Edge modes, degeneracies, and topological numbers in non-Hermitian systems

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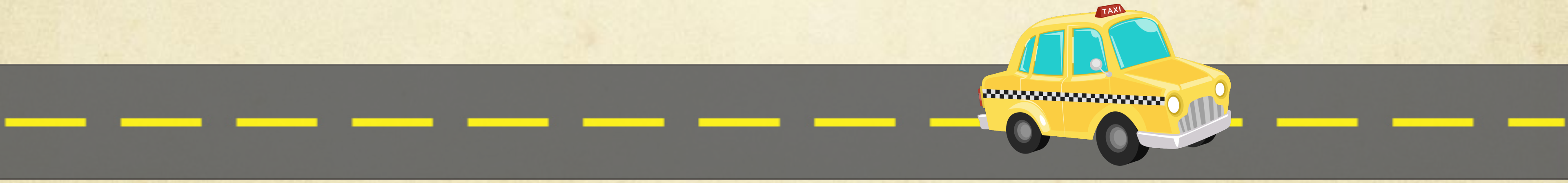

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Motivation

Topological:
quantum Hall effects,
topological insulators,
Dirac points
(topological/Chern/
winding numbers,
Berry curvature)

?

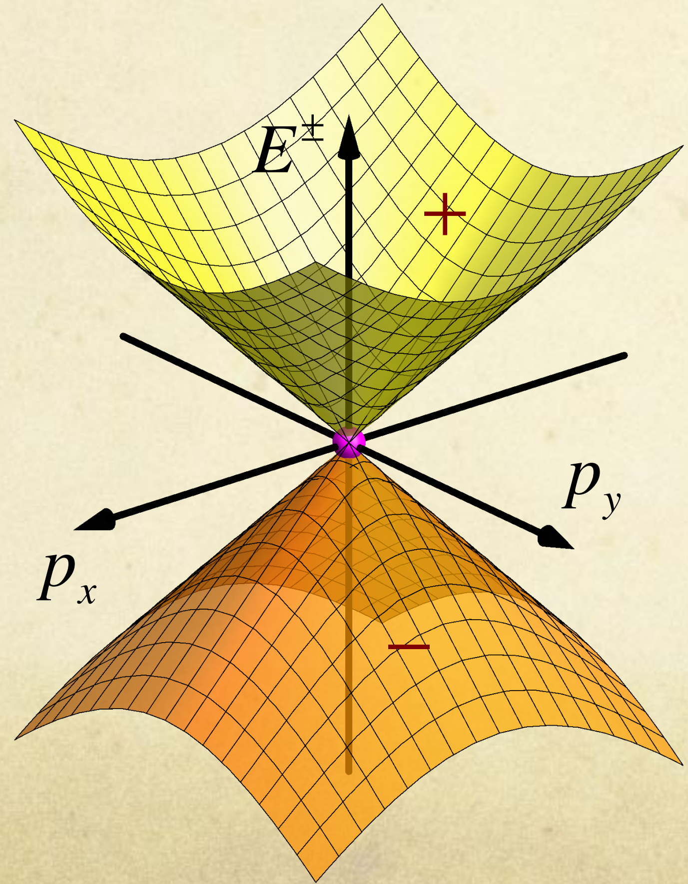
Non-Hermitian:
PT-symmetry
nonreciprocity
anomalous lasing
Exceptional points
(chirality, Berry phase)

Rudner & Levitov *PRL* (2009); Esaki *et al. PRB* (2011); Diehl *et al. Nat. Phys.* (2011); Malzard *et al. PRL* (2015); Zeuner *et al. PRL* (2015); Lee *PRL* (2016)

Degeneracies and topology in Hermitian systems

Hermitian degeneracies

Generic spectral degeneracies in Hermitian systems are conical intersections (Dirac or Weyl points). They have co-dimension of 3, i.e., generically appear in 3D parameter space:



$$E^{\pm} = \pm p = \pm\sqrt{\mathbf{p}\cdot\mathbf{p}}$$

$$\mathbf{p} = \left(p_x, p_y, p_z\right)$$

Hermitian degeneracies

Near the degeneracy in 3D momentum space, Hamiltonian can be written in a Weyl-like form:

$$\hat{H} = s_1 p_x \hat{\sigma}_x + s_2 p_y \hat{\sigma}_y + s_3 p_z \hat{\sigma}_z \equiv \mathbf{B}_{\text{eff}} \cdot \hat{\boldsymbol{\sigma}}$$

where $s_i = \pm 1$.

It generates Berry-curvature monopole in the degeneracy:

$$\mathbf{F}_B^{\pm} = \pm s \frac{\mathbf{p}}{p^3}$$

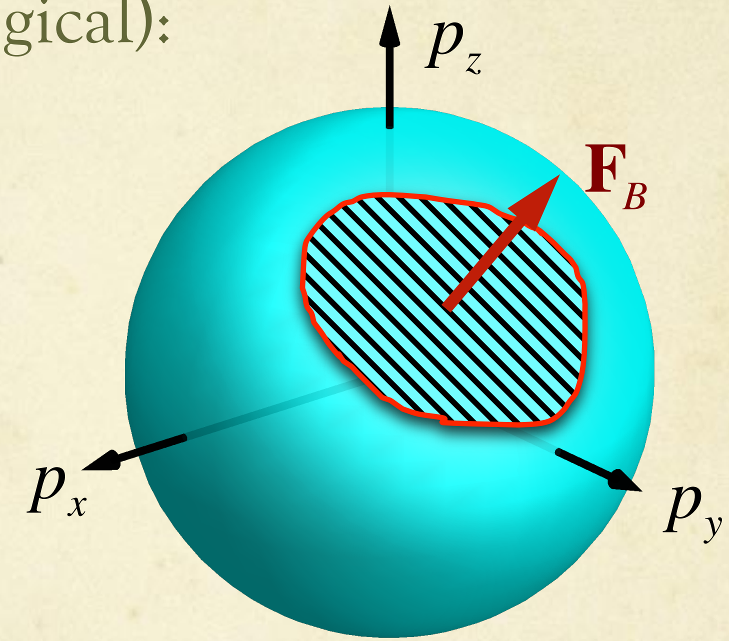
where $s = \operatorname{sgn}(s_1 s_2 s_3)$.

Berry, *Proc. R. Soc. A* (1984)

Berry phase and Chern numbers

The Berry curvature produces Berry phase, which is geometric (not topological):

$$\Phi_B^{\pm} = \iint \mathbf{F}_B^{\pm} d^2\mathbf{p}$$



But the Chern number is topological:

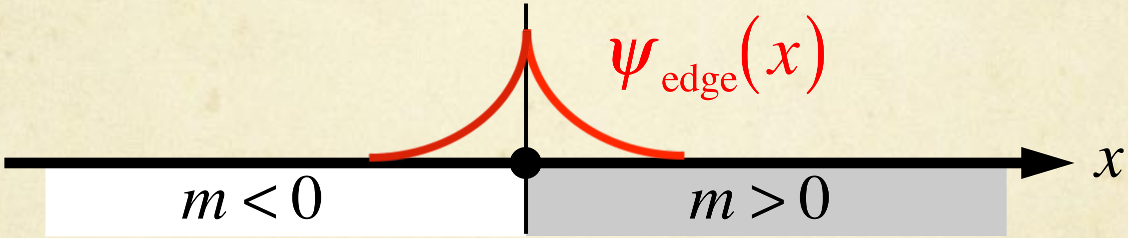
$$C^{\pm} = \frac{1}{2\pi} \oiint \mathbf{F}_B^{\pm} d^2\mathbf{p} = \pm s$$

Topological edge modes

Jackiw–Rebbi example of topological edge modes in a 1D Dirac-like Hamiltonian:

$$\hat{H} = \hat{p}_x \hat{\sigma}_x + m \hat{\sigma}_z \qquad m(x) = \begin{cases} -m_1, & x < 0 \\ +m_2, & x > 0 \end{cases}$$

This system has a gapped bulk spectrum $E^{\pm} = \pm\sqrt{p_x^2 + m^2}$ and a zero-energy chiral edge mode at the interface:



