and  $\sigma(y) = -y$ . Suppose that there exists a solution  $u \neq 0$  in a  $\sigma$ -field L with  $L^{\sigma} = \mathbb{C}(x)^{\sigma} = \mathbb{C}$ .

Then  $\sigma(u^2) = (-u)^2 = u^2$  and  $u^2 \in L^{\sigma} = \mathbb{C}$ . Then  $u \in \mathbb{C}$  and  $\sigma(u) = u$ . Contradiction !

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### For difference systems, one has a kind of dichotomy

• either, one solves in a field extension L BUT  $L^{\sigma} \neq K^{\sigma}$ .

 $K = \mathbb{C}(x), \ \sigma(x) = x + 1$ . For any  $\sigma(Y) = AY$  with  $A \in \mathrm{GL}_n(\mathbb{C}(x))$  there exists a fundamental solution matrix  $U \in \mathrm{GL}_n(\mathrm{Mer}(\mathbb{C}))$ .

• or one solves in  $\sigma$ -rings *L* that might not be interested as L that might not be interested as L that might not be interested as L and L as L

(General Picard Vessiot theory *cf.* van der Put-Singer, Wibmer)

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# Existence and uniqueness of Picard-Vessiot pseudofields

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### Theorem

Let  $(K, \sigma)$  a  $\sigma$ -field with  $K^{\sigma}$  algebraically closed and  $A \in Gl_n(K)$ .

Then, there exists a unique Picard-Vessiot pseudofield  $K_A$  for the system  $\sigma(Y) = AY$ .

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•  $K_A = K(U)$  with U a fundamental solution matrix •  $K_A^{\sigma} = K^{\sigma}$ 



### d all the results for fields hold in this context

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# Let $\mathbb{C}^{\mathbb{N}}$ the ring of $\mathbb{C}\text{-valued}$ sequences with addition and multiplication defined component by component. The morphism

 $\sigma: \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}, (a(0), a(1), \dots, a(n), \dots) \mapsto (a(1), \dots, a(n), \dots)$ 

is not injective Set

 $a \sim b$  iff  $\exists N | a(n) = b(n)$  for all n > N.

Then  $\sigma$  induces on  $S = \mathbb{C}^{\mathbb{N}} / \sim$  an injective morphism. The  $\sigma$ -ring S is called the ring of germs of sequences.

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Picard Vessiot pseudofields : Consider  $\mathbb{C}(x)$  endowed with  $\sigma(f(x)) = f(x+1)$ . The application



$$f \longmapsto (f(0), \ldots, f(n), \ldots)$$

is an injective ring morphism, the identity on  $\mathbb{C},$  commutes with  $\sigma.$ 

The difference field  $(\mathbb{C}(x), \sigma)$  is a  $\sigma$ -subring of S.

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#### Examples

$$K = (\mathbb{C}(x), \sigma(f(x)) = f(x+1)) \subset S$$
 and  $\sigma(y) = -y$ .  
Then  $u = ((-1)^n)_{n \in \mathbb{N}}$  is a fundamental solution matrix. In  $S$ , we have

■ 
$$u - 1 \neq 0$$
 and  $u + 1 \neq 0$  but  $(u - 1)(u + 1) = u^2 - 1 = 0$ .  
■  $\mathbb{C}(x)(u) = L_1 \times L_2 \subset S$  with  $L_1 = \mathbb{C}(x).(u - 1)$  and  $L_2 = \mathbb{C}(x).(u + 1)$ .

A general result : Let  $\mathbb{C}(x) \subset S$  and  $A \in \operatorname{GL}_n(\mathbb{C}(x))$ . There exists a Picard-Vessiot ring  $R_A := \mathbb{C}(x)[U, \frac{1}{\det(U)}] \subset S$ .

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## Theorem (Larson-Tarft-1990)

Let  $u, v \in S$  two sequences each of them satisfying a difference equation over  $\mathbb{C}(x)$  and such that uv = 0. Then, there exist  $u_0, \ldots, u_{t-1}, v_0, \ldots, v_{t-1} \in S$  such that

• u (resp.v) is the interlacing of the  $u_i$ (resp.  $v_i$ )

for all i either  $u_i = 0$  or  $v_i = 0$ 

## Theorem (Wibmer 2012)

et  $u \in S$  satisfying a linear difference equation  $\mathcal{L}$  over  $\mathbb{C}(x)$ . The following are equivalent

Skolem Mahler Lech problem : the set {i|u(i) = 0} is a finite union of arithmetic progressions

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## Application of the pseudofield structure

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> Charlotte Hardouin

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Picard Vession pseudofields :

## Theorem (Larson-Tarft-1990)

Let  $u, v \in S$  two sequences each of them satisfying a difference equation over  $\mathbb{C}(x)$  and such that uv = 0. Then, there exist  $u_0, \ldots, u_{t-1}, v_0, \ldots, v_{t-1} \in S$  such that

- u (resp.v) is the interlacing of the u<sub>i</sub>(resp. v<sub>i</sub>)
- for all *i* either  $u_i = 0$  or  $v_i = 0$

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## Comparing "special functions" and "abstract solutions"

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Then, there exists a Picard-Vessiot extension for  $\sigma(Y) = AY$  containing Z.

#### xamples

 $K = \bigcup_{n \in \mathbb{N}} \mathbb{C}(z^{1/p^n})$  and  $\sigma(f(z)) = f(z^p)$ . Generating series for automatic sequences belong to some  $L = \mathbb{C}((z^{1/p^k}))$ . One has  $L^{\sigma} = \mathbb{C}$ .

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Picard Vession pseudofields :

## Second case : $L^{\sigma} \neq K^{\sigma}$

If Z satisfies an algebraic relation over K, there exists a Picard-Vessiot extension for  $\sigma(Y) = AY$  where a solution vector satisfies the same relation.

#### xamples

 $\mathcal{C} = \mathbb{C}(x) \subset \operatorname{Mer}(\mathbb{C}) = L$  with  $\sigma(f(x)) = f(x+1)$ . Then  $\mathbb{C}(x)^{\sigma} = \mathbb{C} \neq \operatorname{Mer}(\mathbb{C})^{\sigma}$ . If  $\Gamma(x)$  is algebraic over  $\mathbb{C}(x)$ here exists a Picard-Vessiot extension for  $\sigma(y) = xy$ containing a non zero solution algebraic solution.

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