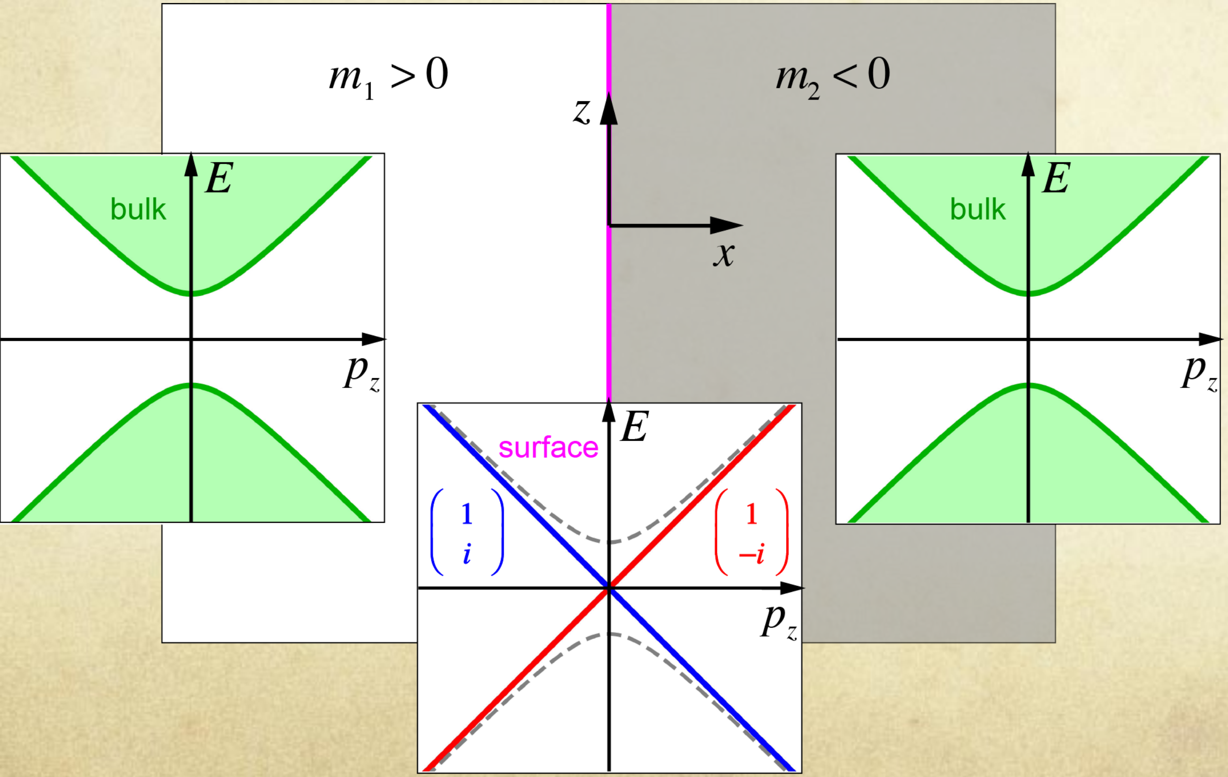
$$\psi_{\text{edge}} \propto \begin{pmatrix} 1 \\ -i \end{pmatrix} \exp(-|mx|), \quad E_{\text{edge}} = 0$$

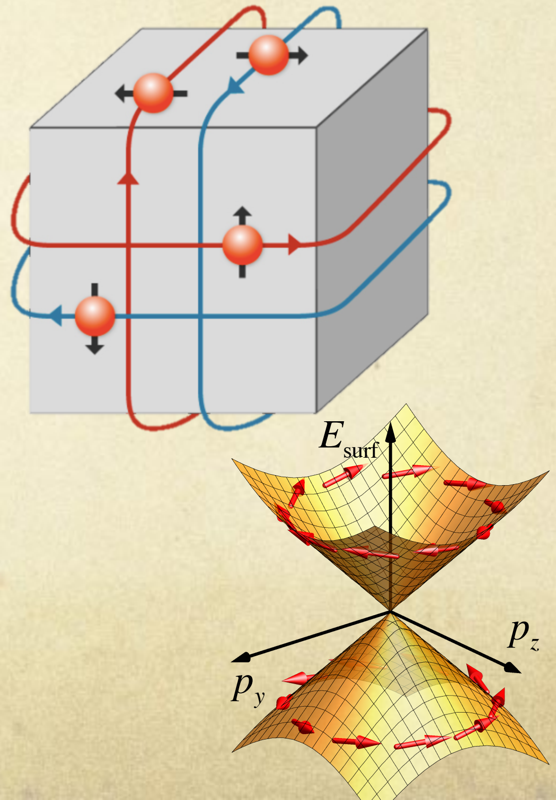
Jackiw & Rebbi *PRD* (1976); Shen et al. *SPIN* (2011); Schnyder *et al. PRB* (2008)

Topological edge modes

Similar edge modes also exist in 2D and 3D versions of the Dirac equation, and these are protected by the topological winding number:

$$C_w = \frac{1}{2}\mathrm{sgn}(m)$$





QSHE and topological insulators

In general, Hermitian quantum Hamiltonians can be divided into 10 classes with various topological edge states and topological numbers:

		TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
Standard	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	−1	0	-	$\mathbb{Z}$	-
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	−1	1	-	-	$\mathbb{Z}$

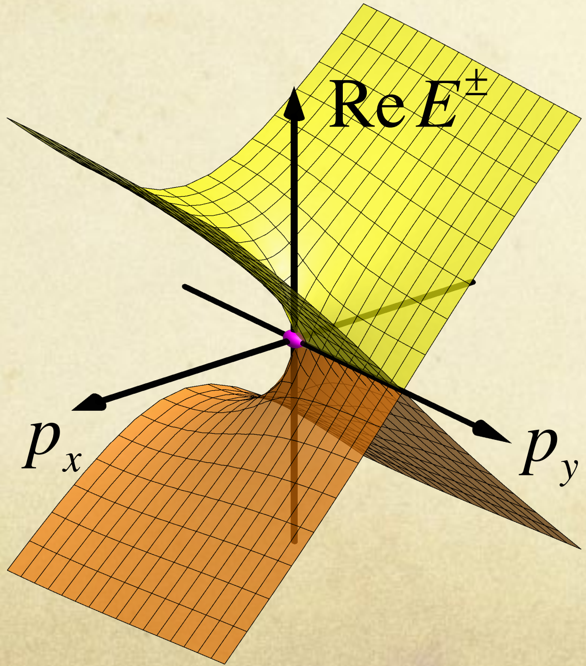
Schnyder *et al. PRB* (2008); Hasan & Kane *RMP* (2010)

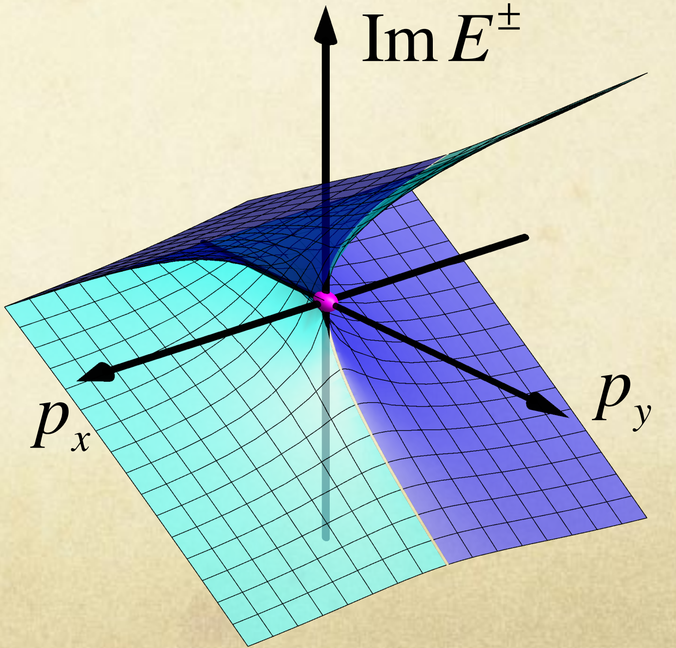
Non-Hermitian degeneracies:
Exceptional points

Non-Hermitian degeneracies

Generic spectral degeneracies in non-Hermitian systems are exceptional points (EP). These are branch points of the complex eigenvalues on 2D parameter space: $\mathbf{p} = (p_x, p_y)$.

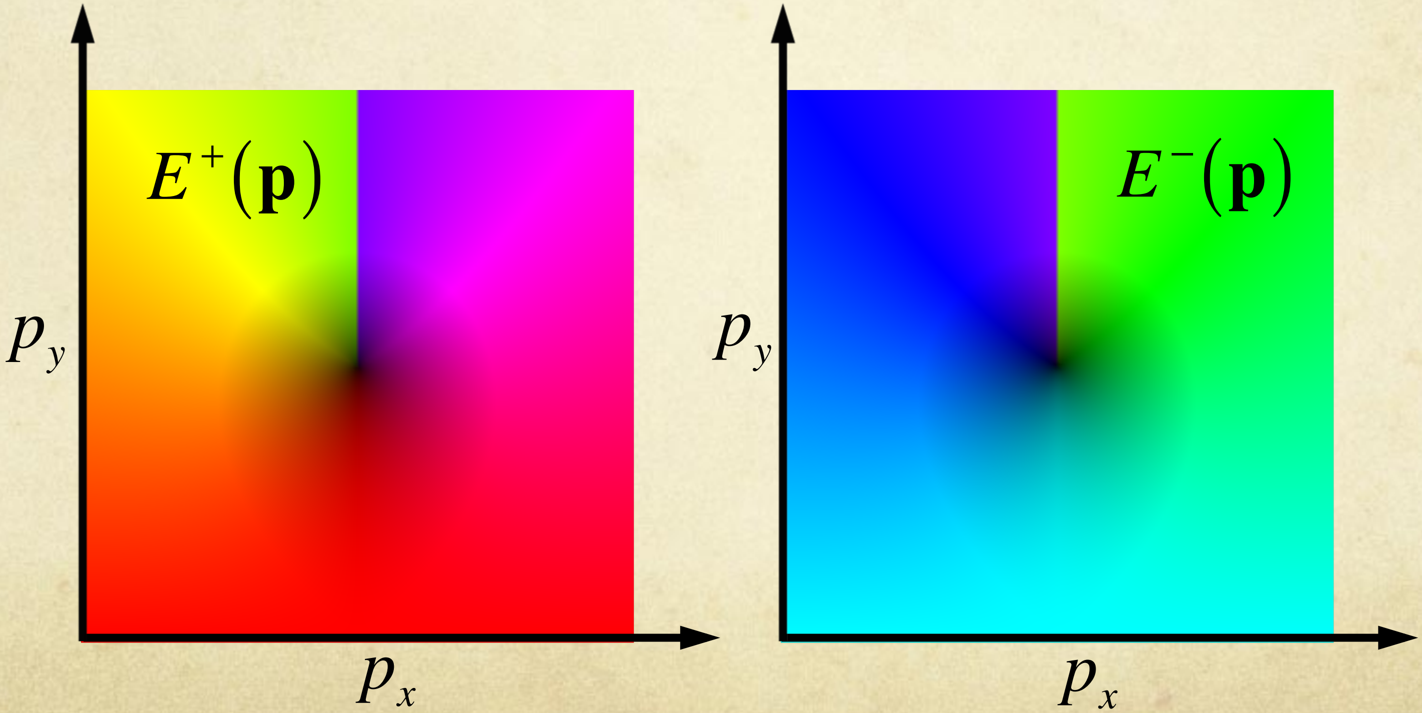
$$E^{\pm} \propto \pm\sqrt{p_x - i s p_y} = \sqrt{|\mathbf{p}|}\exp\left[i s \,\mathrm{Arg}(\mathbf{p})/2 \right] \qquad s = \pm 1$$

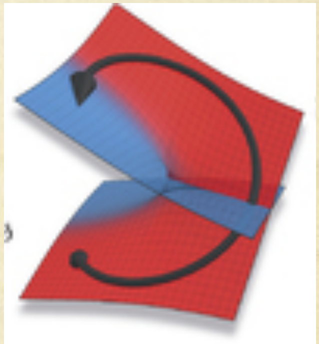




Non-Hermitian degeneracies

The eigenvalues form half-vortices of charge $s/2$ near the EP with π phase jumps to the opposite level. Thus, labeling of two energy levels is ambiguous: encircling the EP leads to the opposite level (branch cut is needed).





Berry, *Czech. J. Phys.* (2004); Heiss, *J. Phys. A* (2012)

Berry phase

Remarkably, continuously encircling the EP twice leads to the original level with π phase shift in the wavefunction:

$$\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \longrightarrow \begin{pmatrix} \pm\psi^- \\ \mp\psi^+ \end{pmatrix} \longrightarrow \begin{pmatrix} -\psi^+ \\ -\psi^- \end{pmatrix}$$

This is the Berry phase. In contrast to Hermitian systems, where the Berry phase is geometric, here it is topological and can provide a topological number similar to the Chern number (e.g., counting the number of degeneracies weighted by their charges).

Heiss *EPJD* (1999); Dembowski *et al. PRE* (2004); Gao et al. *Nature* (2015)

Non-Hermitian model with exceptional points

Non-Hermitian model

Typical non-Hermitian Hamiltonian with EPs is:

$$\hat{H} = \begin{pmatrix} p_x - isp_y & m \\ m & -p_x + isp_y \end{pmatrix}$$

$$\equiv (p_x - isp_y)\hat{\sigma}_z + m\hat{\sigma}_x \equiv \mathbf{B}_{\text{eff}} \cdot \hat{\boldsymbol{\sigma}}$$

$$\mathbf{p}_{\text{EP}} = (0, \pm|m|)$$

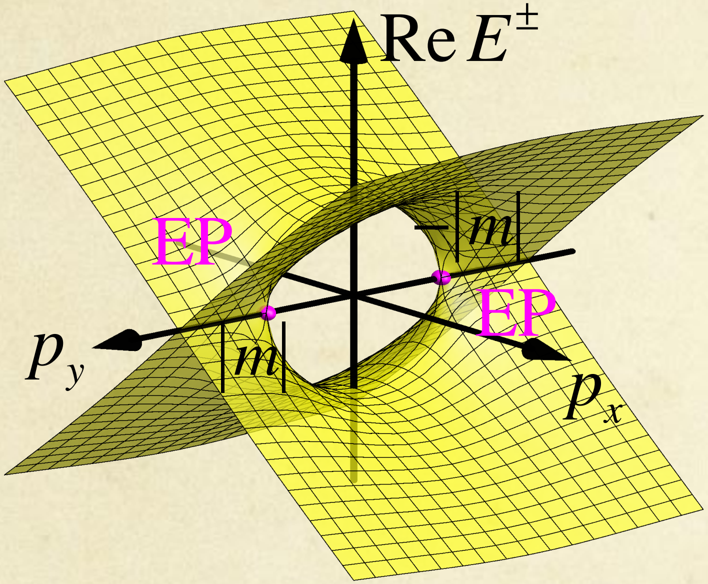
$$\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_z, \hat{\sigma}_x, \hat{\sigma}_y)$$

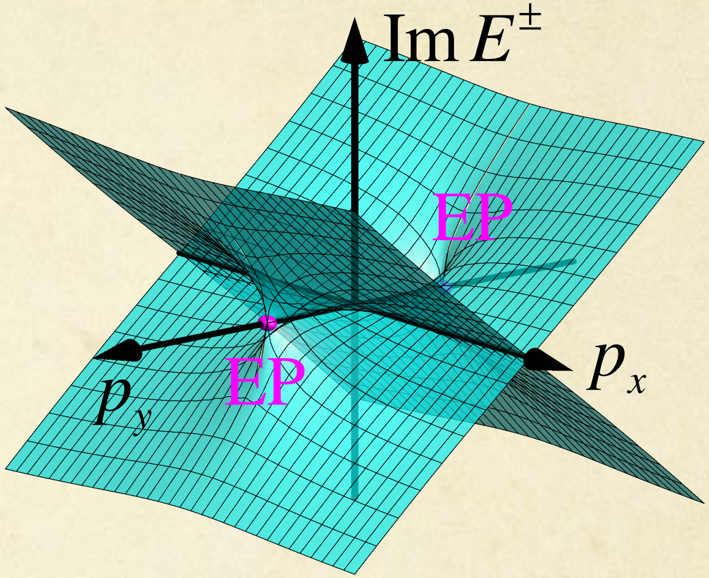
Such Hamiltonians describe many two-level systems, e.g., coupled resonators with loss/gain. However, here we consider this model in momentum space: $\mathbf{p} \to \hat{\mathbf{p}} = -i\nabla$.

Remarkably, Hermitian limit $p_y = 0$ yields the Jackiw-Rebbi 1D Dirac Hamiltonian.

Non-Hermitian model

Complex spectrum: $$E^{\pm} = \pm\sqrt{\left(p_x - i s p_y\right)^2 + m^2}$$





Eigenmodes:

$$\psi^{\pm} \propto \begin{pmatrix} 1 \\ m/\left(p_x - i s p_y + E^{\pm}\right) \end{pmatrix}$$

Chiral mode in the EPs:

$$\psi_{\text{EP}} \propto \begin{pmatrix} 1 \\ \pm i\,\text{sgn}(sm) \end{pmatrix}$$

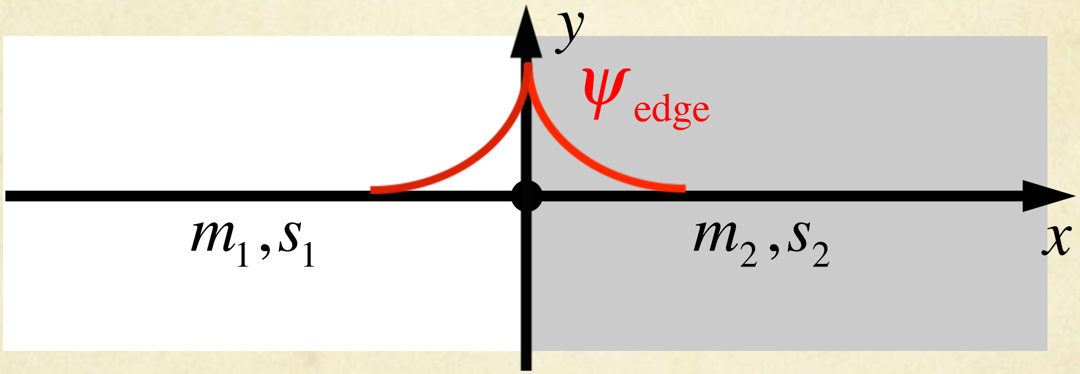
Topological edge modes

Edge modes

We now consider Schrodinger equation with Hamiltonian

$$\hat{H} = \left(\hat{p}_x - is\hat{p}_y\right)\hat{\sigma}_z + m\hat{\sigma}_x$$

Consider an interface between two media with different "masses" m and/or "non-Hermitian charges" s:



We seek edge modes:

$$\psi_{\text{edge}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{cases} \exp\left(iky + \gamma_1 x\right), & \text{Re}\,\gamma_1 < 0,\ x > 0 \\ \exp\left(iky + \gamma_2 x\right), & \text{Re}\,\gamma_2 > 0,\ x < 0 \end{cases}$$

Edge modes

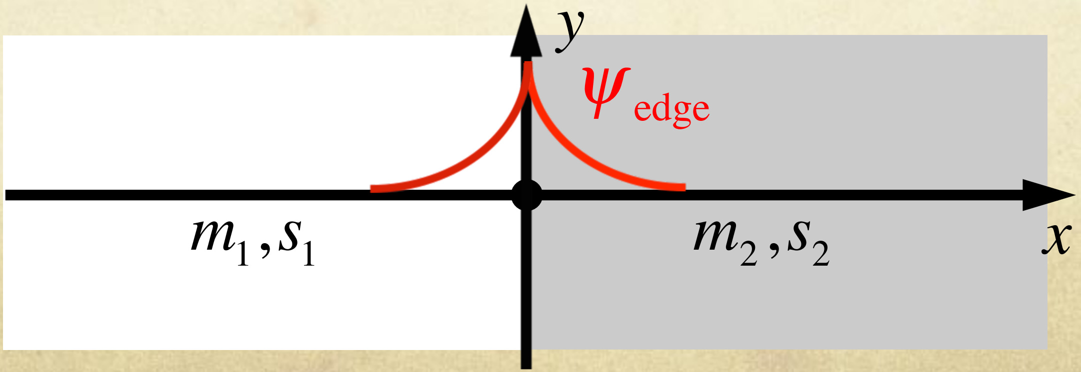
There are chiral zero-energy edge modes:

$$E_{\text{edge}} = 0\,, \quad (\beta / \alpha)_{\text{edge}} = \pm i$$

They exist when two simple real equations are satisfied:

$$-\gamma_1 = s_1 k \pm m_1\,, \quad -\gamma_2 = s_2 k \pm m_2$$

Despite their simplicity, these conditions result in a rather rich and nontrivial structure of edge modes.



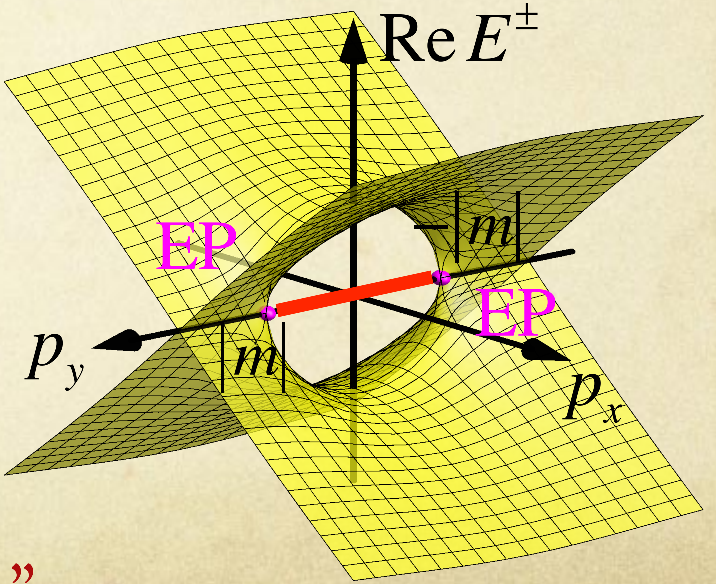
Edge modes

Simple case A (opposite "masses"):

$$s_1 = s_2 = s_1\,,\;\; m_1 = -m_2 = m$$

There is one chiral edge mode in the "gapped" region between the EPs of the bulk spectra:

$$k \in \left(-|m|, |m|\right)$$



In the "Hermitian" point $k = 0$ this becomes the Jackiw–Rebbi edge mode in 1D Dirac system.
We call this mode "Hermitian–like".

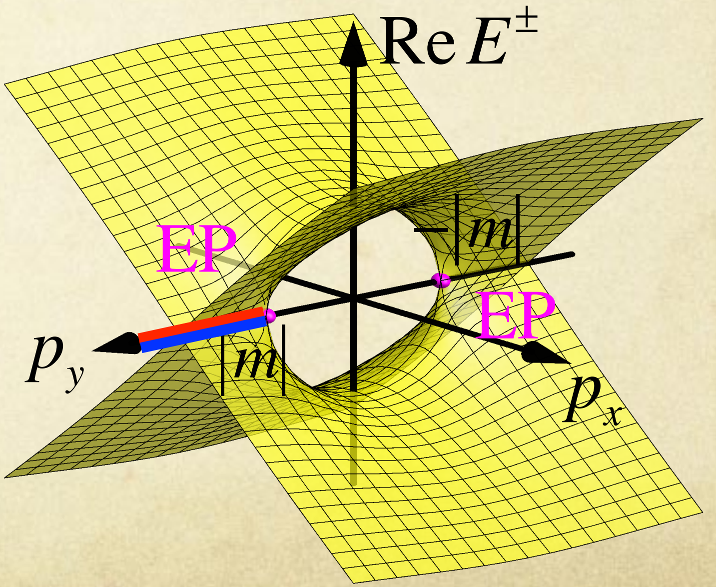
Edge modes

Simple case B (opposite "charges"):

$$s_1 = -s_2 = s_1\,,\quad m_1 = m_2 = m$$

There are two edge modes (with opposite chiralities) in one of the "ungapped" regions of the bulk spectra:

$$k \in \mathrm{sgn}(s)\left(|m|, \infty\right)$$



These modes are essentially "non-Hermitian" and defective, i.e., left eigenvectors do not exist:

$$\psi^{\dagger}_{\mathrm{edge}} \hat{H} \neq 0$$

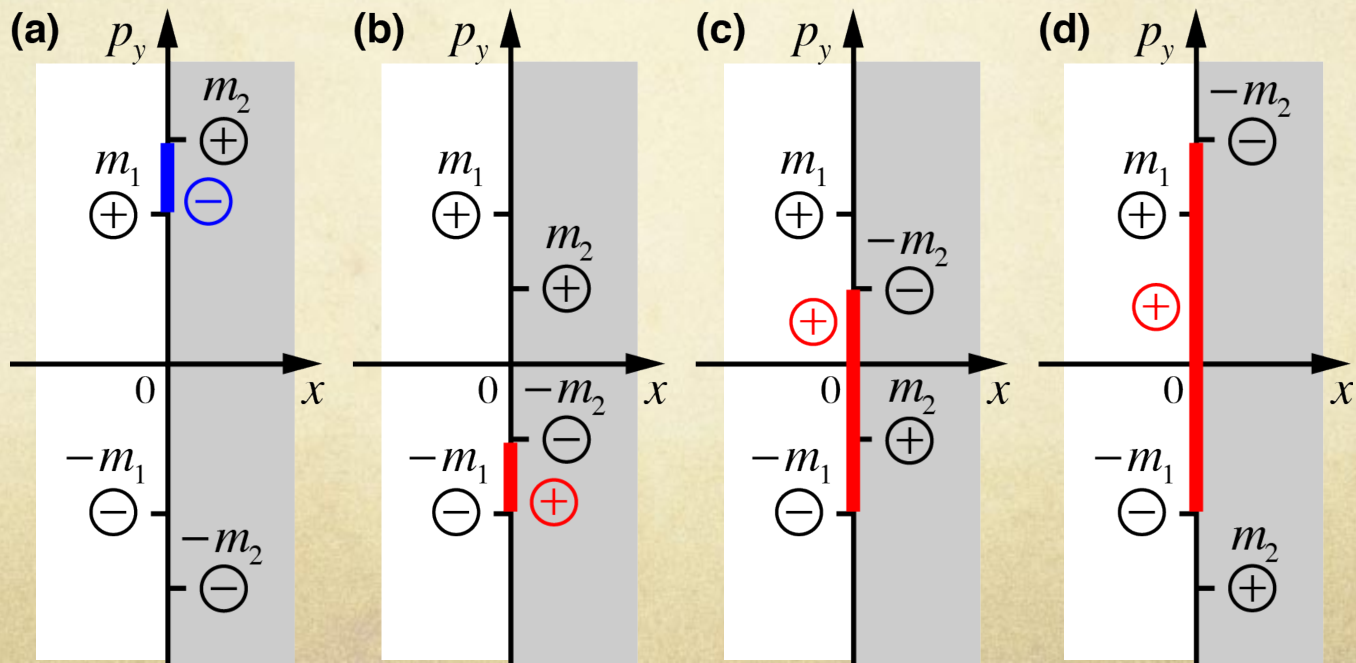
cf. Malzard et al. *PRL* (2015); Lee *PRL* (2016)

Edge modes

When $|m_1| \neq |m_2|$ (EPs of two media do not coincide), the situation is more complicated. Edge modes can also exist in "mixed" regions: "gapped"/"ungapped" bulk spectra.

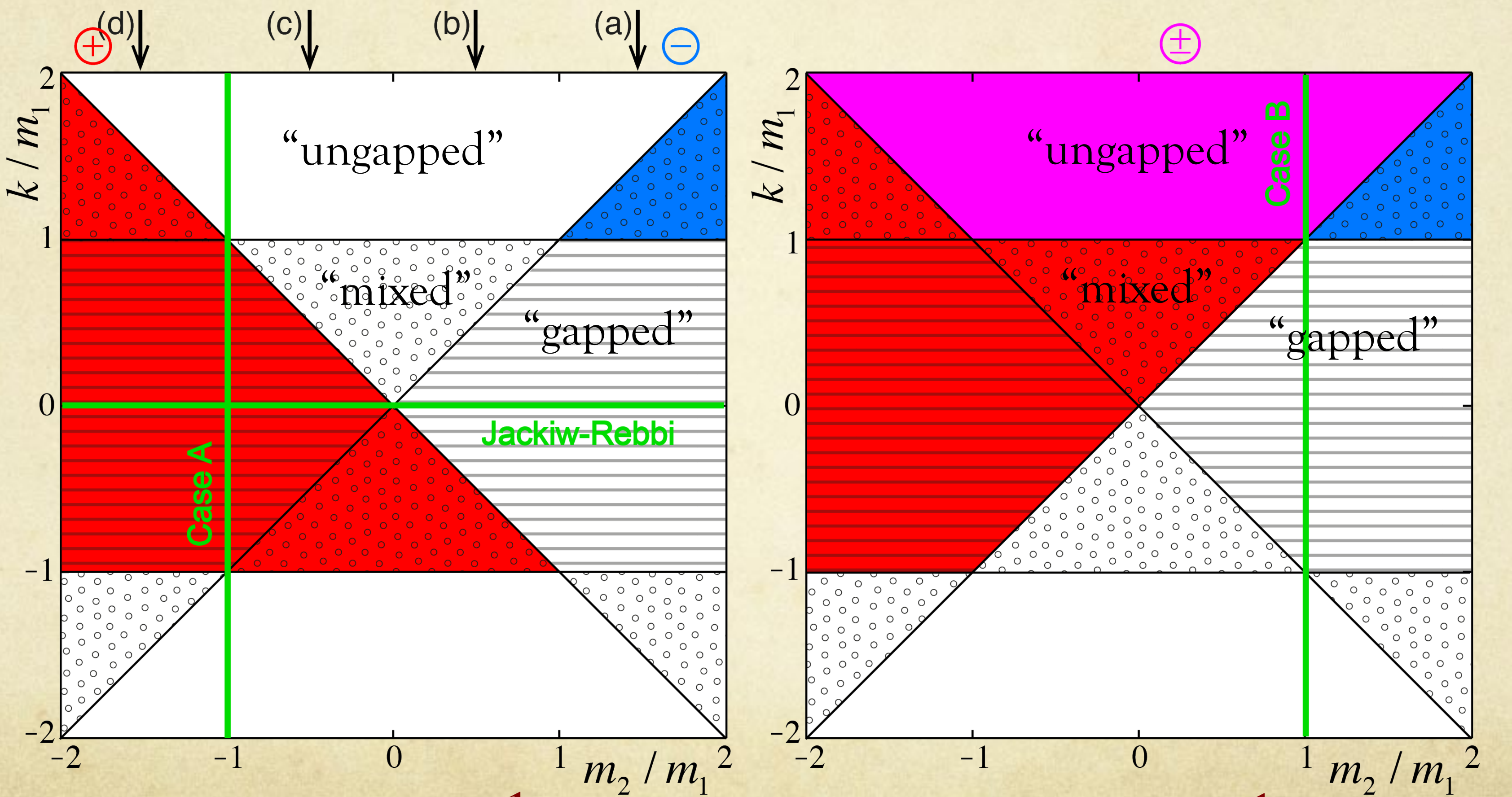
In any case, edge modes occupy regions between the two EPs in the two media:

$$s_1 = s_2 = 1$$
$$m_1 > 0$$



Edge modes

A complete picture of the regions of existence of chiral edge modes is presented in the following phase diagrams:



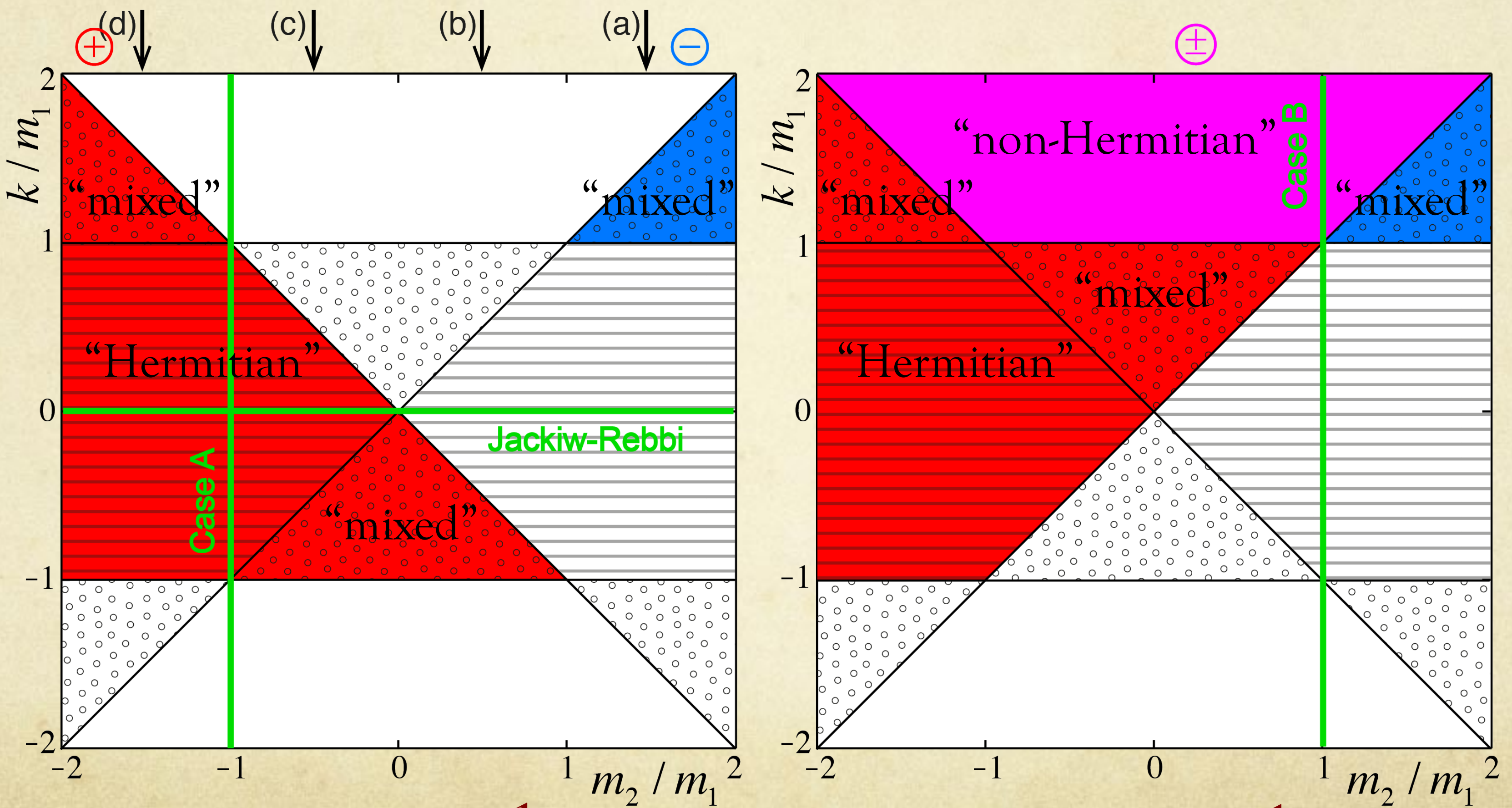
$s_1 = s_2 = 1$

$s_1 = -s_2 = 1$

$m_1 > 0$

Edge modes

Edge modes in the "gapped/ungapped/mixed" regions can be called "Hermitian/non-Hermitian/mixed":



$s_1 = s_2 = 1$ $\quad$ $s_1 = -s_2 = 1$

$m_1 > 0$

Topological winding numbers

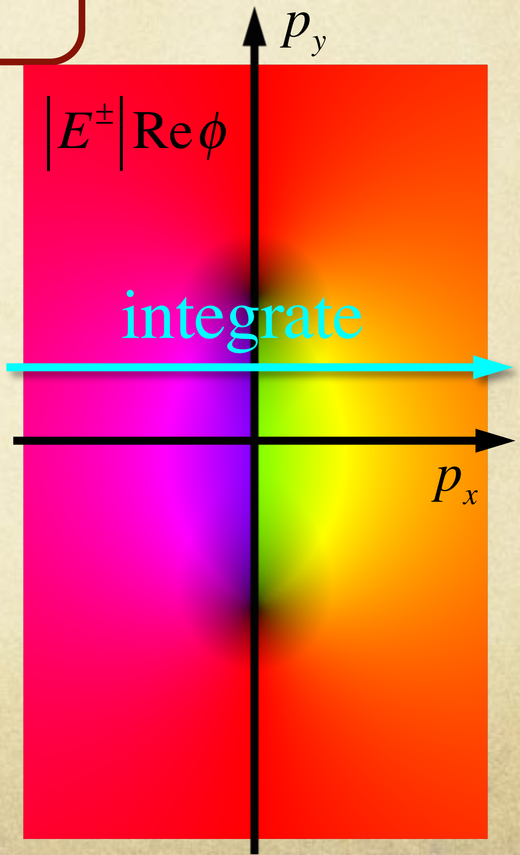
First topological number (Berry phase)

We calculate non-Hermitian version of the Berry phase for complex 2D "magnetic" field (continuous):

$$\mathbf{B}_{\text{eff}} = E^{+}\left(\cos\phi, \sin\phi, 0\right), \quad C_{w1} = \frac{1}{2\pi}\int_{p_x=-\infty}^{p_x=+\infty} d\phi$$

This yields Jackiw-Rebbi winding number describing "Hermitian-like" edge states:

$$C_{w1} = \begin{cases} -\frac{1}{2}\,\text{sgn}(m), & |p_y| < |m| \\ 0, & |p_y| > |m| \end{cases}$$



Garrison & Wright *PLA* (1988); Mailybaev *et al. PRA* (2005); Esaki *et al. PRB* (2011)

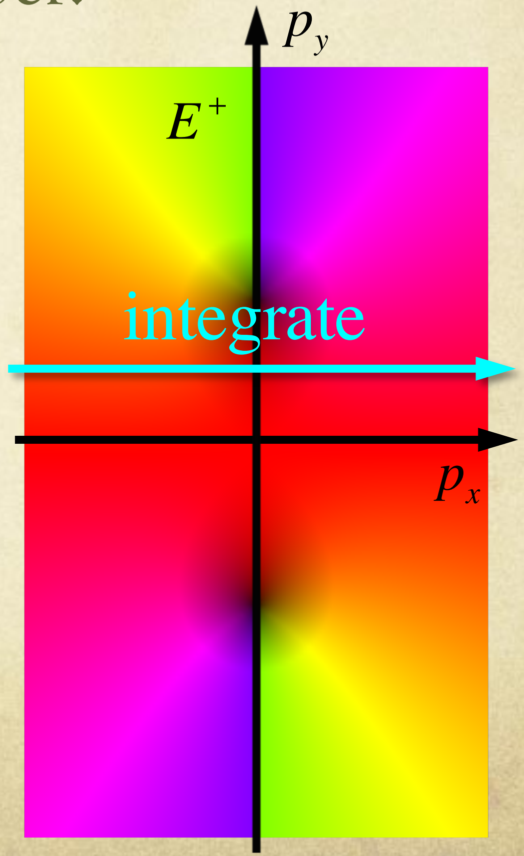
Second topological number

Berry phase stems from the direction of $\mathbf{B}_{\mathrm{eff}}$, i.e., $\phi(\mathbf{p})$. However, in the non-Hermitian case, its length $E^{+}(\mathbf{p})$ is also complex and forms vortices near EPs. This is characterized by the second winding number:

$$C_{w2} = \frac{1}{2\pi}\int_{p_x=-\infty}^{p_x=+\infty} d\,\mathrm{Arg}\left(E^{\pm}\right)$$

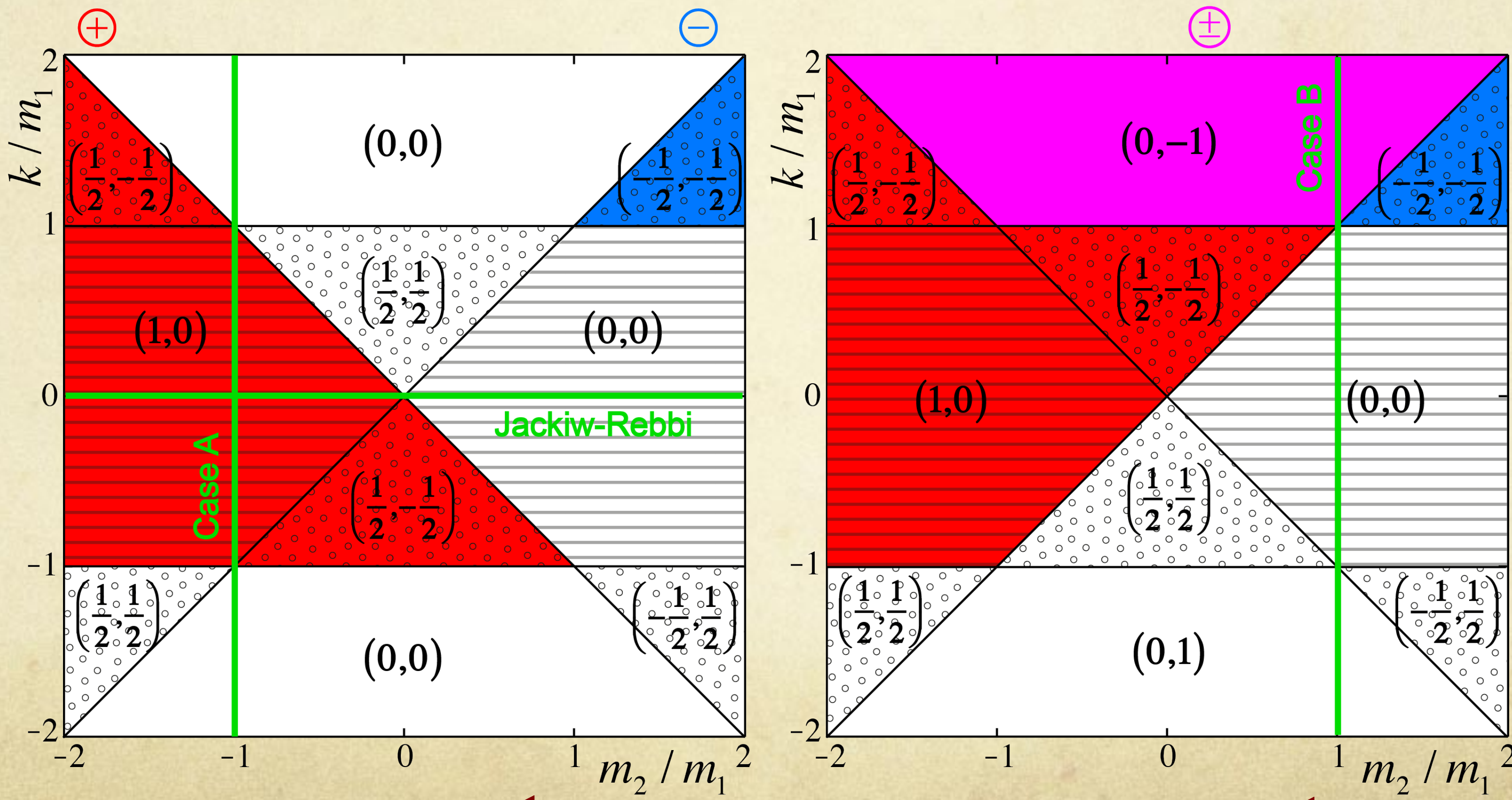
This p_y–asymmetric number describes "non-Hermitian" and "mixed" modes:

$$C_{w2} = \begin{cases} 0, & |s p_y| < |m| \\ -\frac{1}{2}\mathrm{sgn}\left(s p_y\right), & |s p_y| > |m| \end{cases}$$



Phase diagram revisited

The contrasts $(\Delta C_{w1}, \Delta C_{w2})$ at the interface describe all the edge modes and their asymmetry in k. $N = 2|\Delta C_{w2}|$ "non-Hermitian" or "mixed" edge modes exists when $\Delta C_{w2} < 0$.



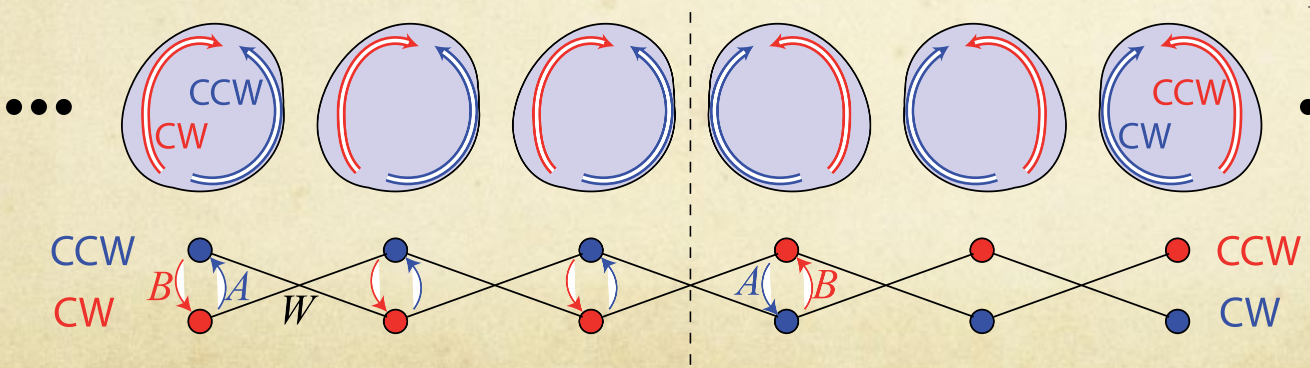
$s_1 = s_2 = 1$ $\qquad$ $s_1 = -s_2 = 1$

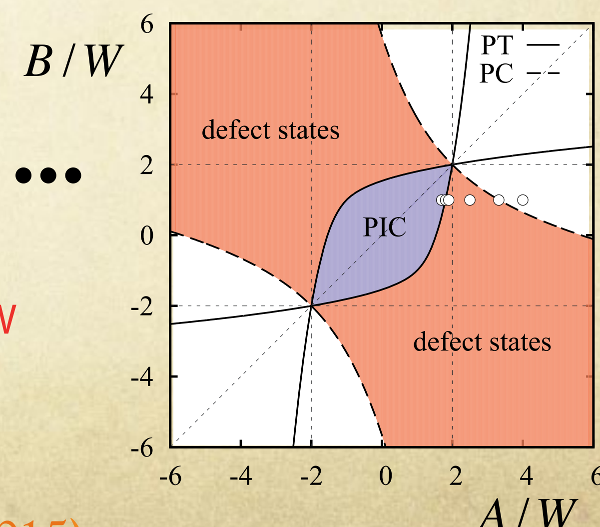
Non-Hermitian lattice systems of coupled resonators

Passive ring resonator chains

Asymmetric backscattering between anti/clockwise modes in a 1D chain of resonators is described by non-Hermitian Hamiltonian with EPs and topological end modes:

$$H(k) = \begin{pmatrix} \Omega & A + 2W\cos k \\ B + 2W\cos k & \Omega \end{pmatrix}$$

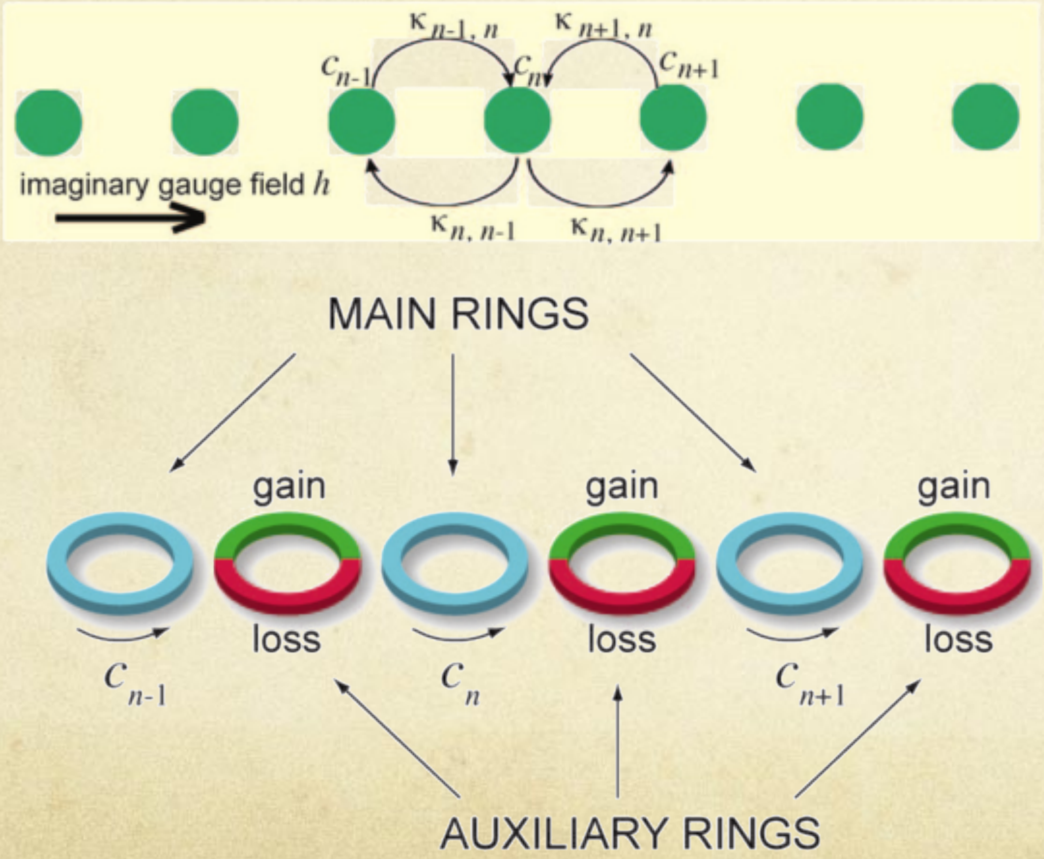




Malzard, Poli, & Schomerus, *PRL* (2015)

Ring resonators with gain/loss

Natural non–Hermiticity appears in systems with gain/loss. E.g., coupled ring resonators with loss and gain:

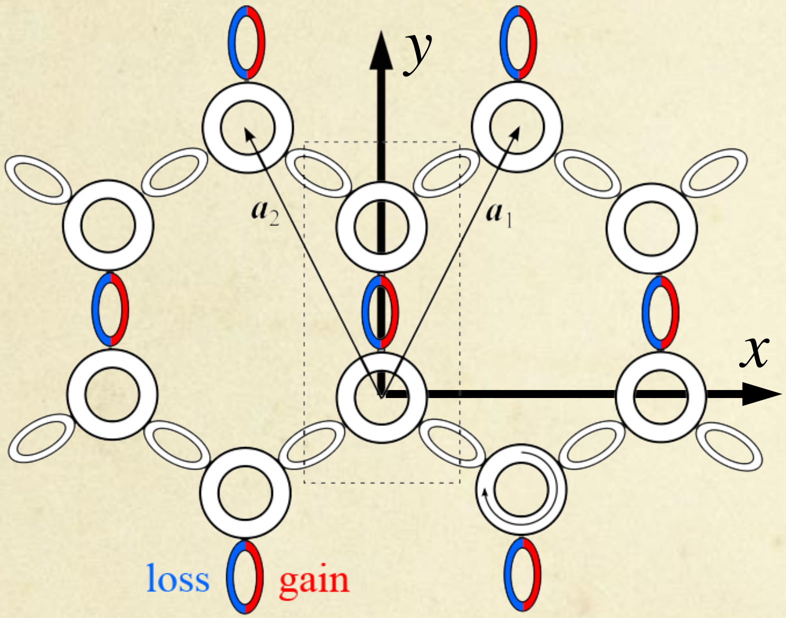


$$\kappa_{n,n+1} = \kappa \exp(-\gamma)$$
$$\kappa_{n,n-1} = \kappa \exp(\gamma)$$

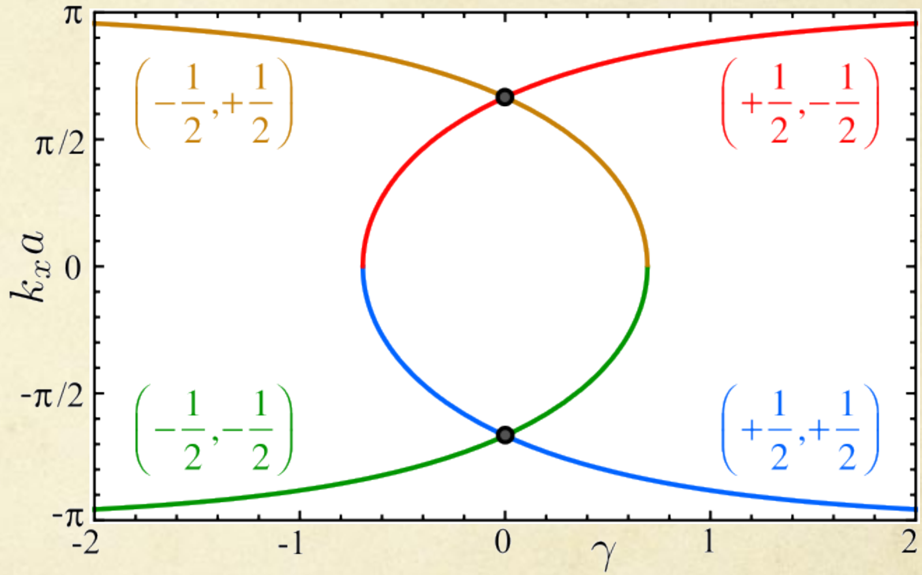
Longhi, Gatti, & Della Valle, *PRB* (2015)

2D honeycomb lattice model

Using this idea, we construct a 2D lattice of ring resonators with non-Hermitian couplings:



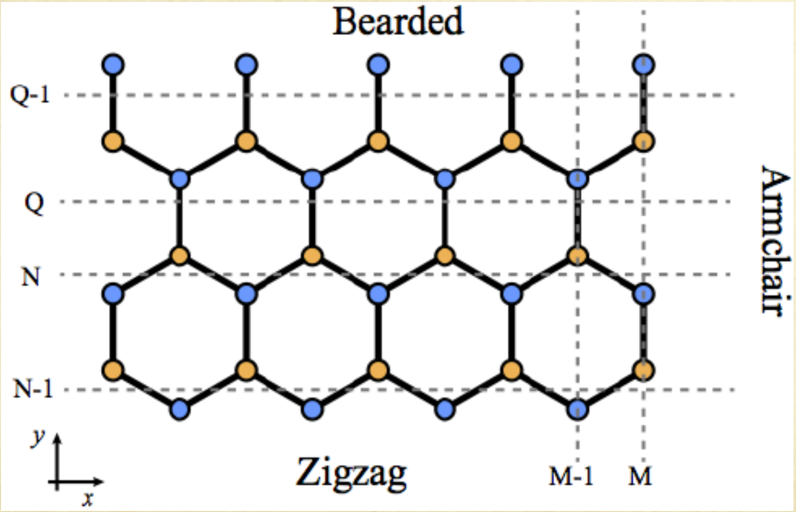
EPs with their "charges"

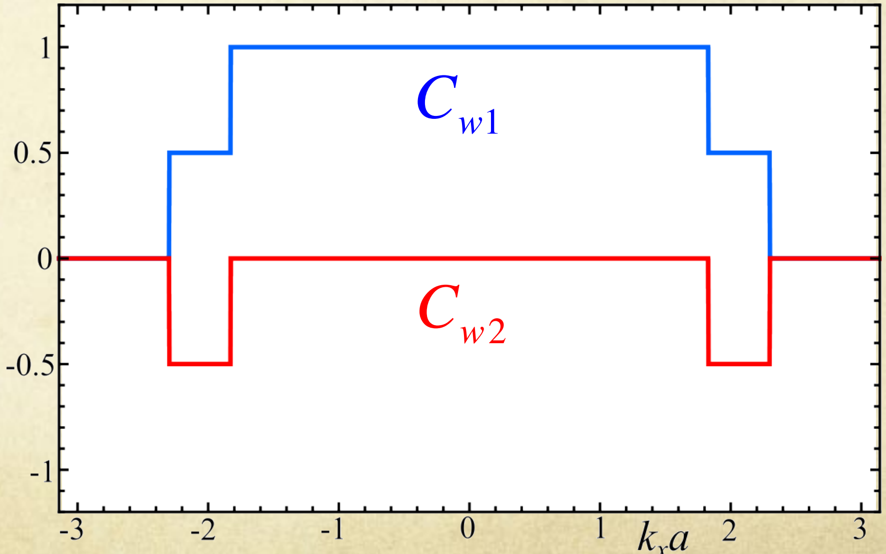


$$\hat{H} = \begin{pmatrix} 0 & \kappa e^{\gamma} + e^{i\mathbf{k}\cdot\mathbf{a}_1} + e^{i\mathbf{k}\cdot\mathbf{a}_2} \\ \kappa e^{-\gamma} + e^{-i\mathbf{k}\cdot\mathbf{a}_1} + e^{-i\mathbf{k}\cdot\mathbf{a}_2} & 0 \end{pmatrix}$$

Winding numbers: bearded edge

Winding numbers are obtained integrating over 1D Brillouin zone perpendicular to edge. Results depend on the edge orientation (cf. Zak phase and edge states in graphene). The two fractional topological numbers describe different types of non–Hermitian edge modes.





Delplace, Ullmo, & Montambaux, PRB (2011), Li et al., CPB (2014)

Conclusions

- ✓ Degeneracies (exceptional points) play a crucial role in topological properties of non-Hermitian systems.
- ✓ In contrast to the Hermitian case, EPs and chiral edge modes are characterized by two topological numbers.
- ✓ These numbers originate from the singularities in the direction and length of the complex field $\mathbf{B}_{\mathrm{eff}}$.
- ✓ They describe different types of chiral edge modes: “Hermitian-like”, “non-Hermitian”, and “mixed”.
- ✓ Non-Hermitian systems are characterized by richer morphology of degeneracies, topological numbers, and chiral edge modes, as compared to the Hermitian case.

Thank you!

Index theorem

Alternatively, we can define a Hermitian Hamiltonian sharing the same zero modes: $\hat{\mathcal{H}} = \hat{H}^\dagger \hat{H}$.

In our model, this yields:

$$\hat{\mathcal{H}} = \left|\hat{\mathbf{p}} - \hat{\sigma}_y s\mathbf{A}(\mathbf{r})\right|^2 + \hat{\sigma}_y B(\mathbf{r})$$

$$\mathbf{A} = (0, m), \;\; B = \partial_x A_y - \partial_y A_x$$

A domain wall in "mass" field $m(\mathbf{r})$ is equivalent to a nonzero "magnetic flux". Zero modes hosted by arbitrary analytic mass fields can be counted using the Aharonov–Casher index theorem.

Aharonov & Casher, *PRA* (1979)